

Ranking Fuzzy Numbers Based on Lexicographical Ordering

B. Farhadinia

Abstract—Although so far, many methods for ranking fuzzy numbers have been discussed broadly, most of them contained some shortcomings, such as requirement of complicated calculations, inconsistency with human intuition and indiscrimination. The motivation of this study is to develop a model for ranking fuzzy numbers based on the lexicographical ordering which provides decision-makers with a simple and efficient algorithm to generate an ordering founded on a precedence. The main emphasis here is put on the ease of use and reliability. The effectiveness of the proposed method is finally demonstrated by including a comprehensive comparing different ranking methods with the present one.

Keywords—Ranking fuzzy numbers, Lexicographical ordering.

I. INTRODUCTION

SINCE ranking and comparing fuzzy numbers plays an important role in many fuzzy optimization problems and decision-making procedure, several kinds of fuzzy rankings have appeared in the literature and various methods [1][2][7][8][9][13] have been proposed to solve such problems. Many of the existing ranking methods have been thoroughly reviewed in [3] and more recently in [4]. The most commonly used technique is to map fuzzy numbers by an appropriate transformation into real numbers and subsequently realize a comparison of them. Up to now, no single existing ranking approach stands obviously superior to the others because almost each method involves different drawbacks such as the lack of discrimination, difficulty in implementing, inconsistency with human intuition and producing counterintuitive ordering. In this concern, any approach has to be judged by its own merit. At present, the specific rational properties proposed in [18] provide a boost to the comparison of fuzzy number ranking approaches. Besides, many other ranking methods have been investigated based on various concepts like the area measurement [9][10][11][14][21], the α -level set [6], the centroid point of fuzzy numbers [9][17], the mean and standard deviation values [12], the radius of gyration [10], the coefficient of variance [7], Hamming distance [19], the total integral values [13], the two crisp maximizing and minimizing barriers [8], the possibility and probability measure of fuzzy events [12], the artificial neural networks [16], the area compensation [11], the area between the centroid and original points [17], the signed distance [21], and so on.

Except a few approaches, others are not simple in calculation procedure and their results are not satisfactory in every case. To overcome such problems, the purpose of this paper is to

present a ranking method based on lexicographical ordering. This present contribution is outlined as follows: Section 2 is devoted to give the definitions of fuzzy numbers and some related results of fuzzy arithmetic on LR fuzzy numbers. In Section 3 the fuzzy lexicographical ordering is introduced and then the satisfaction of some reasonable properties are stated. The latter section is completed by constructing an ordering algorithm. Finally, Section 4 presents the effectiveness of the proposed method by comparing it with the other known ranking approaches.

II. PRELIMINARIES

To start with, it is reviewed some preliminary notions, definitions and results in fuzzy sets theory to be used throughout this article. These are stated as follows.

Definition 1. A fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}} : X \rightarrow [0, 1]$, and denoted by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}.$$

Definition 2. An α -cut or α -level of the set $\tilde{A}$, is the crisp set $[\tilde{A}]_{\alpha} = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 3. The support of a fuzzy set $\tilde{A}$, is the crisp set $\text{Supp}(\tilde{A}) = \{x \in X \mid \mu_{\tilde{A}}(x) > 0\}$.

Definition 4. A fuzzy set $\tilde{A}$ is called a *fuzzy number* if the following conditions are satisfied:

- (i) $\tilde{A}$ is normal. It means that there exists an $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$;
- (ii) $\mu_{\tilde{A}}$ is quasi-convex. It means that for every $x, y \in X$ $\mu_{\tilde{A}}(\gamma x + (1 - \gamma)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$, $\gamma \in [0, 1]$;
- (iii) $\mu_{\tilde{A}}$ is upper semi-continuous.
- (iv) $\text{Supp}(\tilde{A})$ is bounded in X .

Let $\mathcal{F}(R)$ be the set of all fuzzy numbers on R .

It is well known that the α -level set of a fuzzy number is a closed and bounded interval $[\underline{A}(\alpha), \overline{A}(\alpha)]$, where $\underline{A}(\alpha)$ and $\overline{A}(\alpha)$ denote respectively the left- and right-hand endpoints of $[\tilde{A}]_{\alpha}$.

