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(R, S)-Modules and $(1, k)$-Jointly Prime (R, S)-Submodules

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Abstract—We introduced the notions of $(1,k)$-prime ideal and $(1,k)$-jointly prime (R,S)-submodule as a generalization of prime ideal and jointly prime (R,S)-submodule, respectively. We provide a relationship between $(1,k)$-prime ideal and $(1,k)$-jointly prime (R,S)-submodule. Characterizations of $(1,k)$-jointly prime (R,S)-submodules are also given.

Keywords—(R,S)-module, $(1,k)$-prime ideal, $(1,k)$-jointly prime (R,S)-submodule.

I. Introduction

THROUGHOUT this paper, let R and S be rings and M an abelian group.

Definition 1.1: [1] Let R and S be rings and M an abelian group under addition. We say that M is an (R,S)-**module** if there is a function $\_\cdot\_\cdot\_ : R \times M \times S \to M$ satisfying the following properties: for all $r, r_1, r_2 \in R$, $m, n \in M$ and $s, s_1, s_2 \in S$,

(i) $r \cdot (m+n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s$
(ii) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$
(iii) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$
(iv) $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$.

We usually abbreviate $r \cdot m \cdot s$ by rms. We may also say that M is an (R,S)-module under $+$ and $\_\cdot\_\cdot\_$.

An (R,S)-**submodule** of an (R,S)-module M is a subgroup N of M such that $rns \in N$ for all $r \in R$, $n \in N$ and $s \in S$.

Definition 1.2: [1] Let M be an (R,S)-module. A proper (R,S)-submodule P of M is called **jointly prime** if for each left ideal I of R, right ideal J of S and (R,S)-submodule N of M,

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

The structure of an (R,S)-module was created as a generalization of a module structure. The basic results of an (R,S)-module structure have been given by [1] and [2]. Almost all of those results was studied analoguous to a module structure such as the primalities of (R,S)-submodules of (R,S)-modules and left multiplication (R,S)-modules; see [1] and [2].

In this paper, we introduce the notions of $(1,2)$-prime ideal, $(1,k)$-prime ideal, $(1,2)$-jointly prime (R,S)-submodule and $(1,k)$-jointly prime (R,S)-submodule and obtain equivalent conditions for an (R,S)-submodule to be $(1,k)$-jointly prime (R,S)-submodule.

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II. $(1,2)$-Jointly Prime (R,S)-Submodules

In this research, we modify the structure of a jointly prime (R,S)-submodules for more general. Now, we start this section by giving the definition of $(1,2)$-jointly prime (R,S)-submodules.

Definition 2.1: A proper (R,S)-submodule P of M is called $(1,2)$**-jointly prime** if for each left ideal I of R, right ideal J of S and (R,S)-submodule N of M,

$$INJ^2 \subseteq P \text{ implies } IMJ^2 \subseteq P \text{ or } N \subseteq P.$$

By the dual of $(1,2)$-jointly prime, we define $(2,1)$-jointly prime as follow.

*A proper (R,S)-submodule P of M is called $(2,1)$-**jointly prime** if for each left ideal I of R, right ideal J of S and (R,S)-submodule N of M,*

$$I^2NJ \subseteq P \text{ implies } I^2MJ \subseteq P \text{ or } N \subseteq P.$$

It is clear that a jointly prime (R,S)-submodule is $(1,2)$-jointly prime and $(2,1)$-jointly prime. Next, we give a characterization of $(1,2)$-jointly prime and $(2,1)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule of $\mathbb{Z}$ where $r,s \in \mathbb{Z}^+$.

Proposition 2.2: Let $r,s \in \mathbb{Z}^+$ and $p \in \mathbb{Z}_0^+ \backslash \{1\}$. Then

(i) $p\mathbb{Z}$ is an $(1,2)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-module of $\mathbb{Z}$ if and only if $p = 0$, p is a prime integer or $p \mid rs^2$.
(ii) $p\mathbb{Z}$ is a $(2,1)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-module of $\mathbb{Z}$ if and only if $p = 0$, p is a prime integer or $p \mid r^2 s$.

