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Quantum Localization of Vibrational Mirror in Cavity Optomechanics

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***Abstract*—**Recently, cavity-optomechanics becomes an extensive research field that has manipulated the mechanical effects of light for coupling of the optical field with other physical objects specifically with regards to dynamical localization. We investigate the dynamical localization (both in momentum and position space) for a vibrational mirror in a Fabry-Pérot cavity driven by a single mode optical field and a transverse probe field. The weak probe field phenomenon results in classical chaos in phase space and spatio temporal dynamics in position $|\psi(x)^2|$ and momentum space $|\psi(p)^2|$ versus time show quantum localization in both momentum and position space. Also, we discuss the parametric dependencies of dynamical localization for a designated set of parameters to be experimentally feasible. Our work opens an avenue to manipulate the other optical phenomena and applicability of proposed work can be prolonged to turn-able laser sources in the future.

***Keywords*—**Dynamical localization, cavity optomechanics, hamiltonian chaos, probe field.

I. Introduction

CAVITY optomechanics deals with the interaction of confining mirror with an optical field in a resonator [1]. Current experimental advances result in the coupling of nanospheres, cold atoms, mechanical membranes, and BEC with the optomechanical system [2]. Therefore, the development of various devices and sensors in quantum meteorology is made possible due to the playground provided by hybrid optomechanical systems. A central paradigm of field is the cooling of one or several modes of the vibrational mirror in an optical cavity to their quantum mechanical ground state [3], [4]. Rapid progress has made towards this goal using micro and nanofabricated mirrors [5], membranes [6], zippers [7], and optomechanical crystals [8] to develop gravitational wave detectors [9], and measurement of displacement with high precision [10]. Recent studies that associate the theoretical simulation and discussions on high fidelity state transfer, dynamical localization of moving end mirror [11], cavity electro optics [12], bistable behavior of BEC optomechanical system [13], steady-state entanglement of BEC and moving end mirror [14], [15] as well as macroscopic tunneling of an optomechanical membrane [16] lead to open novel avenues in cavity optomechanics.

The dynamical localization appears as an open phenomenon in the time-dependent system. In the intracavity field, atoms are excited to momentum side modes by single mode laser field that act as vibrational mirror formally [17]. The mirror experiences radiation pressure modulation because of induced phase modulation by side modes to the field [18]. It refers to a wave packet dynamically being localized in the angular momentum initially in the same state while propagating inside cavity [19] and exists in a system as a signature of quantum chaos [20], [21]. DL phenomenon has signified through atomic dynamics in periodically driven systems as the hydrogen atom in the microwave field, a motion of Paul trap in the presence of standing wave, and an atom in modulated standing wave field [22].

Many significant work have carried out in the field of optomechanics as relating to dynamical localization since the last decade. Several researchers have focused on dynamical localization of various systems and cavity modes. Blumel et al. [23] reviewed the noise influence of DL in the microwave interaction of Rydberg atoms. They discussed four dynamical regimes which passed as a function of irradiation time in a sequence and temporally separated from each other. Moore et al. [24] provided an insight for a novel testing ground in quantum chaos to observe the DL in atomic momentum transfer. Borgonovi et al. [25] studied dynamical localization (DL) and cantori in the bunimovich stadium and found different localization regimes namely quasi-integrable, ergodic and perturbative. Saif et al. [26] demonstrated the dynamical localization and signatures of classical phase space in their works. A study on control of DL in chaotic kicked rotor systems were carried out by Gong et al. [27] that resulted the enhancement of DL length to explore the quantum fluctuations and correlations in quantum chaos. Aforementioned extensive research provides a review on various aspects of dynamical localization other than optomechanical interaction. Very recently, the researchers have started the novel avenues in the current field that enlights the dynamical localization of cavity optomechanics targeting various aspects specifically quantum chaos. In this regard, Ayub et al. [28] focused on the dynamical localization of Bose-Einstein condensate (BEC) in a Fabry-Perot cavity optomechanical system. Yasir et al. [29] illustrated the exponential localization of moving end mirror in optomechanics both in momentum and position space. Another study by Liu et al. [30] explained the energy-localization-enhanced ground-state cooling of a mechanical resonator from room temperature in optomechanics using a gain cavity. Fu et al. [31], [32] have demonstrated classical dynamical localization in a strongly driven two-mode mechanical system and also claimed the coherent optomechanical switch for motion transduction based on dynamically localized mechanical modes. Further exploration

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on many-body localization theory provides an ideal platform for experimental developments. Following the trend, Major et al. [33] and Wan et al. [34] analyzed single-particle localization in dynamical potentials alongwith controlable photon and phonon localization in optomechanical Lieb lattices respectively. Moreover, this enormous research in the area continues to date. The previous studies merely focused on deduction of DL for the complex systems but restricted themselves to use other physical phenomena than weak probe field for inducing modulation. The applicability of this weak probe field extends to provide experimental feasibility.