Definition 5. Let $L, R : [0, \infty) \rightarrow [0, 1]$ be two upper semi-continuous, non-increasing functions satisfying $L(0) = R(0) = 1$, $L(1) = R(1) = 0$, invertible on $[0, 1]$. Samples of $L(\cdot)$ and $R(\cdot)$ can be found in [22]. Furthermore, let $\underline{a}$ and $\overline{a}$ be real positive numbers. The fuzzy number $\tilde{a} \in \mathcal{F}(R)$ is an LR fuzzy number if

$$\tilde{a}(x) \equiv \mu_{\tilde{a}}(x) = \begin{cases} L(\frac{\underline{a}-x}{\underline{a}}), & x \leq \underline{a}, \\ 1, & \underline{a} \leq x \leq \overline{a}, \\ R(\frac{x-\overline{a}}{\overline{a}}), & x \geq \overline{a}. \end{cases} \quad (1)$$

B. Farhadinia is with the Department of Mathematics, University of Mo-haghegh Ardabili, P. O. Box. 179, Ardabil, Iran, e-mail: farhadinia@uma.ac.ir, bfarhadinia@yahoo.com.au

It is symbolically written $\tilde{a} = (\underline{a}, \bar{a}, \underline{a}, \bar{a})_{LR}$, where $\underline{a}$ and $\bar{a}$ are called the mean values satisfying $\underline{a} \leq \bar{a}$ and $\underline{a}$, $\bar{a}$ are the left and right spreads, respectively.

Definition 6. An LR fuzzy number $\tilde{a} \in \mathcal{F}(R)$ is said to be a *trapezoidal fuzzy number* if the functions L and R are linear. Under the latter assumption, a real-numbered quadruple $(\underline{a}, \bar{a}, \underline{a}, \bar{a})_{LR}$ represents a trapezoidal fuzzy number.

Definition 7. One obtains the so-called *triangular fuzzy number* when the mean values of a trapezoidal fuzzy number fulfilling $\underline{a} = \bar{a} = a$. In this case, triple $(a, \underline{a}, \bar{a})_{LR}$ characterizes the triangular fuzzy number $\tilde{a} \in \mathcal{F}(R)$.

Remark 1. If $\tilde{a} \in \mathcal{F}(R)$ is a trapezoidal fuzzy number then

$$[\tilde{a}]_{\alpha} = [\underline{a} - L^{-1}(\alpha)\underline{a}, \bar{a} + R^{-1}(\alpha)\bar{a}], \quad 0 \leq \alpha \leq 1, \quad (2)$$

needless to say, in the case that $\tilde{a} \in \mathcal{F}(R)$ is a triangular fuzzy number then

$$[\tilde{a}]_{\alpha} = [a - (1 - \alpha)\underline{a}, a + (1 - \alpha)\bar{a}], \quad 0 \leq \alpha \leq 1, \quad (3)$$

when $L(x) = R(x) = 1 - x$.

III. FUZZY LEXICOGRAPHICAL ORDERING

Although problem of ordering of fuzzy numbers has been discussed broadly so far and many approaches have been extensively proposed, they contained some shortcomings and in some situations they may fail to exhibit the consistency of human intuition. Also most of the existing approaches are not simple in calculation procedure.

In order to overcome the mentioned problems, specially the complexity of the computational procedures, it is here presented a ranking method based on lexicographical ordering. A question to ask is why lexicographic order is implemented to compare fuzzy numbers. This is due to it providing decision-makers with a simple and efficient algorithm that formulates an ordering founded on a precedence and also the lexicographic order is a total order on ground terms when the precedence is total. In other words, any two fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{F}(R)$ are comparable in the sense that one has either $\tilde{a} \prec \tilde{b}$ or $\tilde{a} \succ \tilde{b}$ or $\tilde{a} \approx \tilde{b}$.

Before proceeding to present the main results, a number of definitions are required in this stage.

Definition 8. Let $\tilde{a} \in \mathcal{F}(R)$ be a fuzzy number. Define

- (i) $C(\tilde{a}) = \inf\{x \in \text{Supp}(\tilde{a}); \tilde{a}(x) = 1\}$,
- (ii) $L(\tilde{a}) = \inf \text{Supp}(\tilde{a})$,
- (iii) $W(\tilde{a}) = |\text{Supp}(\tilde{a})|$,
- (iv) $S(\tilde{a}) = \int \tilde{a}(x) dx$,
- (v) $V(\tilde{a}) = (C(\tilde{a}), L(\tilde{a}), W(\tilde{a}), S(\tilde{a}))$.