Proof: (i) $(\Rightarrow)$ Assume that $p\mathbb{Z}$ is a $(1,2)$-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-module of $\mathbb{Z}$. Suppose that $p \neq 0$ and p is not a prime integer. Then $p = mn$ for some integer $m,n > 1$. It implies that $(rm\mathbb{Z})(n\mathbb{Z})(s^2\mathbb{Z}) = (rmns^2)\mathbb{Z} \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is $(1,2)$-jointly prime and $p \nmid n$, $(rm\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$. Note that $(r\mathbb{Z})(m\mathbb{Z})(s^2\mathbb{Z}) = (rm\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is $(1,2)$-jointly prime and $p \nmid m$, $(r\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$. Hence $p \mid rs^2$.

$(\Leftarrow)$ If $p = 0$ or p is a prime integer or $p \mid rs^2$, then it is clear that $p\mathbb{Z}$ is $(1,2)$-jointly prime. ■

Now, we already have an example of $(1,2)$-jointly prime but is not jointly prime.

Example 2.3: It is clear that $\mathbb{Z}$ is a $(2\mathbb{Z}, 3\mathbb{Z})$-module. Then $9\mathbb{Z}$ is a $(1,2)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ but $9\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$.

The following is an example showing that $(1,2)$-jointly prime and $(2,1)$-jointly prime are exactly different.

Example 2.4: Recall that $\mathbb{Z}$ is a $(2\mathbb{Z}, 3\mathbb{Z})$-module. Then $4\mathbb{Z}$ is a $(2,1)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ but $4\mathbb{Z}$ is not a $(1,2)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ and $9\mathbb{Z}$

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is a $(1,2)$-jointly prime $(2\mathbb{Z},3\mathbb{Z})$-submodule of $\mathbb{Z}$ but $9\mathbb{Z}$ is not a $(2,1)$-jointly prime $(2\mathbb{Z},3\mathbb{Z})$-submodule of $\mathbb{Z}$.

Moreover, $p\mathbb{Z}$ is both a $(1,2)$-jointly prime and $(2,1)$-jointly prime $(2\mathbb{Z},3\mathbb{Z})$-submodule of $\mathbb{Z}$ if and only if $p\mathbb{Z}$ is a jointly prime $(2\mathbb{Z},3\mathbb{Z})$-submodule of $\mathbb{Z}$.

Note that $(1,2)$-jointly prime and $(2,1)$-jointly prime may be different even if ring R and S are commutative.

We have a question from Example 2.4 that for general, if P is $(1,2)$-jointly prime and $(2,1)$-jointly prime, then can P be a jointly prime (R,S)-submodule? The following is an answers.

Example 2.5: It easy to see that $\mathbb{Z}$ is a $(2\mathbb{Z},4\mathbb{Z})$-module. Then $16\mathbb{Z}$ is both a $(1,2)$-jointly prime and $(2,1)$-jointly prime $(2\mathbb{Z},4\mathbb{Z})$-submodule of $\mathbb{Z}$ but $16\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z},4\mathbb{Z})$-submodule of $\mathbb{Z}$.

Example 2.6: Let $\mathbb{Z}$ be a ring of integer and let

$$R=\left\{\begin{bmatrix}0&0&x\\0&0&y\\0&0&0\end{bmatrix}\;\middle|\; x,y\in\mathbb{Z}\right\},$$

$$S=\left\{\begin{bmatrix}0&0&0\\x&0&0\\y&0&0\end{bmatrix}\;\middle|\; x,y\in\mathbb{Z}\right\}\quad\text{and}$$

M is the set of all 3×3 matrices on integer. Then M is an (R,S)-module. Since $R^2=0=S^2$, all proper (R,S)-submodules of M are both $(1,2)$-jointly prime and $(2,1)$-jointly prime (R,S)-submodule of M. However, 0 is not a jointly prime (R,S)-submodule of M.