Therefore, our study reports the quantum and classical dynamics in momentums as well as position space. Hence, it has shown dispersion suppressed in quantum domain and increased in classical counterpart with time. The rest of the paper is organized as follows: Section II provides an overview of the model of the system, while Section III outlines the derivation of Langevin equations that offers a supportive background for our schematic model. Moreover, this work highlights the central phenomenon of dynamical localization of mirror for momentum and position space in Section IV. Lastly, Section V provides a discussion and conclusion to ease readers and in-spirants for a better understanding of overall work.

II. The Model

We consider a Fabry-Pérot cavity of length L has driven by a single mode optical field having a frequency ω_E while fixed mirror and moving-end mirror possess maximum amplitude of q_0 along with frequency ω_m as shown in Fig. 1. The perturbance of moving end mirror within the system results in nonlinear dynamics that has been inter-generated by the probe field with intensity q_0 and frequency ω_p. The Hamiltonian of the system comprised of four fragments,

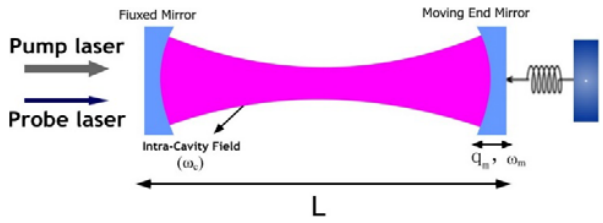


Fig. 1 Schematic model of Cavity-optomechanics. This model is the representative of the Cigar-shaped Fabry- Pérot cavity (length L) optical system with one fixed mirror and one moving end mirror possess maximum amplitude of q_0 driven by an external pump with frequency ω_E and transverse probe field with frequency ω_p

$$\hat{H} = \hat{H}_m + \hat{H}_c + \hat{H}_{\text{int}} + \hat{H}_{lp} \quad (1)$$

$$\hat{H}_m = \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2), \quad (2)$$

$$\hat{H}_c = \hbar\Delta_c \hat{c}^\dagger \hat{c}, \quad (3)$$

$$\hat{H}_{\text{int}} = -g_{mc}\hbar\Delta_c \hat{c}^\dagger \hat{c}, \quad (4)$$

$$\hat{H}_{lp} = -i\hbar\Omega_l(\hat{c} - \hat{c}^\dagger) - i\hbar E_p(\hat{c}e^{i\Delta_p t} - \hat{c}^\dagger e^{-i\Delta_p t}), \quad (5)$$

where $\hat{H}_m$ is related to the harmonic motion of vibrational mirror where $\frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2)$ holds the values for energies linked with the mobility of the end mirror. In this equation, $\hat{q}$ and $\hat{p}$ are representatives of dimensionless momentum and position operators respectively, defined by quantum mechanical commutation relation $[\hat{q}, \hat{p}] = i$, with scaled Planck's constant $\hbar = 1$. $\hat{H}_c$ term denotes intra-cavity optical field energy, while Δ_c is effective detuning of intra-cavity and external pump field. $[\hat{c}^\dagger, \hat{c}]$ describe the total number of the photon within the cavity having a frequency ω_c. The coupling of the intra-cavity field with a vibrational mirror has made by radiation pressure, and its energy is denoted by $\hat{H}_{\text{int}}$. In the intra-cavity optical field mode, the coupling strength has defined by $g_{mc} = \sqrt{2}\left(\omega_c / L\right)x_0$ where L and ω_c indicate length as well as frequency respectively. The formula $x_0 = \sqrt{\hbar / 2m\omega_m}$, is the zero-point motion of the mechanical mirror having frequency ω_m and mass m. The annihilation (creation) operators for the intra-cavity field are $[\hat{c}^\dagger, \hat{c}]$ which interact with a mirror possessing a commutation relation $[\hat{c}^\dagger, \hat{c}] = 1$. The $\hat{H}_{lp}$ has described the intracavity field and its relationship with the input pump and probe field under quantum rotation-wave approximation. The term $i\hbar\Omega_l(\hat{c} - \hat{c}^\dagger)$ defines the association of intra-cavity along with external pump field power $|\Omega_l| = \sqrt{P\kappa / \hbar\omega_E}$, having κ associated intra-cavity field decay rate. It also depicts the pump power where the value of η depends on the external pump field intensity. Similarly, the second term associated with external probe field power $|E_p| = \sqrt{P_p\kappa / \hbar\omega_p}$ where E_p denotes the intensity of the external probe field that is directly proportional to the probe field frequency and Δ_p is far-off detuning of probe field frequency. The total Hamiltonian of the system has been illustrated by:

$$\hat{H} = \frac{\hbar\omega_m}{2}(\hat{p}^2 + \hat{q}^2) + \hbar\Delta_c \hat{c}^\dagger \hat{c} - g_{mc}\hbar\Delta_c \hat{c}^\dagger \hat{c} - i\hbar\Omega_l(\hat{c} - \hat{c}^\dagger) - i\hbar E_p(\hat{c}e^{i\Delta_p t} - \hat{c}^\dagger e^{-i\Delta_p t}), \quad (6)$$

where $\Delta_c = \omega_c - \omega_E$ and $\Delta_p = \omega_p - \omega_E$ are the effective detuning of intra-cavity and probe field frequency ω_p with external pump field frequency ω_E.

III. Langevin Equations

The quantum interaction picture has been used to integrate time dependence and to understand their mutual coupling. Quantum Langevin equations are the most suitable tool for it. We use quantum Langevin formula to incorporate the dissipation effect caused by damping as well as noises correlated to the system (intra-cavity field and vibrational mirror) which can be represented by

$$\frac{d\hat{c}}{dt} = (i\Delta_c + ig_{mc}\hat{q} - \kappa)\hat{c} + \Omega_l + E_p e^{-i\Delta_p t}, \quad (7)$$

$$\frac{d\hat{p}}{dt} = -\omega_m \hat{q} + g_{mc}\hat{c}^\dagger \hat{c} - \gamma_m \hat{p}, \quad (8)$$

$$\frac{d\hat{q}}{dt} = \omega_m \hat{p} \quad (9)$$

In the above equation, the term γ_m has illustrated a mechanical decay rate for the motion of the end mirror. To represent the classical dynamics for finding solutions of steady-state Langevin equations, we have treated the momentum along with positions of the mirror as traditional variables. By fixing the time derivate zero in (7)-(9), we achieve a steady state solution of Langevin equations. The optical field decay is supposed to be at the fastest rate. The steady–state values of operators have been denoted as,

$$c_s = \frac{\Omega_l + E_p e^{-i\Delta_p t}}{\kappa - i(\Delta_c + g_{mc}\hat{q})}, \quad (10)$$

$$p_s = 0, \quad (11)$$

$$q_s = \frac{g_{mc}\hat{c}^\dagger \hat{c}}{\omega_m}, \quad (12)$$

where c_s, q_s, p_s represent steady-state values of the intracavity field, mirror displacement, and position respectively. As supposed, the adiabatic approximation of motion of side-mode mirror and moving end mirror where detainment effects of optical damping have been ignored. Also, by considering the position and momentum as classical variables and setting the time derivative zero in (7)-(9), and plug these equations in the second derivative of (9), the coupled equation of motion can be demonstrated by their nonlinear dynamics via utilizing the Langevin equations, i.e.

$$\frac{d^2\hat{q}}{dt^2} = -\omega_m^2 \hat{q} + \omega_m g_{mc} \frac{\eta^2 + E_p{}^2 + 2\Omega_l E_p \cos(\Delta_p t)}{\kappa^2 + (\Delta_c + g_{mc}\hat{q})^2}, \quad (13)$$

$$\hat{H}_{eff} = K + V \quad (14)$$

$$K = \frac{\hbar\omega_m}{2} p^2, \quad (15)$$

$$V = \frac{\omega_m \hat{q}^2}{2} - \frac{1}{\kappa}(\Omega_l{}^2 + E_p{}^2 + 2\Omega_l E_p \cos(4\tau).\tan^{-1}(\frac{\Delta_c + g_{mc}\hat{q}}{\kappa})), \quad (16)$$

$$\hat{H}_{eff} = \frac{\hbar\omega_m}{2} p^2 - \frac{\omega_m \hat{q}^2}{2} - (\beta + \lambda^2 + 2\beta\lambda^2 \cos(4\tau))\tan^{-1}(\Delta + G\hat{q}), \quad (17)$$

