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Ranking Alternatives in Multi-Criteria Decision Analysis using Common Weights Based on Ideal and Anti-ideal Frontiers

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Abstract**—**One of the most important issues in multi-criteria decision analysis (MCDA) is to determine the weights of criteria so that all alternatives can be compared based on the collective performance of criteria. In this paper, one of popular methods in data envelopment analysis (DEA) known as common weights (CWs) is used to determine the weights in MCDA. Two frontiers named ideal and anti-ideal frontiers, instead of ideal and anti-ideal alternatives, are defined based on two new proposed CWs models. Ideal and anti-ideal frontiers are more flexible than that of alternatives. According to the optimal solutions of these two models, the distances of an alternative from the ideal and anti-ideal frontiers are derived. Then, a relative distance is introduced to measure the value of each alternative. The suggested models are linear and despite weight restrictions are feasible. An example is presented for explaining the method and for comparing to the existing literature.

Keywords**—**Anti-ideal frontier, Common weights (CWs), Ideal frontier, Multi-criteria decision analysis (MCDA)

I. Introduction

HUMAN always has to make decision for selecting an alternative among a bundle of alternatives when there are different criteria. For example, when choosing a job, different criteria are considered including income, social status, creativity, innovation, etc, and decision maker has to evaluate different alternatives based on the criteria. In practice, there are many states for calculating the collective performance of a group of alternatives based on a set of criteria. Related literature is in multi-criteria decision analysis [1] (MCDA). Some of widely used techniques for ranking alternatives are TOPSIS method [2] and analytic hierarchy process (AHP) method [3] and etc.

One other method for ranking alternatives is the use of data envelopment analysis (DEA) in which alternatives can be considered as decision making units (DMUs) with no input or with one input that has the same value of all DMUs. DEA without inputs or outputs was studied by Lovell and Pastor [4]. Because each alternative uses the most desirable weights for calculating its performance, usually there are more than one efficient alternative by DEA. Therefore, it is not possible to rank alternatives. For removing the mentioned problem, we utilize common weights (CWs) method in DEA to gain two sets of weights for criteria by linear models and then rank alternatives.

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Next key factor in MCDA is specifying an index for comparison alternatives. Usually, one ideal alternative is recognized and alternatives closer to the ideal alternative are preferred. Ideal alternative is a hypothetical alternative which has the most desirable of all criteria. Against, there is an anti-ideal alternative with the most undesirable value in all criteria. In this article, using ideal alternative and a CWs method we present a model known as ideal model and acquire a set of weights to make an ideal frontier. A Similar procedure is proposed to make anti-ideal frontier. All alternatives locate under the ideal frontier and above the anti-ideal frontier. Then the distances of each alternative from the ideal and anti-ideal frontiers are calculated and based on them a relative distance is defined to rank the alternatives. The proposed method is compared with the recently published method in this area [5], a method that also use the ideal and anti-ideal alternatives and CWs method for ranking alternatives.

This article includes following sections: In the second section, CWs method in DEA is introduced. The third section presents our proposed method including ideal and anti-ideal models. In the fourth section, we explain the method using a numerical example. Final section includes conclusions.

II. Common weights (CWs) in DEA

Charnes *et al.* [6] proposed CCR model for calculating the relative efficiency of DMUs, which uses several inputs for producing several outputs, as follows:

$$\begin{aligned} \max \ & \sum_{r=1}^{s} u_r y_{pr}, \\ s.t \ & \sum_{i=1}^{k} v_i x_{pi} = 1, \\ & \sum_{r=1}^{s} u_r y_{jr} - \sum_{i=1}^{k} v_i x_{ji} \le 0, \quad j = 1,\ldots,n, \\ & u_r \ge \varepsilon, \quad v_i \ge \varepsilon, \quad i = 1,\ldots,k, \ r = 1,\ldots,s, \end{aligned} \qquad (1)$$

where and are respectively the *rth* output and *ith* input of DMU_j, in which is the weight of the *rth* output and is the weight of the *ith* input and is a non-Archimedes value. In this model, each unit uses the most desirable weights for calculating its performance. Usually, there is more than one efficient unit. Because efficient units have the same efficiency score one, it is not possible to rank them. One of the lucrative methods in DEA for removing this hourly problem is using CWs approach. The following model was proposed by Saati [7] to determine a common set of weights as:

$$\max \ \phi,$$

$$s.t \quad \sum_{r=1}^{s} u_r y_{jr} - \sum_{i=1}^{k} v_i x_{ji} \le 0, \quad j = 1,...,n,$$
$$U_r \phi \le u_r \le U_r(1-\phi), \quad r = 1,...,s,$$
$$V_i \phi \le v_i \le V_i(1-\phi), \quad i = 1,...,k, \tag{2}$$

in which $U_r = 1/\max_j \{y_{jr}\}$ $(r = 1,...,s)$ and $V_i = 1/\max_j \{x_{ji}\}$ $(i = 1,...,m)$. After the optimal weights $(v_1^*,...v_k^*, u_1^*,..,u_s^*)$ are gained, the performance of the *pth* DMU is measured by $\sum_{r=1}^{s} u_r^* y_{pr} \Big/ \sum_{i=1}^{k} v_i^* x_{pi}$.

III. Ideal and Anti-Ideal Frontiers in MCDA

Consider a set of *n* alternatives with *m* criteria, *s* desirable and *k* undesirable criteria, as the following Table I. According to Table I, y_{jr} $(r = 1,...,s)$ and x_{ji} $(i = 1,...,k)$ are respectively the values of the *rth* desirable criterion and the *ith* undesirable criterion for the *jth* $(j = 1,...,n)$ alternative.

Ideal alternative denoted by I^+ has the *rth* $(r = 1,...,s)$ desirable criterion $y_r^{max} = \max_j\{y_{jr}\}$ and the *ith* $(i = 1,...,k)$ undesirable criterion $x_i^{min} = \min_j\{x_{ji}\}$. Anti-ideal alternative denoted by I^- has the *rth* desirable criterion $y_r^{min} = \min_j\{y_{jr}\}$ and the *ith* undesirable criterion $x_i^{max} = \max_j\{x_{ji}\}$.

In the proposed method in this article, we consider alternatives as DMUs which desirable and undesirable criteria are as outputs and inputs of them, respectively. In forthcoming part, an ideal frontier is defined which is a hyperplane passing through the origin and ideal alternative and anti-ideal frontier which is a hyperplane that passes through the origin and anti-ideal alternative. For gaining an ideal frontier, we consider constraint $\sum_{r=1}^{s} u_r y_r^{max} - \sum_{i=1}^{k} v_i x_i^{min} \le 0$ in model (2). Therefore, the transformed model is as follows:

$$\max \ \phi,$$
$$s.t \sum_{r=1}^{s} u_r y_r^{max} - \sum_{i=1}^{k} v_i x_i^{min} \le 0,$$
$$\phi \le u_r y_r^{max} \le (1-\phi), \quad r = 1,...,s,$$
$$\phi \le v_i x_i^{min} \le (1-\phi), \quad i = 1,...,k \tag{3}$$

Theorem 1: Model (3) is always feasible.

Proof: It is obvious when $\phi = 0$, $u_r = \dfrac{1}{s\, y_r^{max}}$ $(r = 1,...,s)$, $v_i = \dfrac{1}{k\, x_i^{min}}$ $(i = 1,...,k)$. ■

TABLE I
Alternatives and Desirable and Undesirable Criteria

Alternative	Desirable criteria (Outputs)			Undesirable criteria (Inputs)		
	1	…	s	1	…	k
1	y_{11}	…	y_{1s}	x_{11}	…	x_{1k}
2	y_{21}	…	y_{2s}	x_{21}	…	x_{2k}
⋮	⋮	…	⋮	⋮	…	⋮
J	y_{j1}	…	y_{js}	x_{1j}	…	x_{jk}
⋮	⋮	…	⋮	⋮	…	⋮
N	y_{n1}	…	y_{ns}	x_{n1}	…	x_{nk}
I^+	y_1^{max}	…	y_s^{max}	x_1^{min}	…	x_k^{min}
I^-	y_1^{min}	…	y_s^{min}	x_1^{max}	…	x_k^{max}

If $\bar{\phi}$, $\bar{u}_r$ $(r = 1,...,s)$, $\bar{v}_i$ $(i = 1,...,k)$ is the optimal solution of model (3), the ideal frontier will be as follows:

$$\sum_{r=1}^{s} \bar{u}_r y_r - \sum_{i=1}^{k} \bar{v}_i x_i = 0$$

Theorem 2: All DMUs (alternatives) locate under the ideal frontier.