Definition 9. For $X, Y \in R^n$, the lexicographical ordering on R^n , denoted by $\prec_{lex}$, is defined by requiring

$$X = (x_1, x_2, \dots, x_n) \prec_{lex} Y = (y_1, y_2, \dots, y_n)$$

if and only if there is $1 \leq i \leq n$ so that

$$x_j = y_j \text{ holds for } j < i \text{ and } x_i < y_i.$$

Furthermore, $\preceq_{lex}$ means that $X \prec_{lex} Y$ or $X = Y$.

On the basis of the latter definitions, the following *fuzzy lexicographic order* is established on $\mathcal{F}(R)$:

- (i) $\tilde{a} \prec \tilde{b}$ if and only if $V(\tilde{a}) \prec_{lex} V(\tilde{b})$,
- (ii) $\tilde{a} \preceq \tilde{b}$ if and only if $V(\tilde{a}) \preceq_{lex} V(\tilde{b})$,
- (iii) $\tilde{a} \approx \tilde{b}$ if and only if $V(\tilde{a}) \preceq_{lex} V(\tilde{b})$ and $V(\tilde{a}) \succeq_{lex} V(\tilde{b})$,
if and only if $V(\tilde{a}) = V(\tilde{b})$.

Obviously, the relations $\tilde{a} \succ \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$ can be viewed as $\tilde{b} \prec \tilde{a}$ and $\tilde{b} \preceq \tilde{a}$, respectively.

It is not hard to see that the fuzzy lexicographic order $\prec$ has the following reasonable properties [18]:

- (i) $\tilde{a} \preceq \tilde{a}$,
- (ii) If $\tilde{a} \prec \tilde{b}$ and $\tilde{b} \prec \tilde{c}$, then $\tilde{a} \prec \tilde{c}$.
- (iii) If $\tilde{a} \prec \tilde{b}$, then $\tilde{b} \prec \tilde{a}$ can not hold.
- (iv) If $\sup \text{Supp}(\tilde{a}) \leq \inf \text{Supp}(\tilde{b})$ then $\tilde{a} \prec \tilde{b}$.
- (v) If $\tilde{a} \preceq \tilde{b}$ and $\tilde{b} \preceq \tilde{a}$ then $\tilde{a} \approx \tilde{b}$.
- (vi) If $\tilde{a} \preceq \tilde{b}$ then $\tilde{a}^* \preceq \tilde{b}^*$ where $\tilde{a}^* = \tilde{a} + \tilde{c} \in \mathcal{F}(R)$ and $\tilde{b}^* = \tilde{b} + \tilde{c} \in \mathcal{F}(R)$.

It is to be noted that unlike different types of the existing ranking orders, the fuzzy lexicographic order is so easy to handle the calculations. As particular case, if $\tilde{a}$ in parametric representation is given by $(\underline{a}, \bar{a}, \underline{a}, \bar{a})_{LR}$ then $C(\tilde{a}) = \underline{a}$, $L(\tilde{a}) = \underline{a}$, $W(\tilde{a}) = \bar{a} - \underline{a} + \bar{a} + \underline{a}$ and $S(\tilde{a}) = \bar{a} - \underline{a} + \frac{1}{2}(\bar{a} + \underline{a})$. Another advantage which arises from this ordering is a simple and fast algorithm to determine stepwise the precedence ordering of the fuzzy numbers. However, the algorithm may terminate successfully at step one while the comparison is complete.

Algorithm 1. (Fuzzy lexicographic order)

- Step 1: Input two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. Then, according to Definition 8 do:
- Step 2: Compare $C(\tilde{a})$ and $C(\tilde{b})$. If $C(\tilde{a}) = C(\tilde{b})$ then goto Step 3. Otherwise stop and the larger $C(*)$ is, the larger corresponding fuzzy number $*$ is.
- Step 3: Compare $L(\tilde{a})$ and $L(\tilde{b})$. If $L(\tilde{a}) = L(\tilde{b})$ then goto Step 4. Otherwise stop and the larger $L(*)$ is, the larger corresponding fuzzy number $*$ is.
- Step 4: Compare $W(\tilde{a})$ and $W(\tilde{b})$. If $W(\tilde{a}) = W(\tilde{b})$ then goto Step 5. Otherwise stop and the larger $W(*)$ is, the larger corresponding fuzzy number $*$ is.
- Step 5: Compare $S(\tilde{a})$ and $S(\tilde{b})$. If $S(\tilde{a}) = S(\tilde{b})$ then stop and output $\tilde{a} \approx \tilde{b}$. Otherwise the larger $S(*)$ is, the larger corresponding fuzzy number $*$ is.