Now, we describe the corresponding effective Hamiltonian as $\hat{H}_{eff} = K + V$ where K is kinetic energy and V shows potential energy being found using (13). It has been supposed that weak coupling for the vibrational mirror with frequency ω_m and $q = q_o \cos\omega_m t$, transforms into harmonic oscillation, where q_0 represents the mean position for maximum displacement; while $\beta = \frac{\Omega_l}{\kappa}$, $\lambda = \frac{E_P}{\kappa}$, $G = \frac{g_{mc}}{\kappa}$ and $\Delta = \frac{\Delta_C}{\kappa}$ are dimensionless parameters. In future, we suppose external field power $P = 0.0164mW$, frequency $\omega_E = 3.8 \times 2\pi \times 10^{14}$ Hz, and wavelength $\lambda_p = 780nm$. Coupling of the pump and the intra-cavity field is $\Omega_l = 18.4 \times 2\pi MHz$, intra-cavity field frequency taken as $\omega_c = 15.3 \times 2\pi \times 10^{14} Hz$ and decay rate $\kappa = 1.3 \times 2\pi kHz$ with cavity length $L = 1.25 \times 10^{-4} m$. The vibrational end mirror of the cavity that oscillates with a frequency $\omega_m = 1.02 \times 2\pi MHz$ along damping $\gamma_m = 1.1 \times 2\pi kHz$ should be a perfect reflector. From these, we observe that the system is found to be at good cavity limit because of the condition, i.e. $\omega_m \gg \kappa$.

IV. Dynamical Localization

We check the classical behaviour of the effective Hamiltonian by solving the equation of motion. Poincare surfaces study the classical dynamics of the mirror's position and momentum. Fig. 2 shows the classical phase space for different modulation. Poincare surfaces are showing the series

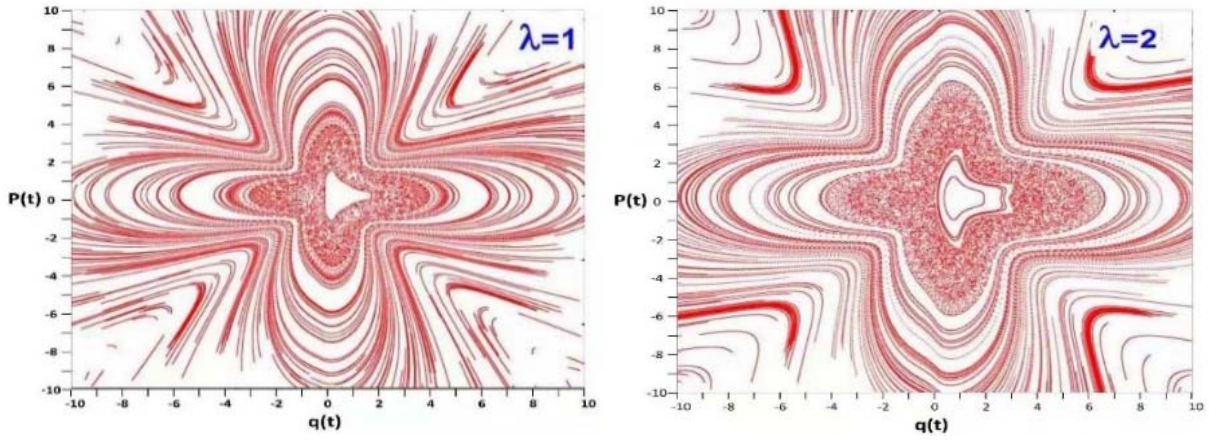
of chaotic regions and stable islands. To discuss the dynamical localization, we solve time-dependent Schrodinger equation for an initially localized mirror whose wave function has been defined as a Gaussian wave packet, $\Psi(x)$ at $\tau = 0$.

$$\Psi(x) = \frac{1}{\sqrt{\sqrt{2\pi}}} \exp(-\left(\frac{x - 0.L/2}{2}\right)^2 \exp\left(-2i\frac{pix}{L}\right), \quad (18)$$

Here, the initial dispersion in momentum along with position has been set to be definite in a way which in turn satisfies the minimum condition for uncertainty. We use the Runge-Kutta integration method for the numerical solution of ODEs. We assume that the system starts from its non-stable maximum where $q_0 = q_{\max}$ and $p = 0$.

