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Quasi-Permutation Representations for the Group $SL(2,q)$ when Extended by a certain Group of order Two

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***Abstract*—** A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix. For a finite group G the minimal degree of a faithful representation of G by quasi-permutation matrices over the rationals and the complex numbers are denoted by $q(G)$ and $c(G)$ respectively. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of G. The purpose of this paper is to calculate $q(G), c(G)$ and $r(G)$ for the group $SL(2,q)$ when extended by a certain group of order two.

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In [10] Wong defined a quasi-permutation group of degree n to be a finite group G of automorphisms of an n-dimensional complex vector space such that every element of G has non-negative integral trace. Also Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [3] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results.

By a quasi-permutation matrix we mean a square matrix over the complex field C with non-negative integral trace. For a given finite group G, let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field Q and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices and finally let $r(G)$ denote the minimal degree of a faithful rational valued character of G. In this paper we will apply the algorithms in [1] to the group $K_2^2(2^n)$, where

$$K_n^2(q) =< SL(n,q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} > .$$

We will prove

Theorem 1: **A)** Let $G = K_2^2(2)$ then $r(G) = 2, c(G) = q(G) = 4$

B) Let $K_2^2(q)$, where $q = 2^n$. Then

1) If $q \equiv -1 \mod 3$ then $r(G) = q-1$, $c(G) = q(G) = 2(q-1)$

2) Otherwise : $r(G) = q$, $c(G) = q(G) = 2q$

Let $SL(n,q)$ denote the special general linear group of a vector space of dimension n over a field with q elements . Let $\theta : SL(n,q) \longrightarrow SL(n,q)$ be the automorphism of $SL(n,q)$ given by $\theta(A) = (A^t)^{-1}$, where A^t denotes the transpose of the matrix $A \in SL(n,q)$. In this case one can define the split extension $SL(n,q). < \theta >$ that following the notations used in [6] is denoted by $K_n^2(q)$. Therefore we have $K_n^2(q) =< SL(n,q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} >$, see[4] .

Now let G denote the group $SL(n,q)$ and let the split extension of G by the cyclic group $< \theta >$ of order 2 be denoted by G^+ . Since $[G^+ : G] = 2$, we have $G^+ = G \cup \theta G$, and elements of G^+ which lie in G are called positive and those outside G are called negative elements . A conjugacy class in G^+ is called positive if it lies in G otherwise it is called negative . We may assume that using [7] one can obtain information about conjugacy classes and complex irreducible characters of G , therefore so far as conjugacy classes of G^+ are concerned one must pay attention to negative conjugacy classes of G^+ .

One can show that there is a one-to-one correspondence between the set of negative conjugacy classes of G^+ and the set of equivalence classes of invertible matrices in G .

Now we begin with a summary of facts relevant to the irreducible complex characters of $K_2^2(q)$.
Complex irreducible characters of G^+ are divided into two kinds . The group $< \theta >$ acts on the set of complex irreducible characters of G as follows . If $\chi \in Irr(G)$, then $\chi^\theta(A) := \chi(\theta^{-1}A\theta)$. If $\chi^\theta = \chi$, then we say that χ is invariant under $< \theta >$ and in this case χ forms an orbit of G^+ acting on $Irr(G)$. Now by standard results that can be found in [8] there exists an irreducible character φ of G^+ such that $\varphi \downarrow_G = \chi$. Since $G^+/G \cong Z_2$ has two linear characters , therefore multiplication of φ with the non-trivial character of G^+/G gives another irreducible character φ' of G^+ such that $\varphi \downarrow_G = \chi$. In this case we say that χ extends to φ and φ' and it is enough to calculate one of them on the negative conjugacy classes of G^+ .
As we mentioned earlier we have $K_2^2(q) = SL(2,q). < \theta > =< SL(2,q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1}, \forall A \in SL(2,q) >$. In the following Lemma we give the structure of $K_2^2(q)$.

Lemma 1: Let $G = K_2^2(q)$. If q is even , then $K_2^2(q) \cong SL(2,q) \times < \theta >$and if q is odd , then $K_2^2(q) \cong SL(2,q) \circ 4$ a central product of $SL(2,q)$ with the cyclic group of order 4 .

Proof. The automorphism $\theta : SL(2,q) \rightarrow SL(2,q)$ is

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given by $\theta(A) = (A^{-1})^t$ for all $A \in SL(2,q)$. If we set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then it is easy to verify that for $A \in SL(2,q)$ we have $J^{-1}AJ = (A^{-1})^t$ and therefore θ is equal to an inner automorphism i_J of $SL(2,q)$ .We have $K_2^2(q) =< SL(2,q), \theta >=< SL(2,q), \theta J >= SL(2,q). < \theta J >$ and since $\theta J \in Z(K_2^2(q))$ hence $K_2^2(q) \cong \frac{SL(2,q)\times <\theta J>}{\{(I,I),(-I,-I)\}}$. If the characteristic of $GF(q)$ is even , then we get $K_2^2(q) = SL(2,q)\times < \theta J > \cong SL(2,q) \times < \theta >$ and if the characteristic of $GF(q)$ is odd we obtain $K_2^2(q) \cong SL(2,q) \circ < \theta J >$ the central product of $SL(2,q)$ with a cyclic group of order 4 .