IV. NUMERICAL EXAMPLES

In this section, the aim is to demonstrate that the results of the fuzzy lexicographic order are generally more reasonable than the outcomes when ranked with the other approaches. In order to confirm the reasonability of the propose ordering, the several experiments performed in the literature [2][5][7][8][9][20][21] are again considered in showing the capability of the approach.

Example 1. Consider the two fuzzy numbers $\tilde{a} = (2, 2, 0.1, 0.1)_{LR}$ and $\tilde{b} = (3, 3, 0.9, 1)_{LR}$. One can observe from Figure 1 that $\tilde{a}$ should be intuitively smaller than $\tilde{b}$. The ranking outcome with the CV index proposed in [7] is $\tilde{a} \succ \tilde{b}$ which is not rational. It is easy to see that the

TABLE I
COMPARATIVE RESULTS OF EXAMPLE 2.

Fuzzy number	Fuzzy lexicographic order	Wang
$\tilde{a} = (6, 6, 1, 1)_{LR}$	$V(\tilde{a}) = (6, 5, -, -)$	0.2500
$\tilde{b} = (6, 6, 0.1, 1)_{LR}$	$V(\tilde{b}) = (6, 5.9, -, -)$	0.5339
$\tilde{c} = (6, 6, 0, 1)_{LR}$	$V(\tilde{c}) = (6, 6, -, -)$	0.5625
Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$
Sign distance P=1	Sign distance P=2	Chu-Tsao
6.12	8.52	3.000
12.45	8.82	3.126
12.50	8.85	3.085
$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{c} \prec \tilde{b}$
Cheng distance	CV index	
6.0210	0.0280	
6.3490	0.0098	
6.3519	0.0089	
$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{c} \prec \tilde{b} \prec \tilde{a}$	

fuzzy lexicographic order leads to $\tilde{a} \prec \tilde{b}$ at the first step of Algorithm 1 resulting $C(\tilde{a}) = 2 < C(\tilde{b}) = 3$.

Example 2. Consider the three fuzzy numbers $\tilde{a} = (6, 6, 1, 1)_{LR}$, $\tilde{b} = (6, 6, 0.1, 1)_{LR}$ and $\tilde{c} = (6, 6, 0, 1)_{LR}$. The corresponding ranking outcomes with the methods in [17] and [1] are as the same as with the fuzzy lexicographic order, that is, $\tilde{a} \prec \tilde{b} \prec \tilde{c}$. Meanwhile, by the methods in [9] and [7] the results are respectively $\tilde{a} \prec \tilde{c} \prec \tilde{b}$ and $\tilde{c} \prec \tilde{b} \prec \tilde{a}$. However, as follows from Figure 2, the two latter relations are not rational and reasonable. More comparative results are available in Table 1.

Example 3. Consider the sets of three fuzzy numbers as follows:

Set 1: $\tilde{a} = (0.5, 0.5, 0.1, 0.5)_{LR}$, $\tilde{b} = (0.7, 0.7, 0.3, 0.3)_{LR}$ and $\tilde{c} = (0.9, 0.9, 0.5, 0.1)_{LR}$.

Set 2: $\tilde{a} = (0.4, 0.7, 0.1, 0.2)_{LR}$, $\tilde{b} = (0.7, 0.7, 0.4, 0.2)_{LR}$ and $\tilde{c} = (0.7, 0.7, 0.2, 0.2)_{LR}$.

Set 3: $\tilde{a} = (0.5, 0.5, 0.2, 0.2)_{LR}$, $\tilde{b} = (0.5, 0.8, 0.2, 0.1)_{LR}$ and $\tilde{c} = (0.5, 0.5, 0.2, 0.4)_{LR}$.

Set 4: $\tilde{a} = (0.4, 0.7, 0.4, 0.1)_{LR}$, $\tilde{b} = (0.5, 0.5, 0.3, 0.4)_{LR}$ and $\tilde{c} = (0.6, 0.6, 0.5, 0.2)_{LR}$.

The fuzzy numbers considered in Set 1-4 are depicted in Figure 3-6, respectively. A comparison with other methods is provided in Table 2.

V. CONCLUSION

The argument of this application explained that how the fuzzy lexicographical ordering makes the method easier to program and the ranked results more intuitively to produce. Particular emphasis was put on the ease of use and reliability. In addition, it was presented a comprehensive experimental study comparing different ranking methods to demonstrate the effectiveness of the proposed method.