Fig. 2 shows the phase-space representation of vibrational mirror for $\lambda = 1, 2$ in Figs. 2 (a) and (b) respectively, where $\beta = 0.8,\ G = 2$ and $\Delta = 0.4$. The Poincare surfaces have determined the classical dynamics of position and momentum for the mirror. Fig. 2 shows the phase space region for the sample size of 100 points in random distribution, on each sample by taking the width of 0.001 over time interval t=0 to 1600. In Figs. 2 (a) and (b), beautiful butterfly wings pattern can be seen as a result of the external probe field (i.e. λ≠0). The marginal distributions behave as probability distributions for phase space. The accuracy of modulation results in mixed phase space that possess regular and chaotic regions. The change in modulation results in the increase of chaotic regions as well as shrinking resonant areas simultaneously — a small difference in the modulation effects the position and momentum of the vibrational mirror. The exact coordinates for a position and momentum cannot be determined by the same location for the second time.

Fig. 3 (a) demonstrates dispersion in momentum space, while (b) shows dispersion in real space. The probability distributions in position and momentum space have calculated for an evolution time. In the Poincar section, nonlinear resonances relate to the peak points both in position and momentum coordinates. The quantum dispersion shows that, compared to the classical one, it experiences a more stable position around its equilibrium. The quantum dispersion oscillates around the origin for larger time and has an indicated localization in real and momentum space. Figs. 3 (c) and (d) describe the real and imaginary part of the mirror wave function for a different time. Blue, red, and yellow colour describe time-dependent behaviour of wave function at $t_0 = 0$, $t_1 = 2$ and $t_2 = 4$, respectively. The wave function is written as $\Psi(x,t) = \Psi_r(x,t) + \Psi_I(x,t)$ where $\Psi_r(x,t)$ and $\Psi_I(x,t)$ show the real and imaginary part of the wave function, respectively. The plots display that wave function oscillates between $[-4,4]$ with time around x=0 where dispersion in position is being found dynamically localized at this time-some states are localized, whereas other remain extended. The simulation shows a straight picture of the mirror's dynamical localization process in momentum and position space. Both plots exhibit that the mirror has fairly localized around its equilibrium point. The remaining parameters are the same. Furthermore, analysis of spatiotemporal behaviour of vibrational mirror reveals apparent dynamical localization in momentum and position space. We demonstrate changing aspects of space-time for both in position and momentum space.



(a) (b)

Fig. 2 Phase space of vibrational mirror for $\lambda = 1, 2$ in (a) and (b) respectively, where $\beta = 0.8,\ G = 2$ and $\Delta = 0.4$

