

# $(\epsilon, \epsilon \vee q)$ -Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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**Abstract**—The aim of this paper is to introduce the concepts of  $(\epsilon, \epsilon \vee q)$ -fuzzy subalgebras,  $(\epsilon, \epsilon \vee q)$ -fuzzy ideals and  $(\epsilon, \epsilon \vee q)$ -fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

**Keywords**—BCI-algebras with operators,  $(\epsilon, \epsilon \vee q)$ -fuzzy subalgebras,  $(\epsilon, \epsilon \vee q)$ -fuzzy ideals,  $(\epsilon, \epsilon \vee q)$ -fuzzy quotient algebras.

## I. INTRODUCTION

THE fuzzy set is a generalization of the classical set and was used afterwards by several authors such as Imai [1], Iseki [2] and Xi [3], in various branches of mathematics. Particularly, in the area of fuzzy topology, after the introduction of fuzzy sets by Zadeh [15], much research has been carried out: the concept of fuzzy subalgebras and fuzzy ideals of BCK-algebras, and their some properties.

BCK-algebras and BCI-algebras are two important classes of logical algebras, which were introduced by Imai and Iseki [1], [2]. In 1991, Xi [3] applied the fuzzy sets to BCK-algebras and discussed some properties about fuzzy subalgebras and fuzzy ideals. From then on, fuzzy BCK/BCI-algebras have been widely investigated by some researchers. Jun et al. [4], [5] raised the notions of fuzzy positive implicative ideals and fuzzy commutative ideals of BCK-algebras. Ming and Ming [12] introduced the neighbourhood structure of a fuzzy point in 1980; Jun et al. [6] introduced the concept of  $(\epsilon, \epsilon \vee q)$ -ideals of BCI-algebras. In 1993, Zheng [7] defined operators in BCK-algebras and introduced the concept of BCI-algebras with operators and gave some isomorphism theorems of it. Then, Liu [9] introduced the university property of direct products of BCI-algebras. In 2002, Liu [8] introduced the notion of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [10] introduced the  $(\alpha, \beta)$ -fuzzy ideals of BCK/BCI-algebras and established the characterizations of  $(\epsilon, \epsilon \vee q)$ -fuzzy ideals. Next, Pan [13] introduced fuzzy ideals of sub-algebra and fuzzy H-ideals of sub-algebra. In 2011, Liu and Sun [11] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigated some basic properties. In 2017, we [14] also introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with

operators.

In this paper, we introduce the concepts of  $(\epsilon, \epsilon \vee q)$ -fuzzy subalgebras,  $(\epsilon, \epsilon \vee q)$ -fuzzy ideals and  $(\epsilon, \epsilon \vee q)$ -fuzzy quotient algebras of BCI-algebras with operators. Moreover, the basic properties were discussed and several results have been obtained.

## II. PRELIMINARIES

Some definitions and propositions were recalled which may be needed.

An algebra  $\langle X; *, 0 \rangle$  of type  $(2, 0)$  is called a BCI-algebra, if for all  $x, y, z \in X$ , it satisfies:

- (1)  $((x * y) * (x * z)) * (z * y) = 0$ ;
- (2)  $(x * (x * y)) * y = 0$ ;
- (3)  $x * x = 0$ ;
- (4)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$ .

We can define  $x * y = 0$  if and only if  $x \leq y$ , and the above conditions can be written as:

1.  $(x * y) * (x * z) \leq z * y$ ;
2.  $x * (x * y) \leq y$ ;
3.  $x \leq x$ ;
4.  $x \leq y$  and  $y \leq x$  imply  $x = y$ .

A BCI-algebra is called a BCK-algebra if it satisfies  $0 * x = 0$ .

**Definition 1.** [5]  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy subset  $A$  of  $X$  is called a fuzzy ideal of  $X$  if it satisfies:

- (1)  $A(0) \geq A(x), \forall x \in X$ ,
- (2)  $A(x) \geq A(x * y) \wedge A(y), \forall x, y \in X$ .

**Definition 2.** [4]  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy subset  $A$  of  $X$  is called a fuzzy subalgebra of  $X$  if it satisfies:

$$A(x * y) \geq A(x) * A(y), \forall x, y \in X.$$

**Definition 3.** [12]  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy subset  $A$  of  $X$  of the form

$$A(y) = \begin{cases} t (\neq 0), & y = x, \\ 0, & y \neq x, \end{cases}$$

is said to be a fuzzy point with support  $x$  and value  $t$ , and is

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denoted by  $x_t$ .

**Definition 4.** [12] If  $x_t$  is a fuzzy point, it is said to belong to (resp. be quasi-coincident with) a fuzzy subset  $A$ , written as  $x_t \in A$  (resp.  $x_t qA$ ) if  $A(x) \geq t$  (resp.  $A(x) + t > 1$ ). If  $x_t \in A$  or  $x_t qA$ , then we write  $x_t \in \vee qA$ . The symbol  $\overline{\in \vee q}$  (resp.  $\overline{\in}$  or  $\overline{q}$ ) means  $\in \vee q$  (resp.  $\in$  or  $q$ ) does not hold.

**Definition 5.** [10]  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy set  $A$  of  $X$  is called an  $(\in, \in \vee q)$ -fuzzy ideal of  $X$  if for all  $t, r \in (0, 1]$  and  $x, y \in X$ , it satisfies:

1.  $x_t \in A \Rightarrow 0_t \in \vee qA$ ,
2.  $(x * y)_{t_{inv}} \in A$  and  $y_r \in A \Rightarrow x_{inv} \in \vee qA$ .