TABLE II
COMPARATIVE RESULTS OF EXAMPLE 3.

Authors/method	Fuzzy number	Set 1	Set 2
Chooibineh-Li	$\tilde{a}$	0.3330	0.4580
	$\tilde{b}$	0.5000	0.5830
	$\tilde{c}$	0.6670	0.6670
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$
Yager	$\tilde{a}$	0.600	0.575
	$\tilde{b}$	0.700	0.650
	$\tilde{c}$	0.800	0.700
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$
Chen	$\tilde{a}$	0.3375	0.4315
	$\tilde{b}$	0.5000	0.5625
	$\tilde{c}$	0.6670	0.6250
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$
Baldwin-Guild	$\tilde{a}$	0.30	0.27
	$\tilde{b}$	0.33	0.27
	$\tilde{c}$	0.44	0.37
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \approx \tilde{b} \prec \tilde{c}$
Chu-Tsao	$\tilde{a}$	0.29900	0.28470
	$\tilde{b}$	0.35000	0.32478
	$\tilde{c}$	0.39930	0.35000
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$
Yao-Wu	$\tilde{a}$	0.600	0.575
	$\tilde{b}$	0.700	0.650
	$\tilde{c}$	0.800	0.700
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$
Sign distance p=1	$\tilde{a}$	1.20	1.15
	$\tilde{b}$	1.40	1.30
	$\tilde{c}$	1.60	1.40
	Results	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$	$\tilde{a} \prec \tilde{b} \prec \tilde{c}$

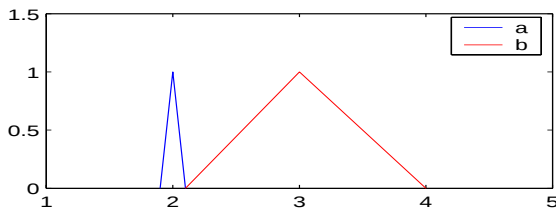


Fig. 1. Fuzzy numbers in Example 1.

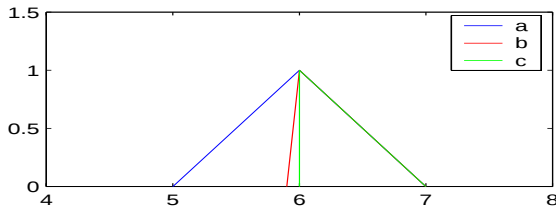


Fig. 2. Fuzzy numbers in Example 2.

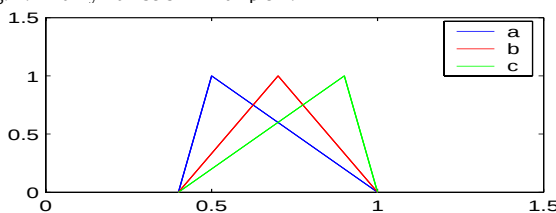


Fig. 3. Fuzzy numbers in Example 3, Set 1.

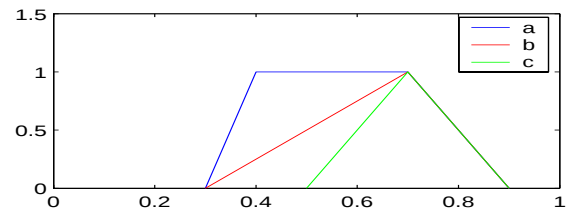


Fig. 4. Fuzzy numbers in Example 3, Set 2.

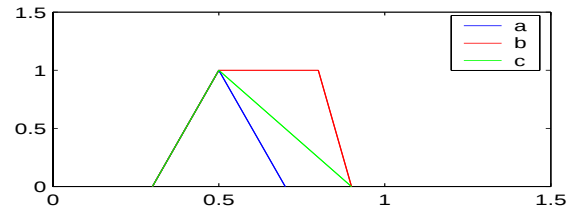


Fig. 5. Fuzzy numbers in Example 3, Set 3.

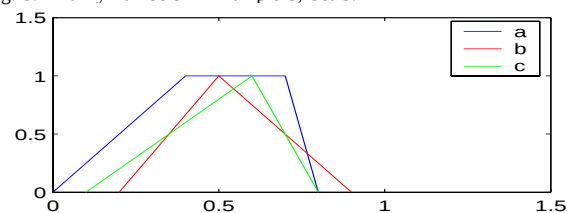


Fig. 6. Fuzzy numbers in Example 3, Set 4.

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