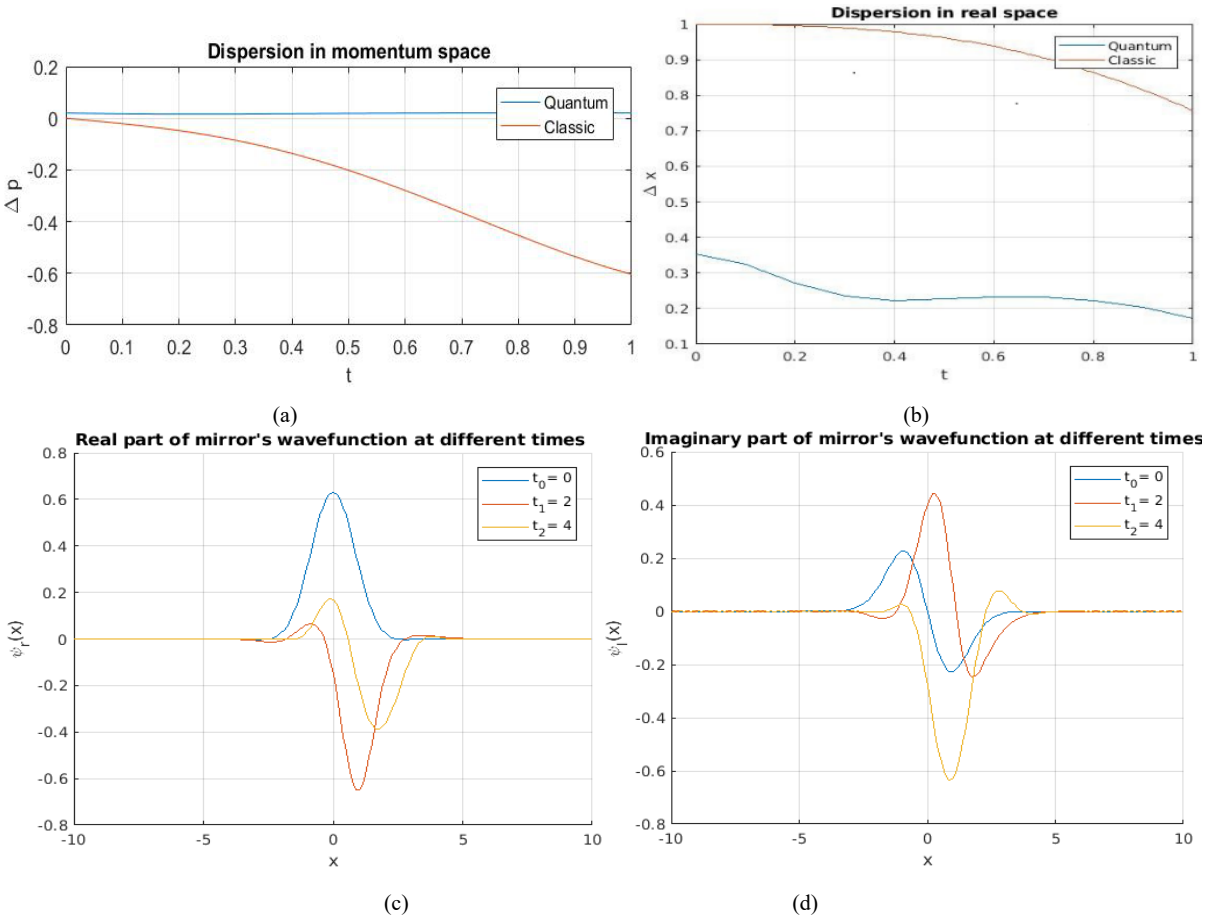


Fig. 3 (a) Dispersion in momentum space and (b) dispersion in real space against time with comparison classical and quantum mechanical behaviour. Furthermore (c) and (d) show the Imaginary and real part of the mirror's wave function at a different time

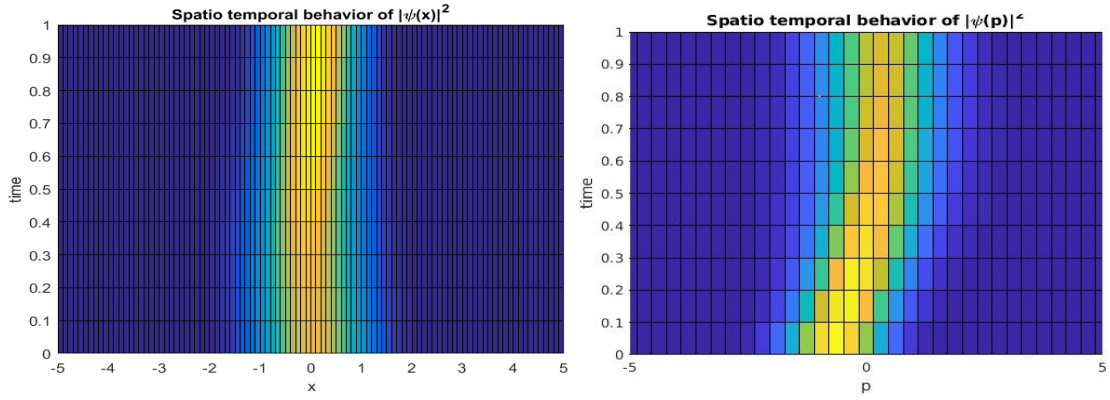


Fig. 4 Spatiotemporal behaviour of the mirror

Fig. 4 has illustrated the spatio-temporal dynamics of the mirror in position $|\psi(x)|^2$ and momentum space $|\psi(p)|^2$ versus time. The localization effects both in position and momentum have shown through the time evolution of probability distribution of quantum wave packet. The maximum value for quantum mechanical probability distribution in momentum and position space has experienced between $[-1,1]$ besides small fluctuations in position along with momentum space, the quantum probability distribution in position and momentum space remains localized. We also conclude that localization of the mirror's wave function has settled around the origin for larger time. Starting with a

Gaussian wave function as an initial condition, one may expect to see dispersion in space. To obtain spatiotemporal behaviour of the intra-cavity system, we can solve Langevin equation. We are permitted to ignore quantum noise effects related to the optomechanical system by considering position operator of the vibrational mirror as classical variable. A completely localized wave function results, whenever a charged particle has to experience an external field (time-harmonic).