**Definition 6.** [10] A fuzzy set  $A$  is an  $(\in, \in \vee q)$ -fuzzy ideal of  $X$  if and only if it satisfies:

- (1)  $A(0) \geq A(x) \wedge 0.5, \forall x \in X$ ,
- (2)  $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X$ .

**Definition 7.** [7]  $\langle X; *, 0 \rangle$  is a BCI-algebra,  $M$  is a non-empty set, if there exists a mapping  $(m, x) \rightarrow mx$  from  $M \times X$  to  $X$  which satisfies

$$m(x * y) = (mx) * (my), \forall x, y \in X, m \in M,$$

then  $M$  is called a left operator of  $X$ ,  $X$  is called BCI-algebra with left operator  $M$ , or  $M$ -BCI-algebra for short.

**Proposition 1.** [6] Let  $\langle X; *, 0 \rangle$  be a BCI-algebra, if  $A$  is an  $(\in, \in \vee q)$ -fuzzy ideal of it, and  $x * y \leq z$ , then

$$A(x) \geq A(y) \wedge A(z) \wedge 0.5, \forall x, y, z \in X.$$

**Definition 8.** [13] Let  $A$  and  $B$  be fuzzy sets of set  $X$ , then the direct product  $A \times B$  of  $A$  and  $B$  is a fuzzy subset of  $X \times X$ , define  $A \times B$  by

$$A \times B(x, y) = A(x) \wedge B(y), \forall x, y \in X.$$

**Definition 9.** [7] Let  $\langle X; *, 0 \rangle$  and  $\langle \bar{X}; *, 0 \rangle$  be two  $M$ -BCI-algebras, if for all  $x \in X, m \in M, f(mx) = mf(x)$ , and  $f$  is a homomorphism from  $\langle X; *, 0 \rangle$  to  $\langle \bar{X}; *, 0 \rangle$ , then  $f$  is called a homomorphism with operators.

**Definition 10.** [13]  $\langle X; *, 0 \rangle$  is an  $M$ -BCI-algebra, let  $B$  be a fuzzy set of  $X$ , and  $A$  be a fuzzy relation of  $B$ , if it satisfies:

$$A_B(x, y) = B(x) \wedge B(y), \forall x, y \in X,$$

then  $A$  is called a strong fuzzy relation of  $B$ .

**Definition 11.** [14] If  $\langle X; *, 0 \rangle$  is an  $M$ -BCI-algebra,  $A$  is a

non-empty subset of  $X$ , and  $mx \in A$  for all  $x \in A, m \in M$ , then  $\langle A; *, 0 \rangle$  is called a  $M$ -subalgebra of  $\langle X; *, 0 \rangle$ .

In this paper,  $X$  always means a  $M$ -BCI-algebra unless otherwise specified.

### III. $(\in, \in \vee q)$ -FUZZY SUBALGEBRAS OF BCI-ALGEBRAS WITH OPERATORS

**Definition 12.**  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy set  $A$  of  $X$  is called a  $M$ - $(\in, \in \vee q)$ -fuzzy subalgebra of  $X$  if for all  $t, r \in (0, 1]$  and  $x, y \in X$ , it satisfies:

1.  $x_t \in A$  and  $y_r \in A \Rightarrow (x * y)_{t_{inv}} \in \vee qA$ ,
2.  $x_t \in A \Rightarrow (mx)_t \in \vee qA$ .

**Proposition 2.**  $\langle X; *, 0 \rangle$  is a BCI-algebra, a fuzzy set  $A$  of  $X$  is an  $M$ - $(\in, \in \vee q)$ -fuzzy subalgebra of  $X$  if and only if it satisfies:

- (1)  $A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, \forall x, y \in X$ ,
- (2)  $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$ .

**Proof.** Suppose that  $A$  is an  $M$ - $(\in, \in \vee q)$ -fuzzy subalgebra of  $X$ . (1) Let  $x, y \in X$ , suppose that  $A(x) \wedge A(y) < 0.5$ , then  $A(x * y) \geq A(x) \wedge A(y)$ , if not, then we have  $A(x * y) < t < A(x) \wedge A(y), \exists t \in (0, 0.5)$ ; it follows that  $x_t \in A$  and  $y_t \in A$ , but  $(x * y)_{t_{inv}} = (x * y)_t \overline{\in \vee qA}$ , which is a contradiction, then whenever  $A(x) \wedge A(y) < 0.5$ . We have  $A(x * y) \geq A(x) \wedge A(y)$ . If  $A(x) \wedge A(y) \geq 0.5$ , then  $(x)_{0.5} \in A$  and  $(y)_{0.5} \in A$ , which implies that  $(x * y)_{0.5} = (x * y)_{0.5 \wedge 0.5} \in \vee qA$ , therefore  $A(x * y) \geq 0.5$ , because if  $A(x * y) < 0.5$ , then  $A(x * y) + 0.5 < 0.5 + 0.5 = 1$ , which is a contradiction, hence

$$A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, \forall x, y \in X.$$

(2) Let  $x \in X$  and assume that  $A(x) < 0.5$ . If  $A(mx) < A(x)$ , then we have  $A(mx) < t < A(x), \exists t \in (0, 0.5)$ , and we have  $x_t \in A$  and  $(mx)_t \overline{\in A}$ , since  $A(mx) + t < 1$ , we have  $(mx)_t \overline{qA}$ ; it follows that  $(mx)_t \overline{\in \vee qA}$ , which is a contradiction, hence  $A(mx) \geq A(x)$ . Now if  $A(x) \geq 0.5$ , then  $x_{0.5} \in A$ , thus  $(mx)_{0.5} \in \vee qA$ , hence  $A(mx) \geq 0.5$ , otherwise  $A(mx) + 0.5 < 0.5 + 0.5 = 1$ , which is a contradiction, consequently,  $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$ . Conversely, assume that  $A$  satisfies condition (1), (2).