By [5] we have two important lemmas as follows

Lemma 2: a) Let $V_i (i = 1, 2)$ be KG-modules . Then the tensor product $V_1 \otimes_K V_2$ over K obviously becomes a $K[G_1 \times G_2]$ module by

$$(v_1 \otimes v_2)(g_1, g_2) = v_1 g_1 \otimes v_2 g_2$$

For $v_i \in V_i, g_i \in G$.
If χ_i is the character of G_i on V_i , then the character τ of $G_1 \times G_2$ on $V_1 \otimes V_2$ is given by

$$\tau((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$$

For $g_i \in G_i$.

b) Let $\chi_1, ..., \chi_h$ be the irreducible characters of G_i over C and $\psi_1, ..., \psi_k$ the irreducible characters of G_2 over C . Then the t_{ij} defined by $t_{ij}((g_1, g_2)) = \chi_i(g_1)\psi_j(g_2)$ where $i = 1, ..., h$ and $j = 1, ..., k$ are all the irreducible characters of $G_1 \times G_2$.

Lemma 3: Let F be the finite field of $q = 2^n$ elements , and let ν be a generator of the cyclic group $F^* = F - 0$. Denote

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

in $G = SL(2, F)$. G contains an element b of order $q + 1$. For any $x \in G$, let (x) denote the conjugacy class of G containing x . Then G has exactly $q + 1$ conjugacy classes $(1), (c), (a), (a^2), ..., (a^{(q-2)/2}), (b), ..., (b^{q/2})$, where

Table (1)
Conjugacy Classes of $SL(2, 2^n)$

x	1	c	a^l	b^m
$\|(x)\|$	1	$q^2 - 1$	$q(q+1)$	$q(q-1)$

for $1 \le l \le (q-2)/2, 1 \le m \le q/2$.
let $\rho \in C$ be a primitive $(q-1)$-th root of 1 , table of G over C is

Table (2)
Character Table of $SL(2, 2^n)$

	1	c	a^l	b^m
1_G	1	1	1	1
ψ	q	0	l	-1
χ_i	$q+1$	1	$\rho^{il} + \rho^{-il}$	0
θ_j	$q-1$	-1	0	$-(\sigma^{jm} + \sigma^{-jm})$

for $1 \le i \le (q-2)/2, 1 \le j \le q/2, 1 \le l \le (q-2)/2, 1 \le m \le q/2$.

Remark 1: In the case of q even $< \theta J >$ has order 2 and its irreducible characters are denoted by μ_0 and μ_1 where μ_0 is the identity character . Regarding the structure of $K_2^2(q)$ and Lemmas $2, 3$ the irreducible characters of $K_2^2(q)$ in the case of q even are $\mu_k 1_G, \mu_k \psi, \mu_k \chi_i$ and $\mu_k \theta_j$ where $k = 0, 1$ and $1 \le i \le \frac{q-2}{2}, 1 \le j \le \frac{q}{2}$.

Lemma 4: Let $G = SL(2, q)$, if q is a power of 2 then the Schur index of any irreducible character of G over the rational numbers Q is 1 .

Proof. See [9] .

By [9] it is easy to see that :

Lemma 5: Let $G = H \times K$ and $\psi \in Irr(H)$ and $\theta \in Irr(K)$. Let $\chi = \psi \times \theta$ and let $F \subseteq C$.
a) $m_F(\chi)$ divides $m_F(\psi)m_F(\theta)$.
b) Equality occurs in (a) provided $(m_F(\psi), \theta(1)|F(\theta) : F|) = 1$ and
$(m_F(\theta), \psi(1)|F(\psi) : F|) = 1$

Lemma 6: Let G be a finite group . If the Schur index of each non-principal irreducible character is equal to m , then $q(G) = mc(G)$.

Proof. See [1],Corollary 3.15 .

We can see all the following statements in [1],[2].

Definition 1: Let χ be a complex charater of G, such that, $\ker \chi = 1$. Then define

1) $d(\chi) = |\Gamma(\chi)|\chi(1)$

2) $m(\chi) = \begin{cases} 0 & \chi = 1_G \\ |\min\{\sum_{\alpha\in\Gamma(\chi)} \chi^\alpha(g) : g \in G\}| & otherwise \end{cases}$

3) $c(\chi) = \sum_{\alpha\in\Gamma(\chi)} \chi^\alpha + m(\chi)1_G$.

Now according to Corollary 3.11 of [1] and above statements the following lemma is useful for calculation of $r(G), c(G)$ and $q(G)$.

Lemma 7: Let G be a finite group with a unique minimal normal subgroup. Then

1) $r(G) = \min\{d(\chi) : \chi$ is a faithful irreducible complex character of $G\}$

2) $c(G) = \min\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$

3) $q(G) = \min\{m_Q(\chi)c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$.

By [2] we have the following lemmas .