V. Conclusion and Discussion

The optomechanical system consists of a cigar-shaped high-finesse cavity having a single vibrational mirror that can run through the single-mode optical field along with cavity axis and a transverse probe field. This field has length $L = 1.25 \times 10^{-4} m$, with a vibrational mirror that has driven via single mode external field of power $P = 0.0164 mW$, frequency $\omega_p = 3.8 \times 2\pi \times 1014 Hz$ and wavelength $\lambda_p = 780 nm$. The vibrational mirror must act as perfect reflector performing oscillations due to intra-cavity field radiation pressure having a frequency $\omega_m = 1.02 \times 2\pi MHz$ with coupling field $\xi = 3.8 MHz$. While the intracavity field possesses frequency as $\omega_c = 15.3 \times 2\pi \times 10^{14} Hz$ having a cavity decay rate $\kappa = 1.3 \times 2\pi kHz$.

This work concludes dynamical localization in position and momentum space in the above-mentioned cavity optomechanical system. The existing dynamical localization within the system has been designated as a signature of chaos. In addition, spatiotemporal behaviour of $|\psi(p)|^2$ and $|\psi(x)|^2$ for momentum space and position space respectively also authenticate our results. Perhaps in a hybrid optomechanical system, Dynamical localization is the best-known phenomenon in quantum chaos. Such a phenomenon somehow behaves differently from the dynamical localization exhibited by ultra-cold atoms present within the modulated optical field momentum space only [22]. The exploitation of current parameters has determined the phenomenon of dynamical localization being reliable in momentum and position space. This research can be limited in terms of experimental feasibility, as one should be well informed about the external heat reservoir hazed associated with the quantum domain of nanomechanical systems. This hazed is capable of limiting the application of any laser-induced mechanical system. Additionally to maintain quality factor the detuning of the system to a particular range of frequencies is required to experiment precisely. In future, the applicability of this work will extend to turn-able laser sources.

References

[1] Kippenberg, Tobias J., and Kerry J. Vahala. "Cavity optomechanics: back-action at the mesoscale." *science* 321, no. 5893 (2008): 1172-1176.

[2] Liu, Ke, Lei Tan, Chun-Hai Lv, and Wu-Ming Liu. "Quantum phase transition in an array of coupled dissipative cavities." *Physical Review A* 83, no. 6 (2011): 063840.

[3] Sun, Qing, Xing-Hua Hu, An-Chun Ji, and W. M. Liu. "Dynamics of a degenerate Fermi gas in a one-dimensional optical lattice coupled to a cavity." *Physical Review A* 83, no. 4 (2011): 043606.

[4] Yixiang, Yu, Yu Chen, Jinwu Ye, and Wuming Liu. "Goldstone and Higgs modes of photons inside an cavity and their detections." In *APS Meeting Abstracts*. 2013.

[5] Arcizet, Olivier, P-F. Cohadon, Tristan Briant, Michel Pinard, and Antoine Heidmann. "Radiation-pressure cooling and optomechanical instability of a micromirror." *Nature* 444, no. 7115 (2006): 71.

[6] Thompson, J. D., B. M. Zwickl, A. M. Jayich, Florian Marquardt, S. M. Girvin, and J. G. E. Harris. "Strong dispersive coupling of a high-finesse cavity to a micromechanical membrane." *Nature* 452, no. 7183 (2008): 72.

[7] Lin, Qiang, Jessie Rosenberg, Darrick Chang, Ryan Camacho, Matt Eichenfield, Kerry J. Vahala, and Oskar Painter. "Coherent mixing of mechanical excitations in nano-optomechanical structures." *Nature Photonics* 4, no. 4 (2010): 236.

[8] Eichenfield, Matt, Jasper Chan, Ryan M. Camacho, Kerry J. Vahala, and Oskar Painter. "Optomechanical crystals." *Nature*462, no. 7269 (2009): 78.

[9] Zotos, Euaggelos E. "Escapes in Hamiltonian systems with multiple exit channels: part I." *Nonlinear Dynamics* 78, no. 2 (2014): 1389-1420.

[10] Braginsky, V. B., and S. P. Vyatchanin. "Low quantum noise tranquilizer for Fabry–Perot interferometer." *Physics Letters A*293, no. 5-6 (2002): 228-234.

[11] Yasir, Kashif Ammar, Muhammad Ayub, and Farhan Saif. "Exponential localization of moving end mirror in optomechanics." *Journal of Modern Optics* 61, no. 16 (2014): 1318-1323.

[12] Khorasani, Sina. "Higher-Order Interactions in Quantum Optomechanics: Revisiting Theoretical Foundations." *Applied Sciences* 7, no. 7 (2017): 656.

[13] Zhang, Kuang, W. Chen, M. Bhattacharya, and P. Meystre. "Hamiltonian chaos in a coupled BEC–optomechanical-cavity system." *Physical Review A* 81, no. 1 (2010): 013802.