For each (R,S)-submodule P of M and $k\in\mathbb{Z}^+$, let

$$(P:M)_{R;S^k}=\{r\in R\mid rMS^k\subseteq P\}.$$

Proposition 2.7: Let P be an (R,S)-submodule of an (R,S)-module M and $k\in\mathbb{Z}^+$. The followings hold.

(i) $(P:M)_{R;S^k}$ is a subgroup of R under addition.
(ii) $(P:M)_{R;S^k}\subseteq(P:M)_{R;S^{k+1}}$.
(iii) If $S^2=S$, then $(P:M)_{R;S^k}$ is an ideal of R.

Proof: The proof is straightforward. ■

Next, we introduced a particular nonempty subset of R which play a role in this research.

Let R be a ring and T a proper ideal of R. Then T is said to be a $(1,2)$**-prime ideal** of R if for each ideal A and B of R, if $AB^2\subseteq T$, then $A\subseteq T$ or $B^2\subseteq T$. A prime ideal of R is a $(1,2)$-prime ideal of R but the converse is not true. We show by observing the following example.

Example 2.8: Let p be an integer. If $p=0$ or p is a prime integer or $p=q^2$ where q is a prime integer, then $p\mathbb{Z}$ is a $(1,2)$-prime ideal of $\mathbb{Z}$.

It clear that $4\mathbb{Z}$ is a $(1,2)$-prime ideal of $\mathbb{Z}$ but is not a prime ideal of $\mathbb{Z}$.

Proposition 2.9: Let P be an (R,S)-submodule of an (R,S)-module M such that $(P:M)_{R;S^2}$ is a proper ideal of R. If P is a $(1,2)$-jointly prime (R,S)-submodule of M, then $(P:M)_{R;S^2}$ is a $(1,2)$-prime ideal of R.

Proof: Assume that P is a $(1,2)$-jointly prime (R,S)-submodule of M. Let A and B be ideals of R such that $AB^2\subseteq(P:M)_{R;S^2}$. Hence $(AB^2)MS^2S^2\subseteq(AB^2)MS^2\subseteq P$. Thus $A(B^2MS^2)S^2\subseteq P$. Since P is $(1,2)$-jointly prime, $AMS^2\subseteq P$ or $B^2MS^2\subseteq P$. Therefore $A\subseteq(P:M)_{R;S^2}$ or $B^2\subseteq(P:M)_{R;S^2}$. This means that $(P:M)_{R;S^2}$ is a $(1,2)$-prime ideal of R. ■

The converse of Proposition 2.9 is invalid. For example, $6\mathbb{Z}$ is a $(\mathbb{Z},2\mathbb{Z})$-submodule of $\mathbb{Z}$. We see that $(6\mathbb{Z}:\mathbb{Z})_{\mathbb{Z};(2\mathbb{Z})^2}=3\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$, of course, $3\mathbb{Z}$ is a $(1,2)$-prime ideal of $\mathbb{Z}$ but $6\mathbb{Z}$ is not a (1,2)-jointly prime $(\mathbb{Z},2\mathbb{Z})$-submodule of $\mathbb{Z}$.

III. $(1,k)$-Jointly Prime (R,S)-Submodules

In this section, we extend the notion of $(1,2)$-jointly prime to $(1,k)$-jointly prime where $k\in\mathbb{Z}^+$. Similarly, we also extend the notion of $(2,1)$-jointly prime to $(k,1)$-jointly prime where $k\in\mathbb{Z}^+$.

Definition 3.1: Let $k\in\mathbb{Z}^+$ and M be an (R,S)-module. A proper (R,S)-submodule P of M is called $(1,k)$**-jointly prime** if for each left ideal I of R, right ideal J of S and (R,S)-submodule N of M,

$$INJ^k\subseteq P \text{ implies } IMJ^k\subseteq P \text{ or } N\subseteq P.$$

Dually, a proper (R,S)-submodule P of M is called $(k,1)$**-jointly prime** if for each left ideal I of R, right ideal J of S and (R,S)-submodule N of M,

$$I^kNJ\subseteq P \text{ implies } I^kMJ\subseteq P \text{ or } N\subseteq P.$$

Note here that jointly prime and $(1,1)$-jointly prime are identical.