Proof: At ideal alternative, we have the following for desirable criteria:

$$\begin{bmatrix} y_1^{max} \\ y_2^{max} \\ \vdots \\ y_s^{max} \end{bmatrix} \ge \begin{bmatrix} y_{j1} \\ y_{j2} \\ \vdots \\ y_{js} \end{bmatrix}, \quad j = 1,...,n$$

$$\Rightarrow \begin{bmatrix} \bar{u}_1 y_1^{max} \\ \bar{u}_2 y_2^{max} \\ \vdots \\ \bar{u}_s y_s^{max} \end{bmatrix} \ge \begin{bmatrix} \bar{u}_1 y_{j1} \\ \bar{u}_2 y_{j2} \\ \vdots \\ \bar{u}_s y_{js} \end{bmatrix}, \quad j = 1,...,n$$

$$\Rightarrow \sum_{r=1}^{s} \bar{u}_r y_r^{max} \ge \sum_{r=1}^{s} \bar{u}_r y_{jr}, \quad j = 1,...,n \tag{4}$$

Similarly, for undesirable criteria, we have:

$$\begin{bmatrix} x_1^{min} \\ x_2^{min} \\ \vdots \\ x_k^{min} \end{bmatrix} \le \begin{bmatrix} x_{j1} \\ x_{j2} \\ \vdots \\ x_{jk} \end{bmatrix}, \quad j = 1,...,n$$

$$\Rightarrow \begin{bmatrix} -\bar{v}_1 x_1^{\min} \\ -\bar{v}_2 x_2^{\min} \\ \vdots \\ -\bar{v}_k x_k^{\min} \end{bmatrix} \geq \begin{bmatrix} -\bar{v}_1 x_{j1} \\ -\bar{v}_2 x_{j2} \\ \vdots \\ -\bar{v}_k x_{jk} \end{bmatrix}, \quad j=1,\ldots,n$$

$$\Rightarrow -\sum_{i=1}^{k} \bar{v}_i x_i^{\min} \geq -\sum_{i=1}^{k} \bar{v}_i x_{ji}, \quad j=1,\ldots,n \tag{5}$$

Summing two inequalities (4) and (5), we have:

$$\sum_{r=1}^{s} \bar{u}_r y_{jr} - \sum_{i=1}^{k} \bar{v}_i x_{ji} \leq$$

$$\sum_{r=1}^{s} \bar{u}_r y_r^{\max} - \sum_{i=1}^{k} \bar{v}_i x_i^{\min} \leq 0, \; j=1,\ldots,n$$

Therefore all DMUs locate under the hyperplane $\sum_{r=1}^{s} \bar{u}_r y_r - \sum_{i=1}^{k} \bar{v}_i x_i = 0$ and the proof is completed.■

To determine the ideal frontier, we used hourly CWs model (2) based on ideal alternative. Similarly, for determining anti-ideal frontier, we use model (2) and only anti-ideal alternative and a model is presented as follows:

$$\begin{aligned} &\max \ \phi, \\ &s.t \sum_{r=1}^{s} u_r y_r^{\min} - \sum_{i=1}^{k} v_i x_i^{\max} \leq 0, \\ &\quad \phi \leq u_r y_r^{\min} \quad \leq (1-\phi), \quad r=1,\ldots,s, \\ &\quad \phi \leq v_i x_i^{\max} \quad \leq (1-\phi), \quad i=1,\ldots,k \end{aligned} \tag{6}$$

Theorem 3: Model (6) is always feasible.