[14] Wang, Ying-Dan. "Y.-D. Wang and AA Clerk, Phys. Rev. Lett. 108, 153603 (2012)." *Phys. Rev. Lett.* 108 (2012): 153603.

[15] Singh, S., H. Jing, E. M. Wright, and P. Meystre. "Quantum-state transfer between a Bose-Einstein condensate and an optomechanical mirror." *Physical Review A* 86, no. 2 (2012): 021801.

[16] Buchmann, L. F. "LF Buchmann, L. Zhang, A. Chiruvelli, and P. Meystre, Phys. Rev. Lett. 108, 210403 (2012)." *Phys. Rev. Lett.* 108 (2012): 210403.

[17] Meystre, Pierre. "A short walk through quantum optomechanics." *Annalen der Physik* 525, no. 3 (2013): 215-233.

[18] Moore, M., and R. Blümel. "Quantum manifestations of order and chaos in the Paul trap." *Physical Review A* 48, no. 4 (1993): 3082.

[19] Frahm, Klaus M., and Dima L. Shepelyansky. "Quantum localization in rough billiards." *Physical review letters* 78, no. 8 (1997): 1440.

[20] Saif, Farhan. "Classical and quantum chaos in atom optics." *Physics Reports* 419, no. 6 (2005): 207-258.

[21] Pillet, Pierre. "Special issue on Optics and Interferometry with Atoms." *J. Phys. II* 4 (1994): 1877.

[22] Saif, Farhan. "Dynamical localization and signatures of classical phase space." *Physics Letters A* 274, no. 3-4 (2000): 98-103.

[23] Blümel, Reinhold, Andreas Buchleitner, Robert Graham, Leszek Sirko, Uzy Smilansky, and Herbert Walther. "Dynamical localization in the microwave interaction of Rydberg atoms: The influence of noise." *Physical Review A*44, no. 7 (1991): 4521.

[24] Moore, F. L., J. C. Robinson, C. Bharucha, P. E. Williams, and M. G. Raizen. "Observation of dynamical localization in atomic momentum transfer: A new testing ground for quantum chaos." *Physical review letters* 73, no. 22 (1994): 2974.

[25] Borgonovi, Fausto, Paolo Conti, Daniela Rebuzzi, Bambi Hu, and Baowen Li. "Cantori and dynamical localization in the Bunimovich stadium." *Physica D: Nonlinear Phenomena* 131, no. 1-4 (1999): 317-326.

[26] Saif, Farhan. "Dynamical localization and signatures of classical phase space." *Physics Letters A* 274, no. 3-4 (2000): 98-103.

[27] Gong, Jiangbin, Hans Jakob Wörner, and Paul Brumer. "Control of dynamical localization." *Physical Review E* 68, no. 5 (2003): 056202.

[28] Ayub, Muhammad, Kashif Ammar Yasir, and Farhan Saif. "Dynamical localization of matter waves in optomechanics." *Laser Physics* 24, no. 11 (2014): 115503.

[29] Yasir, Kashif Ammar, Muhammad Ayub, and Farhan Saif. "Exponential localization of moving end mirror in optomechanics." *Journal of Modern Optics* 61, no. 16 (2014): 1318-1323.

[30] Liu, Yu-Long, and Yu-xi Liu. "Energy-localization-enhanced ground-state cooling of a mechanical resonator from room temperature in optomechanics using a gain cavity." *Physical Review A* 96, no. 2 (2017): 023812.
[31] Fu, Hao, Zhi-cheng Gong, Li-ping Yang, Tian-hua Mao, Chang-pu Sun, Su Yi, Yong Li, and Geng-yu Cao. "Classical dynamical localization in a strongly driven two-mode mechanical system." *arXiv preprint arXiv:1706.06254* (2017).
[32] Fu, Hao, Zhi-cheng Gong, Li-ping Yang, Tian-hua Mao, Chang-pu Sun, Su Yi, Yong Li, and Geng-yu Cao. "Coherent Optomechanical Switch for Motion Transduction Based on Dynamically Localized Mechanical Modes." *Physical Review Applied* 9, no. 5 (2018): 054024.
[33] Major, Jan, Giovanna Morigi, and Jakub Zakrzewski. "Single-particle localization in dynamical potentials." *arXiv preprint arXiv:1808.07509* (2018).
[34] Wan, Liang-Liang, Xin-You Lü, Jin-Hua Gao, and Ying Wu. "Controllable photon and phonon localization in optomechanical Lieb lattices." *Optics express* 25, no. 15 (2017): 17364-17374.