(1) Let  $x, y \in X$  and  $t_1, t_2 \in (0, 1]$  be such that  $x_{t_1} \in A$  and  $y_{t_2} \in A$ , then  $A(x) \geq t_1$  and  $A(y) \geq t_2$ . Suppose that  $A(x * y) < t_1 \wedge t_2$ , if  $A(x) \wedge A(y) < 0.5$ , then  $A(x * y) \geq A(x) \wedge A(y) \wedge 0.5 = A(x) \wedge A(y) \geq t_1 \wedge t_2$ , this is a contradiction, so we have  $A(x) \wedge A(y) \geq 0.5$ , it follows that

$$A(x * y) + t_1 \wedge t_2 > 2A(x * y) \geq 2(A(x) \wedge A(y) \wedge 0.5) = 1,$$

so that  $(x * y)_{t_1 \wedge t_2} \in \vee qA$ .

(2) Let  $x \in X$  and  $t \in (0, 1]$  be such that  $x_t \in A$ , then we have  $A(x) \geq t$ . Suppose that  $A(mx) < t$ , if  $A(x) < 0.5$ , then  $A(mx) \geq A(x) \wedge 0.5 = A(x) \geq t$ , this is a contradiction, hence we know that  $A(x) \geq 0.5$ , and we have

$$A(mx) + t > 2A(mx) \geq 2(A(x) \wedge 0.5) = 1,$$

then  $(mx)_t \in \vee qA$ . Consequently,  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra.

**Example 1.** If  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ , then  $X_A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ , define  $X_A$  by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

**Proof.** (1) For all  $x, y \in X$ , if  $x, y \in A$ , then  $x * y \in A$ , then we have

$$X_A(x * y) = 1 \geq X_A(x) \wedge X_A(y) \wedge 0.5,$$

if there exists at least one which does not belong to  $A$  between  $x$  and  $y$ , for example  $x \notin A$ , thus

$$X_A(x * y) \geq 0 = X_A(x) \wedge X_A(y) \wedge 0.5.$$

(2) For all  $x \in X, m \in M$ , if  $x \in A$ , then  $mx \in A$ , therefore

$$X_A(mx) = 1 \geq X_A(x) \wedge 0.5,$$

if  $x \notin A$ , then  $X_A(mx) \geq 0 = X_A(x) \wedge 0.5$ , therefore  $X_A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ .

**Proposition 3.**  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$  if and only if  $A_t$  is an  $M -$ subalgebra of  $X$ , where  $A_t$  is a non-empty set, define  $X_A$  by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 0.5].$$

**Proof.** Suppose  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ ,  $A_t$  is a non-empty set,  $t \in [0, 0.5]$ , then we have  $A(x * y) \geq A(x) \wedge A(y) \wedge 0.5$ . If  $x \in A_t, y \in A_t$ , then  $A(x) \geq t, A(y) \geq t$ , thus

$$A(x * y) \geq A(x) \wedge A(y) \wedge 0.5 \geq t,$$

then we have  $x * y \in A_t$ . If  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ , then  $A(mx) \geq A(x) \wedge 0.5 \geq t, \forall x \in X, m \in M$ , then we have  $mx \in A_t$ . Therefore  $A_t$  is an  $M -$ subalgebra of  $X$ . Conversely, suppose  $A_t$  is an  $M -$ subalgebra of  $X$ , then we have  $x * y \in A_t$ . Let  $A(x) = t$ , then

$$A(x * y) \geq t = A(x) \geq A(x) \wedge A(y) \wedge 0.5.$$

If  $A_t$  is an  $M -$ subalgebra of  $X$ , then we have

$$A(mx) \geq t = A(x) \geq A(x) \wedge 0.5, \forall x \in X, m \in M,$$

therefore  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ .

**Proposition 4.** Suppose  $X, Y$  are  $M -$ BCI-algebras,  $f$  is a mapping from  $X$  to  $Y$ , if  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of the  $Y$ , then  $f^{-1}(A)$  is a  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ .

**Proof.** Let  $y \in Y$ , suppose  $f$  is an epimorphism, and we have  $y = f(x), \exists x \in X$ . If  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $Y$ , then we have

$$A(x * y) \geq A(x) \wedge A(y) \wedge 0.5, A(mx) \geq A(x) \wedge 0.5.$$

For all  $x, y \in X, m \in M$ , we have

$$(1) f^{-1}(A)(x * y) = A(f(x) * f(y)) \geq A(f(x)) \wedge A(f(y)) \wedge 0.5 = f^{-1}(A)(x) \wedge f^{-1}(A)(y) \wedge 0.5;$$

$$(2) f^{-1}(A)(mx) = A(f(mx)) = A(mf(x)) \geq A(f(x)) \wedge 0.5 = f^{-1}(A)(x) \wedge 0.5.$$

Then  $f^{-1}(A)$  is an  $M - (\in, \in \vee q)$ -fuzzy subalgebra of  $X$ .