Proof: It is obvious that $\phi = 0$, $u_r = \dfrac{1}{s\, y_r^{\min}}$ $(r=1,\ldots,s)$, $v_i = \dfrac{1}{k\, x_i^{\max}}$ $(i=1,\ldots,k)$ is feasible solution of model (6) and so the model is always feasible. ■

If $\underline{\phi}$, $\underline{u_r}$ $(r=1,\ldots,s)$, $\underline{v_i}$ $(i=1,\ldots,k)$ is optimal solution of model (6), anti-ideal frontier is as:

$$\sum_{r=1}^{s} \underline{u_r} y_r - \sum_{i=1}^{k} \underline{v_i} x_i = 0$$

Theorem 4: If $\underline{\phi}$, $\underline{u_r}$ $(r=1,\ldots,s)$, $\underline{v_i}$ $(i=1,\ldots,k)$ is the optimal solution of model (6), then decision making units (alternatives) locate on top of the anti-ideal frontier.

Proof: Proof is similar to Theorem 2. ■

Definition 1: Suppose $H = \{x \,|\, a^t x = b\}$ is a hyperplane in an n-dimensional space. The distance between point *y* and this hyperplane is calculated as follows:

$$l = \frac{|a^t y - b|}{\|a\|}$$

To calculate the distance of DMU_j $(j=1,\ldots,n)$ from the ideal frontier, we use the following formula:

$$\theta_{I^+ j} = \frac{\left| \sum_{r=1}^{s} \bar{u}_r y_{jr} - \sum_{i=1}^{k} \bar{v}_i x_{ji} \right|}{\sqrt{\sum_{r=1}^{s} \bar{u}_r^2 + \sum_{i=1}^{k} \bar{v}_i^2}}, \quad j=1,\ldots,n \tag{7}$$

We denote $\theta_{I^- j}$ $(j=1,\ldots,n)$ as the distance of DMU_j from the anti-ideal frontier and is computed by:

$$\theta_{I^- j} = \frac{\left| \sum_{r=1}^{s} \underline{u_r} y_{jr} - \sum_{i=1}^{k} \underline{v_i} x_{ji} \right|}{\sqrt{\sum_{r=1}^{s} \underline{u_r}^2 + \sum_{i=1}^{k} \underline{v_i}^2}}, \quad j=1,\ldots,n \tag{8}$$

The index $\theta_j = \dfrac{\theta_{I^- j}}{\theta_{I^- j} + \theta_{I^+ j}}$ is introduced for evaluating the performance of DMU_j $(j=1,\ldots,n)$. We have $0 \leq \theta_j \leq 1$. The more the above ratio is close to 1, the better is alternative rank.

IV. Numerical Example

Consider 10 cars. We want to rank them by six criteria out of which three of them are desirable (maximum speed (km) y_1, car power (cv) y_2, area (m^2) y_3) and the other three are undesirable criteria (intercity gas consumption x_1, gas consumption at 120 km/h x_2, price (francs) x_3). The information is presented in Table II. This data was previously studied by Kao [5]. Ranking alternatives, in this example, is summarized in the following Table based on the proposed method by Kao [5], in which s_j is the distance of the *jth* car from the ideal car.The proposed models (3) and (6) (ideal and anti-ideal models), using the data in Table I, are expressed as follows:

$$\begin{aligned} &\max \ \phi, \\ &s.t \ 182u_1 + 13u_2 + 8.47u_3 - 7.2v_1 - 6.75v_2 - 24.8v_3 \leq 0, \\ &\quad \phi \leq 182u_1 \leq (1-\phi), \\ &\quad \phi \leq 13u_2 \leq (1-\phi), \\ &\quad \phi \leq 8.47u_3 \leq (1-\phi), \\ &\quad \phi \leq 7.2v_1 \leq (1-\phi), \\ &\quad \phi \leq 6.75v_2 \leq (1-\phi), \\ &\quad \phi \leq 24.8v_3 \leq (1-\phi) \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\max \ \phi, \\ &s.t \ 117u_1 + 3u_2 + 5.11u_3 - 14.5v_1 - 12.95v_2 - 75.7v_3 \leq 0, \end{aligned}$$