Proposition 3.2: Let $r,s,k\in\mathbb{Z}^+$ and $p\in\mathbb{Z}_0^+\backslash\{1\}$. Then

(i) $p\mathbb{Z}$ is a $(1,k)$-jointly prime $(r\mathbb{Z},s\mathbb{Z})$-module of $\mathbb{Z}$ if and only if $p=0$, p is a prime integer or $p\,|\,rs^k$.
(ii) $p\mathbb{Z}$ is a $(k,1)$-jointly prime $(r\mathbb{Z},s\mathbb{Z})$-module of $\mathbb{Z}$ if and only if $p=0$, p is a prime integer or $p\,|\,r^ks$.

Proof: $(\Rightarrow)$ Assume that $p\mathbb{Z}$ is a $(1,k)$-jointly prime $(r\mathbb{Z},s\mathbb{Z})$-module of $\mathbb{Z}$. Suppose that $p\neq 0$ and p is not a prime integer. Then $p=mn$ for some integer $m,n>1$. It implies that $(rm\mathbb{Z})(n\mathbb{Z})(s^k\mathbb{Z})=(rmns^k)\mathbb{Z}\subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is $(1,k)$-jointly prime and $p\nmid n$, $(rm\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z})\subseteq p\mathbb{Z}$. Note that $(r\mathbb{Z})(m\mathbb{Z})(s^k\mathbb{Z})=(rm\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z})\subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is $(1,k)$-jointly prime and $p\nmid m$, $(r\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z})\subseteq p\mathbb{Z}$. Hence $p\,|\,rs^k$.

$(\Leftarrow)$ If $p=0$ or p is a prime integer or $p\,|\,rs^k$, then it is clear that $p\mathbb{Z}$ is $(1,k)$-jointly prime. ■

Proposition 3.3: Let $k\in\mathbb{Z}^+$ and M be an (R,S)-module. Then

(i) If P is $(1,k)$-jointly prime, then P is $(1,k+1)$-jointly prime.
(ii) If P is $(k,1)$-jointly prime, then P is $(k+1,1)$-jointly prime.

Proof: Assume that P is a $(1,k)$-jointly prime (R,S)-submodule of M. Let I be a left ideal of R, N an (R,S)-submodule of M and J be a right ideal of S such that $INJ^{k+1}\subseteq P$. Note that $I(INJ)J^k\subseteq I^2NJ^{k+1}\subseteq INJ^{k+1}\subseteq P$. Since P is $(1,k)$-jointly prime, $IMJ^k\subseteq P$ or $INJ\subseteq P$. Note that $J^{k+1}\subseteq J^k\subseteq J$. If $IMJ^k\subseteq P$, then $IMJ^{k+1}\subseteq P$. Assume that $INJ\subseteq P$. Then $INJ^k\subseteq P$. Since P is $(1,k)$-jointly prime, $IMJ^k\subseteq P$ or $N\subseteq P$. ■

The following example shows that the converse of Proposition 3.3 is false in general.

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Example 3.4: Recall that $\mathbb{Z}$ is a $(2\mathbb{Z}, 3\mathbb{Z})$-module. Then $27\mathbb{Z}$ and $54\mathbb{Z}$ are $(1,3)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$ but $27\mathbb{Z}$ and $54\mathbb{Z}$ are not $(1,2)$-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$-submodule of $\mathbb{Z}$.

