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Propagation of a Generalized Beam in ABCD System

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***Abstract*—**For a generalized Hermite sinosiodal / hyperbolic Gaussian beam passing through an ABCD system with a finite aperture, the propagation properties are derived using the Collins integral. The results are obtained in the form of intensity graphs indicating that previously demonstrated rules of reciprocity are applicable, while the existence of the aperture accelerates this transformation.

***Keywords*—**Optical communications, Hermite-Gaussian beams, ABCD system.

I. Introduction

UP TO NOW, we have investigated the characteristics of cosh-Gaussian [1], cos-Gaussian [2] Hermite cosh-Gaussian [3], Hermite cos-Gaussian [4] and Hermite-sinh-Gaussian [5] beams propagating in turbulent atmosphere. In this particular study, we take a generalized Hermite sinosiodal / hyperbolic Gaussian source beam, and pass it through an ABCD system with a finite aperture by applying the Collins integral eventually to obtain the receiver field.

Such beams have been studied for on-axis and also for off-axis situations. For instance, [6]-[10] deal with the propagation properties of these beams for axial (on-axis) cases, while [11]-[13] cover the off-axis cases as well as the less general beams such as Hermite-Gaussian type.

It is known that this kind of studies contribute to optical communications, laser radar, imaging and remote sensing applications [14].

II. Formulation

The propagation geometry, illustrated in Fig. 1, is arranged as source and receiver planes laying perpendicular to the axis of propagation, z. The filling medium is free space, i.e., no turbulent atmosphere. The x and y coordinates on the source plane are denoted by the vector $\boldsymbol{s}$ decomposed as $\boldsymbol{s} = (s_x, s_y)$. At the receiver plane, $\boldsymbol{p}$ is the transverse position vector decomposed as $\boldsymbol{p} = (p_x, p_y)$. In this setup, $(\boldsymbol{s})$ or $(\boldsymbol{p},\ z = 0)$ will mean a location on the source plane, whereas $(\boldsymbol{p},\ z = L)$ will point to a location on the receiver plane.

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If optical elements are placed along the propagation axis, z, their combined effects will be described by a 2x2 ABCD ray matrix. For instance, the simple case of a thin lens with focal length f, will have the following representation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \tag{1}$$

Normally, the series of elements (or parts of the medium) encountered on the propagation path will be accounted for by the multiplication of the individual matrices in the reverse order, that is from receiver towards source. For our present purposes however, it is sufficient to carry on with a single ABCD matrix, also noting that, for line of sight, i.e., for free space, the related matrix will take the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{2}$$

where L, measures the distance from the source plane to the receiver plane.

The source field, $u_s(\boldsymbol{s}) = u_s(s_x, s_y)$ is expressed as

$$u_s(\boldsymbol{s}) = A_C H_n(a_x s_x + b_x) H_m(a_y s_y + b_y) \exp\left[-0.5\left(s_x^2/\alpha_{sx}^2 + s_y^2/\alpha_{sy}^2\right)\right] \times\left\{l_1 \exp\left[j\left(V_x s_x + V_y s_y\right)\right] + l_2 \exp\left[j\left(Y_x s_x + Y_y s_y\right)\right]\right\} \tag{3}$$

Here, A_C is the amplitude normalization term and for convenience will be set to unity from this point onwards, $j = (-1)^{0.5}$, α_{sx} and α_{sy} refer to source sizes of the Gaussian beam in x and y directions. Hermite polynomials, $H_n(a_x s_x + b_x)$ and $H_m(a_y s_y + b_y)$, denote the field distributions of order n, width a_x and displacement b_x for s_x axis and of order m, width a_y and displacement b_y for s_y axis. V_x, V_y, Y_x, Y_y are the complex displacement parameters of the related axes, similarly l_1 and l_2 are the complex amplitude parameters such that the appropriate settings of V_x, V_y, Y_x, Y_y, l_1 and l_2 will, as explained in [5], successively produce cos-Gaussian, cosh-Gaussian, sine-Gaussian and sinh-Gaussian beams. In our terminology, (3) is regarded to represent the generalized Hermite sinosiodal / hyperbolic Gaussian beam types, shortened as generalized beam within the scope of this study.