$$\phi \le 117u_1 \le (1-\phi),$$
$$\phi \le 3u_2 \le (1-\phi),$$

TABLE II
DATA FOR TEN CARS WITH SIX CRITERIA

Car	Desirable criteria(Outputs)			Undesirable criteria (Inputs)		
	y_1	y_2	y_3	x_1	x_2	x_3
1	7.88	10	173	49.5	10.01	11.4
2	7.96	11	176	46.7	10.48	12.3
3	5.65	5	142	32.1	7.30	8.2
4	6.15	7	148	39.15	9.61	10.5
5	8.06	13	178	64.7	11.05	14.5
6	8.47	13	180	75.7	10.40	13.6
7	7.81	11	182	68.593	12.26	12.7
8	8.38	11	145	55	12.95	14.3
9	5.11	7	161	35.2	8.42	8.6
10	5.81	3	117	24.8	6.75	7.2
I^+	8.47	13	182	24.8	6.75	7.2
I^-	5.11	3	117	75.7	12.95	14.5

$$\phi \le 5.11u_3 \le (1-\phi),$$
$$\phi \le 14.5v_1 \le (1-\phi),$$
$$\phi \le 12.95v_2 \le (1-\phi),$$

TABLE III
RANKING ALTERNATIVES BASED ON KAO'S METHOD

Car	s_j	Ranking
1	0.2615	2
2	0.2151	1
3	0.4558	8
4	0.4457	7
5	0.3123	5
6	0.3596	6
7	0.3004	4
8	0.5763	9
9	0.2921	3
10	0.6525	10

$$\phi \le 75.7v_3 \le (1-\phi), \qquad (10)$$

Optimal solution of models (9) and (10) are represented in Table IV.

Considering the optimal solution of the ideal model presented in the second column of Table IV, ideal frontier will be gained as follows:

$$(0.0027)Y_1 + (0.0385)Y_2 + (0.0590)Y_3$$
$$-(0.0694)X_1 - (0.0741)X_2 - (0.0202)X_3 = 0$$

Also by the weights in the third column of Table IV, anti-ideal frontier is as:

$$(0.0043)Y_1 + (0.1667)Y_2 + (0.0978)Y_3$$
$$-(0.0345)X_1 - (0.0386)X_2 - (0.0066)X_3 = 0$$

The distances of the first car from the ideal and anti-ideal frontiers are calculated as follows:

$$\theta_{I^+1} = \frac{|1.2061|}{\sqrt{(0.0027)^2 + (0.0385)^2 + (0.0590)^2 + (0.0694)^2 + (0.0741)^2 + (0.0202)^2}} = 9.6293$$

$$\theta_{I^-1} = \frac{|2.0705|}{\sqrt{(0.0043)^2 + (0.1667)^2 + (0.0978)^2 + (0.0345)^2 + (0.0386)^2 + (0.0066)^2}} = 10.3403$$

Other results have been provided in Table V. In the second, third, fourth and fifth columns are respectively shown the distance of alternatives from ideal frontier, the distance of alternative from anti-ideal frontier, assessment index, and the rank of alternatives.

TABLE IV
OPTIMAL SOLUTION OF MODELS (9) AND (10)

Weights	Ideal model	Anti-ideal model
U_1	0.0027	0.0043
U_2	0.0385	0.1667
U_3	0.0590	0.0978
V_1	0.0694	0.0345
V_2	0.0741	0.0386
V_3	0.0202	0.0066

In the following Table, car 3 is ranked as 4, whereas the rank is assigned to car 7 using Kao's method. On one hand, car 9 ranked as 3 using both methods. If we compare criteria of car 3 and 9 as well as car 7 and 9, we will have that the criteria of car 9 is closer to car 3 and so rank 4 is appropriate to car 3, no for car 7.

V. CONCLUSION

In this article, a ranking method was proposed for alternatives in MCDA based on introducing ideal and anti-ideal frontiers. The method does not need pre-determined weights, so results are more convincible because they are a reflection of data. The proposed models are also linear and they are feasible despite weight restrictions in the models. In comparison with using ideal and anti-ideal alternative for ranking alternatives, the proposed method is flexible and realistic. The reason is that in this method each alternative has a unique projection on the ideal frontier whereas the projection of all alternatives by the classic method is ideal alternative. There is similar explanation about utilizing anti-ideal frontier and anti-ideal alternative. As regards different alternatives have different amounts of the corresponding criteria, considering different projections for alternatives is more reasonable than same projection for them. In future research, we try to extend the proposed method in this paper for group decision making.

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