#### IV. $(\in, \in \vee q)$ -FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

**Definition 13.**  $\langle X; *; 0 \rangle$  is a BCI-algebra, a fuzzy set  $A$  of  $X$  is called an  $M - (\in, \in \vee q)$ -fuzzy ideal of  $X$  if for all  $t, r \in (0, 1]$  and  $x, y \in X$ , it satisfies:

1.  $x_t \in A \Rightarrow 0_t \in \vee qA$ ,
2.  $(x * y)_t \in A$  and  $y_r \in A \Rightarrow x_{t \wedge r} \in \vee qA$ ,
3.  $x_t \in A \Rightarrow (mx)_t \in \vee qA$ .

**Proposition 5.** [13]  $\langle X; *; 0 \rangle$  is a BCI-algebra, a fuzzy set  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy ideal of  $X$  if and only if it satisfies:

- (1)  $A(0) \geq A(x) \wedge 0.5, \forall x \in X$ ,
- (2)  $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X$ ,
- (3)  $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$ .

**Proof.** Suppose that  $A$  is an  $M - (\in, \in \vee q)$ -fuzzy ideal of  $X$ .

(1) Let  $x \in X$  and assume that  $A(x) < 0.5$ . If  $A(0) < A(x)$ , then we have  $A(0) < t < A(x), \exists t \in (0, 0.5)$ , and we have  $x_t \in A$  and  $0_t \in \bar{A}$ , since  $A(0) + t < 1$ , we have  $0_t \in \bar{qA}$ , it follows that  $0_t \in \overline{\vee qA}$ , which is a contradiction, then  $A(0) \geq A(x)$ . Now if  $A(x) \geq 0.5$ , then  $x_{0.5} \in A$ , then we have  $0_{0.5} \in \vee qA$ , hence  $A(0) \geq 0.5$ , otherwise,  $A(0) + 0.5 < 0.5 + 0.5 = 1$ , which is a contradiction, consequently,

$$A(0) \geq A(x) \wedge 0.5, \forall x \in X.$$

(2) Let  $x, y \in X$  and suppose that  $A(x * y) \wedge A(y) < 0.5$ , then  $A(x) \geq A(x * y) \wedge A(y)$ , if not, then we have  $A(x) < t < A(x * y) \wedge A(y), \exists t \in (0, 0.5)$ , it follows that  $(x * y)_t \in A$  and  $y_t \in A$ , but  $x_{t \wedge t} = x_t \in \overline{\vee qA}$ , which is a contradiction, hence whenever  $A(x * y) \wedge A(y) < 0.5$ , we have  $A(x) \geq A(x * y) \wedge A(y)$ . If  $A(x * y) \wedge A(y) \geq 0.5$ , then  $(x * y)_{0.5} \in A$  and  $y_{0.5} \in A$ , which implies that  $x_{0.5} = x_{0.5 \wedge 0.5} \in \vee qA$ , therefore  $A(x) \geq 0.5$ , because if  $A(x) < 0.5$ , then  $A(x) + 0.5 < 0.5 + 0.5 = 1$ , which is a contradiction, then

$$A(x) \geq A(x * y) \wedge A(y) \wedge 0.5, \forall x, y \in X.$$

(3) Let  $x \in X$  and assume that  $A(x) < 0.5$ . If  $A(mx) < A(x)$ , then we have  $A(mx) < t < A(x), \exists t \in (0, 0.5)$ , and we have  $x_t \in A$  and  $(mx)_t \in \bar{A}$ , since  $A(mx) + t < 1$ , we have  $(mx)_t \in \bar{qA}$ , it follows that  $(mx)_t \in \overline{\vee qA}$ , which is a contradiction, then  $A(mx) \geq A(x)$ . Now if  $A(x) \geq 0.5$ , then  $x_{0.5} \in A$ , thus  $(mx)_{0.5} \in \vee qA$ , hence  $A(mx) \geq 0.5$ , otherwise  $A(mx) + 0.5 < 0.5 + 0.5 = 1$ , which is a contradiction, consequently,  $A(mx) \geq A(x) \wedge 0.5, \forall x \in X$ . Conversely, suppose that  $A$  satisfies (1), (2), (3) of the Proposition 5, then we have

(1) Let  $x \in X$  and  $t \in (0, 1]$  be such that  $x_t \in A$ , then we have  $A(x) > t$ , suppose that  $A(0) < t$ , if  $A(x) < 0.5$ , then  $A(0) \geq A(x) \wedge 0.5 = A(x) \geq t$ , which is a contradiction, then we know that  $A(x) \geq 0.5$ , and we have  $A(0) + t > 2A(0) \geq 2(A(x) \wedge 0.5) = 1$ , thus  $0_t \in \vee qA$ .

(2) Let  $x, y \in X$  and  $t_1, t_2 \in (0, 1]$  be such that  $(x * y)_{t_1} \in A$  and  $y_{t_2} \in A$ , then  $A(x * y) \geq t_1$  and  $A(y) \geq t_2$ , suppose that  $A(x) < t_1 \wedge t_2$ , if  $A(x * y) \wedge A(y) < 0.5$ , then

$$A(x) \geq A(x * y) \wedge A(y) \wedge 0.5 = A(x * y) \wedge A(y) \geq t_1 \wedge t_2,$$

This is a contradiction, so we have  $A(x * y) \wedge A(y) \geq 0.5$ , it

follows that

$$A(x) + t_1 \wedge t_2 > 2A(x) \geq 2(A(x * y) \wedge A(y) \wedge 0.5) = 1,$$

so that  $x_{t_1 \wedge t_2} \in \vee qA$ .