We obtain the following diagram from Proposition 3.3. Note that the order pair (m,n) means P is a (m,n)-jointly prime (R,S)-submodule of M.

$$\begin{array}{ccccccccc}
(1,1) & \to & (1,2) & \to & \ldots & \to & (1,n) & \to & \ldots \\
\downarrow & & & & & & & & \\
(2,1) & & & & & & & & \\
\downarrow & & & & & & & & \\
\cdot & & & & & & & & \\
\cdot & & & & & & & & \\
\cdot & & & & & & & & \\
\downarrow & & & & & & & & \\
(m,1) & & & & & & & & \\
\downarrow & & & & & & & & \\
\cdot & & & & & & & & \\
\cdot & & & & & & & & \\
\cdot & & & & & & & &
\end{array}$$

In this point, we present a generalization of a (1,2)-prime ideal of R which is called $(1,k)$-prime ideal of R where $k \in \mathbb{Z}^+$. Let R be a ring and T a proper ideal of R and $k \in \mathbb{Z}^+$. Then T is said to be a $(1,k)$**-prime ideal** of R if for each ideal A and B of R, if $AB^k \subseteq T$, then $A \subseteq T$ or $B^k \subseteq T$.

Proposition 3.5: Let $k \in \mathbb{Z}^+$ and P be an (R,S)-submodule of an (R,S)-module M such that $(P:M)_{R;S^k}$ is a proper ideal of R. If P is a $(1,k)$-jointly prime (R,S)-submodule of M, then $(P:M)_{R;S^k}$ is a $(1,k)$-prime ideal of R.

Note that $(X)_l$ and $(X)_r$ is the left ideal generated by X and the right ideal generated by X, respectively, for any subset X of a ring R. $\langle Y \rangle$ is the (R,S)-submodule generated by Y for any (R,S)-submodule Y of an (R,S)-module M Next result needs the following lemma.

Lemma 3.6: Let M be an (R,S)-module and $k \in \mathbb{Z}^+$. The following statments hold:

(i) For all left ideal I of R, left ideal J of S and (R,S)-submodule N and P of M,
$$INJ^k \subseteq P \text{ implies } I^2N(J)_r^k \subseteq P.$$

(ii) For all right ideal I of R, left ideal J of S and (R,S)-submodule N and P of M,
$$INJ^k \subseteq P \text{ implies } (I)_lN(J^2)^k \subseteq P.$$

(iii) For all right ideal I of R, left ideal J of S and (R,S)-submodule P of M,
$$(I)_lM(J^2)^k \subseteq P \text{ implies } (I)_l\langle IMJ^k\rangle J^k \subseteq P.$$

(iv) For all right ideal I of R, right ideal J of S and (R,S)-submodule N and P of M,
$$INJ^k \subseteq P \text{ implies } I^2N(J)_l^k \subseteq P.$$

(v) For all right ideal I of R, right ideal J of S and (R,S)-submodule P of M,
$$I^2M(J)_l^k \subseteq P \text{ implies } I\langle IMJ^k\rangle (J)_l^k \subseteq P.$$

(vi) For all left ideal I of R, right ideal J of S and (R,S)-submodule N and P of M,
$$INJ^k \subseteq P \text{ implies } (I)_rN(J^2)^k \subseteq P.$$

(vii) For all left ideal I of R, right ideal J of S and (R,S)-submodule P of M,
$$(I)_rM(J^2)^k \subseteq P \text{ implies } (I)_r\langle IMJ^k\rangle J^k \subseteq P.$$

Proof: (i) Let I be a left ideal of R, J a left ideal of S, N and P be (R,S)-submodules of M. Assume that $INJ^k \subseteq P$. Then
$$\begin{aligned}
I^2N(J)_r^k &= I^2N(J+JS)^k \\
&\subseteq I^2N(J^k + J^kS) \\
&\subseteq INJ^k + I(INJ^k)S \\
&\subseteq P.
\end{aligned}$$

(ii) Let I be a right ideal of R, J a left ideal of S, N and P be (R,S)-submodules of M. Assume that $INJ^k \subseteq P$. Then
$$\begin{aligned}
(I)_lN(J^2)^k &= (I+RI)N(J^k)^2 \\
&= (I+RI)NJ^kJ^k \\
&\subseteq INJ^kJ^k + RINJ^kJ^k \\
&\subseteq INJ^k + R(INJ^k)J^k \\
&\subseteq P.
\end{aligned}$$