Lemma 8: Let ε be a primitive n-th root of unity in C. Then $\varepsilon + \varepsilon^{-1}$ is rational if and only if $n = 1, 2, 3, 4, 6$. The values which occur are as follows:

Table (3)

n	1	2	3	4	6
$+^{-1}$	2	-2	-1	0	1

Lemma 9: Let ε be a primitive n-th root of unity in C and $m \in Z$. If $\varepsilon + \varepsilon^{-1}$ is rational , then so $\varepsilon^m + \varepsilon^{-m}$.

Lemma 10: Let ε be a primitive n-th root of unity. Then $\varepsilon^j + \varepsilon^{-j}, 1 \le j \le n$ is rational if and only if $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$.

Lemma 11: Let $\chi \in Irr(G), \chi \ne 1_G$. Then $c(\chi)(1) \ge d(\chi) + 1 \ge \chi(1) + 1$.

Proof. From Definition 1 it follows that $c(\chi)(1)$ is a non-negative rational valued character of G so by [1], Lemma 3.2 , $m(\chi) \ge 1$. Now the result follows from Definition 1 .

Lemma 12: Let $\chi \in Irr(G)$. Then

(1) $c(\chi)(1) \ge d(\chi) \ge \chi(1)$;

(2) $c(\chi)(1) \le 2d(\chi)$. Equality occurs if and only if $Z(\chi)/ker\chi$ is of even order .

Proof. (1) follows from the definition of $c(\chi)(1)$ and $d(\chi)$.(2) follows from [1] Lemma 3.13 .

Lemma 13: Let $G = SL(2, q)$ where $q = 2^n$ and $n \ge 2$. Then for each j , $1 \le j \le q/2$,

(1) θ_j is rational if and only if $q \equiv -1$ mod 3 and $j = \frac{q+1}{3}$;

(2) $d(\theta_j) \ge q - 1$ and equality holds if θ_j is rational ;

(3) $c(\theta_j) \ge q + 1$ and equality holds if θ_j is rational .

Proof . As $1 \le j \le \frac{q}{2} < \frac{q+1}{2}$ and as σ is a primitive $(q + 1)$-th root of unity , Lemmas 9 and 10 implies that θ_j is rational if and only if $j = \frac{q+1}{6}, \frac{q+1}{4}, \frac{q+1}{3}$. Since $q + 1$ is odd , $\frac{q+1}{6}$ and $\frac{q+1}{4}$ are not integers . Thus , $\sigma^j + \sigma^{-j} \in Q$ if and only if $3|q + 1$ and $j = \frac{q+1}{3}$. This proves (1) . If θ_j is not rational , then $|\Gamma| \ge 2$ where $\Gamma = \Gamma(Q(\theta_j) : Q)$ so that $c(\theta_j)(1) \ge d(\theta_j) \ge 2(q - 1) > q + 1$ by Lemma 12 . On the other hand if $3|q + 1$, then $8 \le q$, so that $3 \le \frac{q}{2}$; but $\theta_{\frac{q+1}{3}}(b_3) = -2 \le \theta_{\frac{q+1}{3}}(g)$ for all $g \in G$ so that $m(\theta_{\frac{q+1}{3}}) = 2$. Thus $d(\theta_{\frac{q+1}{3}}) = q - 1$ and $c(\theta_{\frac{q+1}{3}})(1) = q + 1$. This completes the proofs of (2) and (3) .

Theorem 2: Let $G = K_2^2(2)$ then $r(G) = 2, c(G) = q(G) = 4$

Proof. By Lemmas 4, 5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have $c(G) = q(G)$. Since the only faithful irreducible character of G is $\mu_1\psi$ so by the character table of the group $K_2^2(2)$ the result follows .

Theorem 3: Let $G = K_2^2(q)$, where $q = 2^n$. Then

1) If $q \equiv -1$ mod 3 then $r(G) = q - 1$, $c(G) = q(G) = 2(q - 1)$

2) Otherwise $r(G) = q$, $c(G) = q(G) = 2q$.

Proof. By Lemmas 4, 5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have $c(G) = q(G)$.

By Lemma 3 we have the irreducible characters of $SL(2, q)$ and by Remark 1 we have the irreducible characters of $K_2^2(2^n)$, now by definition of $d(\chi)$, $c(\chi)$ and character table of $K_2^2(q)$ we obtain the Table (4) as follows

Table (4)

χ (faithful)	$d(\chi)$	$c(\chi)(1)$
$\mu_1\psi$	q	$2q$
$\mu_1\chi_i$	$\ge q + 1$	$\ge 2(q + 1)$
$\mu_1\theta_j$	$\ge (q - 1)$	$\ge 2(q - 1)$

Now by Lemma 7 , Lemma 13 and Table (4) when $q \equiv -1$ mod3 we have

min $\{d(\chi) : \chi$ is a faithful irreducible complex character of $G\} = q - 1$ and

min $\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\} = 2(q - 1)$.

Otherwise $d(\mu_1\theta_j) > q-1$ and so in this case min $d(\chi) = q$ and min $c(\chi) = 2q$.

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