(3) Let  $x \in X$  and  $t \in (0, 1]$  be such that  $x_t \in A$ , then  $A(x) \geq t$ , suppose that  $A(mx) < t$ , if  $A(x) < 0.5$ , then  $A(mx) \geq A(x) \wedge 0.5 = A(x) \geq t$ , which is a contradiction, then we know that  $A(x) \geq 0.5$ , and we have  $A(mx) + t > 2A(mx) \geq 2(A(x) \wedge 0.5) = 1$ , thus  $(mx)_t \in \vee qA$ . Consequently,  $A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal.

**Example 2.** If  $A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ , then  $X_A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ , define  $X_A$  by

$$X_A : X \rightarrow [0, 1], X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

**Proof.** (1) For all  $x, y \in X$ , if  $x, y \in A$ , then  $x * y \in A$ , thus

$$\begin{aligned} X_A(0) &= 1 \geq X_A(x) \wedge 0.5, \\ X_A(x) &= 1 \geq X_A(x * y) \wedge X_A(y) \wedge 0.5, \end{aligned}$$

if there exists at least one between  $x$  and  $y$  which does not belong to  $A$ , for example  $x \notin A$ , thus

$$\begin{aligned} X_A(0) &= 1 \geq X_A(x) \wedge 0.5, \\ X_A(x) &\geq X_A(x * y) \wedge X_A(y) \wedge 0.5 = 0, \end{aligned}$$

therefore  $X_A$  is a  $(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

(2) For all  $x \in X, m \in M$ , if  $x \in A$ , then  $mx \in A$ , therefore  $X_A(mx) = 1 \geq X_A(x) \wedge 0.5$ . If  $x \notin A$ , then  $X_A(mx) = 0 = X_A(x) \wedge 0.5$ , therefore  $X_A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

**Proposition 6.**  $A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$  if and only if  $A_t$  is an  $M$ -ideal of  $X$ , where  $A_t$  is non-empty set, define  $A_t$  by

$$A_t = \{x \mid x \in X, A(x) \geq t\}, \forall t \in [0, 0.5].$$

**Proof.** Suppose  $A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ ,  $A_t$  is non-empty set,  $t \in [0, 0.5]$ , then we have  $A(0) \geq A(x) \wedge 0.5 \geq t$ , then we have  $0 \in A_t$ . If  $x * y \in A_t, y \in A_t$ , then  $A(x * y) \geq t, A(y) \geq t$ , thus  $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5 \geq t$ , then we have  $x \in A_t$ . For all  $x \in X, m \in M$ , if  $A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ , hence  $A(mx) \geq A(x) \wedge 0.5 \geq t$ , thus  $mx \in A_t$ , therefore  $A_t$  is an  $M$ -ideal of  $X$ . Conversely, suppose  $A_t$  is

an  $M$  – ideal of  $X$ , then we have  $0 \in A, A(0) \geq t$ . Let  $A(x) = t$ , thus  $x \in A$ , we have  $A(0) \geq t = A(x)$ , suppose there is no  $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5$ , then there exist  $x_0, y_0 \in X$ , we have  $A(x_0) < A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$ , let  $t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$ , then  $A(x_0) < t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$ , if  $x_0 * y_0 \in A_{t_0}, y_0 \in A_{t_0}$ , then we have  $x_0 \in A_{t_0}$ , then  $A(x_0) \geq t_0$ , which is inconsistent with  $A(x_0) < t_0 = A(x_0 * y_0) \wedge A(y_0) \wedge 0.5$ , then we have  $A(x) \geq A(x * y) \wedge A(y) \wedge 0.5$ . If  $A$  is an  $M$  – ideal of  $X$ , then we have  $A(mx) \geq t \wedge 0.5 = A(x) \wedge 0.5, \forall x \in X, m \in M$ , therefore  $A$  is an  $M$  –  $(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

**Proposition 7.** Suppose  $X, Y$  are  $M$  – BCI-algebras,  $f$  is a mapping from  $X$  to  $Y$ ,  $A$  is an  $M$  –  $(\in, \in \vee q)$ -fuzzy ideal of  $Y$ , then  $f^{-1}(A)$  is an  $M$  –  $(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

**Proof.** Let  $y \in Y$ , suppose  $f$  is an epimorphism, then we have  $y = f(x), \exists x \in X$ . If  $A$  is an  $M$  –  $(\in, \in \vee q)$ -fuzzy ideal of  $Y$ , then we have

$$\begin{aligned} A(0) &\geq A(x) \wedge 0.5, \\ A(x) &\geq A(x * y) \wedge A(y) \wedge 0.5, \\ A(mx) &\geq A(x) \wedge 0.5. \end{aligned}$$

For all  $x, y \in X, m \in M$ , we have

- (1)  $f^{-1}(A)(0) = A(f(0)) = A(0) \geq A(f(x)) \wedge 0.5 = f^{-1}(A)(x) \wedge 0.5;$
- (2)  $f^{-1}(A)(x) = A(f(x)) \geq A(f(x) * f(y)) \wedge A(f(y)) \wedge 0.5 = A(f(x * y)) \wedge A(f(y)) \wedge 0.5 = f^{-1}(A)(x * y) \wedge f^{-1}(A)(y) \wedge 0.5;$
- (3)  $f^{-1}(A)(mx) = A(f(mx)) = A(mf(x)) \geq A(f(x)) \wedge 0.5 = f^{-1}(A)(x) \wedge 0.5.$