(iii) Let I be a right ideal of R, J a left ideal of S, N and P be (R,S)-submodules of M. Assume that $(I)_lM(J^2)^k \subseteq P$. Then
$$\begin{aligned}
(I)_l\langle IMJ^k\rangle J^k &= (I)_l(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)J^k \\
&\subseteq (I)_l(\mathbb{Z}(IMJ^k))J^k + (I)_l(R(IMJ^k)S)J^k \\
&\subseteq \mathbb{Z}(I)_lIM(J^k)^2 + (I)_lRIMJ^kSJ^k \\
&\subseteq \mathbb{Z}(I)_lM(J^2)^k + (I)_lM(J^2)^k \\
&\subseteq P.
\end{aligned}$$

(iv) Let I be a right ideal of R, J a right ideal of S, N and P be (R,S)-submodules of M. Assume that $INJ^k \subseteq P$. Then
$$\begin{aligned}
I^2N(J)_l^k &= I^2N(J+SJ)^k \\
&\subseteq I^2N(J^k + SJ^k) \\
&\subseteq INJ^k + I(INS)J^k \\
&\subseteq INJ^k + INJ^k \\
&\subseteq P + P \\
&\subseteq P.
\end{aligned}$$

(v) Let I be a right ideal of R, J a right ideal of S, N and P be (R,S)-submodules of M. Assume that $I^2M(J)_l^k \subseteq P$. Then
$$\begin{aligned}
I\langle IMJ^k\rangle (J)_l^k &= I(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)(J)_l^k \\
&\subseteq \mathbb{Z}(I^2MJ^k(J)_l^k) + IRIMJ^kS(J)_l^k \\
&\subseteq \mathbb{Z}(I^2M(J)_l^k) + I^2M(J)_l^k \\
&\subseteq P.
\end{aligned}$$

(vi) Let I be a left ideal of R, J a right ideal of S, N and P be (R,S)-submodules of M. Assume that $INJ^k \subseteq P$. Then

$$\begin{aligned}(I)_r N(J^2)^k &= (I+IR)NJ^kJ^k \\ &\subseteq INJ^kJ^k + (IR)NJ^kJ^k \\ &\subseteq INJ^k + I(RNJ^k)J^k \\ &\subseteq INJ^k \\ &\subseteq P.\end{aligned}$$

(vii) Let I be a left ideal of R, J a right ideal of S, N and P be (R,S)-submodules of M. Assume that $(I)_r M(J^2)^k \subseteq P$. Then

$$\begin{aligned}(I)_r\langle IMJ^k\rangle J^k &= (I)_r(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)J^k \\ &\subseteq \mathbb{Z}((I)_r IMJ^kJ^k) + (I)_r RIMJ^kSJ^k \\ &\subseteq \mathbb{Z}(I)_r M(J^2)^k + (I)_r M(J^2)^k \\ &\subseteq P+P. \\ &\subseteq P.\end{aligned}$$

■

Next, we obtain equivalent conditions for an (R,S)-submodule to be $(1,k)$-jointly prime (R,S)-submodules.

Theorem 3.7: Let M be an (R,S)-module and P a proper (R,S)-submodule of M and $k \in \mathbb{Z}^+$. The following statments are equivalent:

(i) P is an $(1,k)$-jointly prime (R,S)-submodule of M.

(ii) For all left ideal I of R, left ideal J of S and (R,S)-submodule N of M,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

(iii) For all right ideal I of R, left ideal J of S and (R,S)-submodule N of M,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

(iv) For all right ideal I of R, right ideal J of S and (R,S)-submodule N of M,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

(v) For all left ideal I of R, right ideal J of S and $m \in M$,

$$I\langle m\rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

(vi) For all left ideal I of R, left ideal J of S and $m \in M$,

$$I\langle m\rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

(vii) For all right ideal I of R, left ideal J of S and $m \in M$,

$$I\langle m\rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

(viii) For all right ideal I of R, right ideal J of S and $m \in M$,

$$I\langle m\rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

Proof: This follows from Lemma 3.6. ■

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