Therefore  $f^{-1}(A)$  is an  $M$  –  $(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

V.  $(\in, \in \vee q)$  - FUZZY QUOTIENT BCI-ALGEBRAS WITH OPERATORS

**Definition 14.** Let  $A$  be an  $M$  –  $(\in, \in \vee q)$ -fuzzy ideal of  $X$ , for all  $a \in X$ , fuzzy set  $A_a$  on  $X$  defined as:  $A_a : X \rightarrow [0, 1]$

$$A_a(x) = A(a * x) \wedge A(x * a) \wedge 0.5, \forall x \in X.$$

Denote  $X/A = \{A_a : a \in X\}$ .

**Proposition 8.** Let  $A_a, A_b \in X/A$ , then  $A_a = A_b$  if and only if

$$A(a * b) \wedge A(b * a) \wedge 0.5 = A(0) \wedge 0.5.$$

**Proof.** Let  $A_a = A_b$ , then we have  $A_a(b) = A_b(b)$ , thus

$$A(a * b) \wedge A(b * a) \wedge 0.5 = A(b * b) \wedge A(b * b) \wedge 0.5 = A(0) \wedge 0.5,$$

that is  $A(a * b) \wedge A(b * a) \wedge 0.5 = A(0) \wedge 0.5$ . Conversely, suppose

that  $A(a * b) \wedge A(b * a) \wedge 0.5 = A(0) \wedge 0.5$ . For all  $x \in X$ , since

$$(a * x) * (b * x) \leq a * b, (x * a) * (x * b) \leq b * a.$$

It follows from Proposition 1 that

$$\begin{aligned} A(a * x) &\geq A(b * x) \wedge A(a * b) \wedge 0.5, \\ A(x * a) &\geq A(x * b) \wedge A(b * a) \wedge 0.5. \end{aligned}$$

Hence

$$\begin{aligned} A_a(x) &= A(a * x) \wedge A(x * a) \wedge 0.5 \\ &\geq A(b * x) \wedge A(x * b) \wedge A(a * b) \wedge A(b * a) \wedge 0.5 \\ &= A(b * x) \wedge A(x * b) \wedge A(0) \wedge 0.5 \\ &= A(b * x) \wedge A(x * b) \wedge 0.5 = A_b(x), \end{aligned}$$

that is  $A_a \geq A_b$ . Similarly, for all  $x \in X$ , since

$$(b * x) * A(a * x) \leq b * a, (x * b) * A(x * a) \leq a * b.$$

It follows from Proposition 1 that

$$\begin{aligned} A(b * x) &\geq A(a * x) \wedge A(b * a) \wedge 0.5, \\ A(x * b) &\geq A(x * a) \wedge A(a * b) \wedge 0.5. \end{aligned}$$

Hence

$$\begin{aligned} A_b(x) &= A(b * x) \wedge A(x * b) \wedge 0.5 \\ &\geq A(a * x) \wedge A(x * a) \wedge A(b * a) \wedge A(a * b) \wedge 0.5 \\ &= A(a * x) \wedge A(x * a) \wedge A(0) \wedge 0.5 \\ &= A(a * x) \wedge A(x * a) \wedge 0.5 = A_a(x), \end{aligned}$$

that is  $A_b \geq A_a$ . Therefore,  $A_a = A_b$ . We complete the proof.

**Proposition 9.** Let  $A_a = A_{a'}, A_b = A_{b'}$ , then  $A_{a * b} = A_{a' * b'}$ .

**Proof.** Since

$$\begin{aligned} ((a * b) * (a' * b')) * (a * a') &= ((a * b) * (a * a')) * (a' * b') \\ &\leq (a' * b) * (a' * b') \leq b' * b, \\ ((a' * b') * (a * b)) * (b * b') &= ((a' * b') * (b * b')) * (a * b) \\ &\leq (a' * b) * (a * b) \leq a' * a. \end{aligned}$$

Hence

$$\begin{aligned} A((a * b) * (a' * b')) &\geq A(a * a') \wedge A(b' * b) \wedge 0.5, \\ A((a' * b') * (a * b)) &\geq A(b * b') \wedge A(a' * a) \wedge 0.5. \end{aligned}$$

Therefore

$$\begin{aligned} A((a * b) * (a' * b')) &\wedge A((a' * b') * (a * b)) \wedge 0.5 \\ &= A(a * a') \wedge A(a' * a) \wedge 0.5 \wedge A(b * b') \wedge A(b' * b) \wedge 0.5 \wedge 0.5 \\ &= A(0) \wedge 0.5, \end{aligned}$$

it follows from Proposition 8. that  $A_{a * b} = A_{a' * b'}$ . We completed the proof.

Let  $A$  be an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ . The operation "\*" of  $R/A$  is defined as:  $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$ . By Proposition 8, the above operation is reasonable.

**Proposition 10.**  $A$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ , then  $R/A = \{R/A; *, A_0\}$  is an  $M$ -BCI-algebra.

**Proof.** For all  $A_x, A_y, A_z \in R/A$ , we have

$$\begin{aligned} ((A_x * A_y) * (A_x * A_z)) * (A_z * A_y) &= A_{((x*y)*(x*z))*(z*y)} = A_0; \\ (A_x * (A_x * A_y)) * A_y &= A_{(x*(x*y))*y} = A_0; \\ A_x * A_x &= A_{x*x} = A_0; \end{aligned}$$

if  $A_x * A_y = A_0, A_y * A_x = A_0$ , then  $A_{x*y} = A_0, A_{y*x} = A_0$ , it follows from Proposition 8 that  $A(x*y) = A(0), A(y*x) = A(0)$ , hence  $A(x*y) \wedge A(y*x) \wedge 0.5 = A(0) \wedge 0.5$ , then we have  $A_x = A_y$ . Therefore  $R/A = \{R/A; *, A_0\}$  is a BCI-algebra. For all  $A_x \in R/A, m \in M$ , we define  $mA_x = A_{mx}$ . Firstly, we verify that  $mA_x = A_{mx}$  is reasonable. If  $A_x = A_y$ , then we verify  $mA_x = mA_y$ , that is to verify  $A_{mx} = A_{my}$ . We have

$$\begin{aligned} A(mx * my) \wedge 0.5 &= A(m(x * y)) \wedge 0.5 \geq A(x * y) \wedge 0.5, \\ A(my * mx) \wedge 0.5 &= A(m(y * x)) \wedge 0.5 \geq A(y * x) \wedge 0.5, \end{aligned}$$

so we have

$$A(mx * my) \wedge A(my * mx) \wedge 0.5 \geq A(x * y) \wedge A(y * x) \wedge 0.5 = A(0) \wedge 0.5,$$

then  $A(mx * my) \wedge A(my * mx) \wedge 0.5 = A(0) \wedge 0.5$ , that is  $A_{mx} = A_{my}$ .

In addition, for all  $m \in M, A_x, A_y \in R/A$ , we have

$$\begin{aligned} m(A_x * A_y) &= mA_{x*y} = A_{m(x*y)} \\ &= A_{(mx)*(my)} = A_{mx} * A_{my} = mA_x * mA_y. \end{aligned}$$

Therefore  $R/A = \{R/A; *, A_0\}$  is an  $M$ -BCI-algebra.

**Definition 15.** Let  $\mu$  be an  $M-(\in, \in \vee q)$ -fuzzy subalgebra of  $X$ , and  $A$  be an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ , we define a fuzzy set of  $X/A$  as follows:

$$\mu/A : X/A \rightarrow [0, 1], \quad \mu/A(A_i) = \sup_{A_i=A_j} \mu(x) \wedge 0.5, \quad \forall A_i \in X/A.$$

**Proposition 11.**  $\mu/A$  is an  $M-(\in, \in \vee q)$ -fuzzy subalgebra of  $X/A$ .

**Proof.** For all  $A_x, A_y \in X/A$ , we have

$$\begin{aligned} \mu/A(A_x * A_y) &= \mu/A(A_{x*y}) = \sup_{A_i=A_{x*y}} \mu(z) \wedge 0.5 \\ &\geq \sup_{A_i=A_x, A_j=A_y} \mu(s * t) \wedge 0.5 \geq \sup_{A_i=A_x, A_j=A_y} \mu(s) \wedge \mu(t) \wedge 0.5 \\ &= \sup_{A_i=A_x} \mu(s) \wedge \sup_{A_j=A_y} \mu(t) \wedge 0.5 \\ &= \mu/A(A_x) \wedge \mu/A(A_y) \wedge 0.5. \end{aligned}$$

For all  $m \in M, A_x \in R/A$ , we have

$$\begin{aligned} \mu/A(A_{mx}) &= \sup_{A_{mc}=A_{mx}} \mu(mz) \wedge 0.5 \\ &\geq \sup_{A_i=A_x} \mu(z) \wedge 0.5 = \mu/A(A_x) \wedge 0.5. \end{aligned}$$

Therefore  $\mu/A$  is an  $M-(\in, \in \vee q)$ -fuzzy subalgebra of  $X/A$ .

#### VI. DIRECT PRODUCTS OF $(\in, \in \vee q)$ -FUZZY IDEALS OF BCI-ALGEBRAS WITH OPERATORS

**Proposition 12.** Suppose  $A$  and  $B$  are  $M-(\in, \in \vee q)$ -fuzzy ideals of  $X$ , then  $A \times B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ .

**Proof.** (1) Let  $(x, y) \in X \times X$ , then

$$\begin{aligned} A \times B(0, 0) &= A(0) \wedge B(0) \geq A(x) \wedge 0.5 \wedge B(y) \wedge 0.5 \\ &= A(x) \wedge B(y) \wedge 0.5 = A \times B(x, y) \wedge 0.5, \end{aligned}$$

then  $A \times B(0, 0) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X$ ;

(2) For all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} A \times B((x_1, x_2) * (y_1, y_2)) &\wedge A \times B(y_1, y_2) \wedge 0.5 \\ &= A \times B(x_1 * y_1, x_2 * y_2) \wedge A \times B(y_1, y_2) \wedge 0.5 \\ &= (A(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge A(y_1) \wedge B(y_2) \wedge 0.5 \\ &= (A(x_1 * y_1) \wedge A(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)) \wedge 0.5 \\ &\leq A(x_1) \wedge B(x_2) = A \times B(x_1, x_2), \end{aligned}$$

then for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$A \times B(x_1, x_2) \geq A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5;$$

(3) For all  $(x, y) \in X \times X$ , we have

$$\begin{aligned} A \times B(m(x, y)) &= A \times B(mx, my) = A(mx) \wedge B(my) \\ &\geq A(x) \wedge 0.5 \wedge B(y) \wedge 0.5 = A(x) \wedge B(y) \wedge 0.5 \\ &= A \times B(x, y) \wedge 0.5, \end{aligned}$$

then we have

$$A \times B(m(x, y)) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X.$$

Therefore  $A \times B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ .

**Proposition 13.** Suppose  $A$  and  $B$  are fuzzy sets of  $X$ , if  $A \times B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ , then  $A$  or  $B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

**Proof.** Suppose  $A$  and  $B$  are  $M-(\in, \in \vee q)$ -fuzzy ideals of  $X$ , then for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} A \times B(x_1, x_2) &\geq A \times B((x_1, x_2) * (y_1, y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5 \\ &= A \times B((x_1 * y_1), (x_2 * y_2)) \wedge A \times B(y_1, y_2) \wedge 0.5, \end{aligned}$$

if  $x_1 = y_1 = 0$ , then

$$A \times B(0, x_2) \geq A \times B(0, x_2 * y_2) \wedge A \times B(0, y_2) \wedge 0.5,$$

then we have

$$A \times B(0, x) = A(0) \wedge B(x) = B(x),$$

thus  $B(x_2) \geq B(x_2 * y_2) \wedge B(y_2) \wedge 0.5$ . If  $A \times B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ , then

$$A \times B(m(x, y)) \geq A \times B(x, y) \wedge 0.5, \forall (x, y) \in X \times X,$$

let  $x = 0$ , then

$$\begin{aligned} A \times B(m(x, y)) &= A \times B(mx, my) = A(mx) \wedge B(my) = B(my) \\ &\geq A(x) \wedge B(y) \wedge 0.5 = A(0) \wedge B(y) \wedge 0.5 \\ &= B(y) \wedge 0.5, \end{aligned}$$

then we have

$$B(my) \geq B(y) \wedge 0.5, \forall y \in X, m \in M.$$

Therefore  $B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

**Proposition 14.** If  $B$  is a fuzzy set,  $A$  is a strong fuzzy relation  $A_B$  of  $B$ , then  $B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$  if and only if  $A_B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ .

**Proof.** If  $B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideals of  $X$ , then for all  $(x, y) \in X \times X$ , we have

$$\begin{aligned} A_B(0, 0) &= B(0) \wedge B(0) \geq B(x) \wedge 0.5 \wedge B(y) \wedge 0.5 \\ &= A_B(x, y) \wedge 0.5; \end{aligned}$$

for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} A_B(x_1, x_2) &= B(x_1) \wedge B(x_2) \\ &\geq (B(x_1 * y_1) \wedge B(y_1) \wedge 0.5) \wedge (B(x_2 * y_2) \wedge B(y_2) \wedge 0.5) \\ &= (B(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge (B(y_1) \wedge B(y_2)) \wedge 0.5 \\ &= A_B(x_1 * y_1, x_2 * y_2) \wedge A_B(y_1, y_2) \wedge 0.5 \\ &= A_B((x_1, x_2) * (y_1, y_2)) \wedge A_B(y_1, y_2) \wedge 0.5; \end{aligned}$$

for all  $(x, y) \in X \times X$ , we have

$$\begin{aligned} A_B(m(x, y)) &= A_B(mx, my) = B(mx) \wedge B(my) \\ &\geq B(x) \wedge 0.5 \wedge B(y) \wedge 0.5 = A_B(x, y) \wedge 0.5. \end{aligned}$$

Therefore  $A_B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ .

Conversely, suppose  $A_B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ , for all  $(x_1, x_2) \in X \times X$ , we have

$$B(0) \wedge B(0) = A_B(0, 0) \geq A_B(x, x) \wedge 0.5 = B(x) \wedge B(x) \wedge 0.5,$$

for all  $(x_1, x_2), (y_1, y_2) \in X \times X$ , we have

$$\begin{aligned} B(x_1) \wedge B(x_2) &= A_B(x_1, x_2) \\ &\geq A_B((x_1, x_2) * (y_1, y_2)) \wedge A_B(y_1, y_2) \wedge 0.5 \\ &= A_B(x_1 * y_1, x_2 * y_2) \wedge A_B(y_1, y_2) \wedge 0.5 \\ &= (B(x_1 * y_1) \wedge B(x_2 * y_2)) \wedge (B(y_1) \wedge B(y_2)) \wedge 0.5 \\ &= (B(x_1 * y_1) \wedge B(y_1)) \wedge (B(x_2 * y_2) \wedge B(y_2)) \wedge 0.5, \end{aligned}$$

let  $x_2 = y_2 = 0$ , then

$$B(x_1) \wedge B(0) \geq (B(x_1 * y_1) \wedge B(y_1)) \wedge B(0) \wedge 0.5,$$

if  $A_B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X \times X$ , then

$$A_B(m(x, y)) \geq A_B(x, y), \forall x, y \in X \times X, m \in M,$$

We have

$$B(mx) \wedge B(my) = A_B(mx, my) \geq A_B(x, y) \wedge 0.5 = B(x) \wedge B(y) \wedge 0.5,$$

if  $x = 0$ , then

$$B(0) \wedge B(my) = A_B(0, my) \geq A_B(0, y) \wedge 0.5 = B(0) \wedge B(y) \wedge 0.5,$$

namely,  $B(my) \geq B(y) \wedge 0.5$ . Therefore  $B$  is an  $M-(\in, \in \vee q)$ -fuzzy ideal of  $X$ .

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