The field at the receiver plane, $u_r(\boldsymbol{p}, z = L) = u_r(p_x, p_y, z = L)$ is to be calculated via

Collins integral (also known as extended Hugens-Fresnel integral)

$$u_r(\boldsymbol{p}, z=L) = jk/(2\pi B)\int_{t_{y1}}^{t_{y2}}\int_{t_{x1}}^{t_{x2}} \boldsymbol{d^2 s} u_s(\boldsymbol{s})$$
$$\times\exp\left\{-0.5jk\left[A\left(s_x^2+s_y^2\right)-2\left(s_x p_x + s_y p_y\right)\right.\right.$$
$$\left.\left.+D\left(p_x^2+p_y^2\right)\right]/B\right\} \quad (4)$$

where A and B are the elements belonging to ABCD matrix, $k = 2\pi/\lambda$ is the wave number with λ being the wavelength of operation and $u_s(\boldsymbol{s})$ is the source field as defined by (3). The limits of the integration, t_{x1}, t_{x2}, t_{y1} and t_{y2}, in (4) refer to the dimensions of the aperture along s_x and s_y respectively (see Figs. 1 and 4). In writing (4), we have omitted the phase term which is of no significance to the present aim of this paper.

After substitution for $u_s(\boldsymbol{s})$ in (4), from (3), expanding all Hermite polynomials, performing a term by term integration based on the derivations given in the Appendix, the following result is achieved.

$$u_r(\boldsymbol{p}, z=L) = \left[jk/(2\pi B)\right]\exp\left[-0.5jkD\left(p_x^2+p_y^2\right)/B\right]$$
$$\times\left\{l_1\exp\left[-0.5\alpha_{sx}^2 Q_{Vx}^2/(BQ_{Ax})\right]\exp\left[-0.5\alpha_{sy}^2 Q_{Vy}^2/\left(BQ_{Ay}\right)\right]S_{Vx}S_{Vy}\right.$$
$$\left.+l_2\exp\left[-0.5\alpha_{sx}^2 Q_{Yx}^2/(BQ_{Ax})\right]\exp\left[-0.5\alpha_{sy}^2 Q_{Yy}^2/\left(BQ_{Ay}\right)\right]S_{Yx}S_{Yy}\right\} \quad (5)$$

where

$$S_{Vx} = \sum_{l_{x1}=0}^{[n/2]}(-1)^{l_{x1}}\,2^{n-l_{x1}}\,T_{lx1}\binom{n}{2l_{x1}}\sum_{l_{nx1}=0}^{n-2l_{x1}}\left(b_x\right)^{n-2l_{x1}-l_{nx1}}\binom{n-2l_{x1}}{l_{nx1}}$$
$$\times\left(a_x\right)^{l_{nx1}}\left(j\alpha_{sx}^2 Q_{Vx}/Q_{Ax}\right)^{l_{nx1}}\frac{l_{nx1}!}{2\left[0.5Q_{Ax}/\left(B\alpha_{sx}^2\right)\right]^{0.5}}$$
$$\times\sum_{k_{x1}=0}^{l_{nx1}}\frac{\left(0.5jQ_{Vx}/B\right)^{-k_{x1}}\left[0.5Q_{Ax}/\left(B\alpha_{sx}^2\right)\right]^{0.5k_{x1}}}{k_{x1}!\left(l_{nx1}-k_{x1}\right)!}$$
$$\times\left(\sum_{k_{x2}=0}^{[0.5(k_{x1}-1)]}\frac{\left(k_{x1}-1\right)!!}{\left(k_{x1}-1-2k_{x2}\right)!!\,2^{k_{x2}}}\right.$$
$$\times\left\{g_{x1}^{k_{x1}-1}\exp\left[\frac{-0.5}{B\alpha_{sx}^2 Q_{Ax}}\left(t_{x1}Q_{Ax}-j\alpha_{sx}^2 Q_{Vx}\right)^2\right]\right.$$
$$\times\left[\frac{0.5}{B\alpha_{sx}^2 Q_{Ax}}\left(t_{x1}Q_{Ax}-j\alpha_{sx}^2 Q_{Vx}\right)^2\right]^{0.5(k_{x1}-1)-k_{x2}}$$
$$-g_{x2}^{k_{x1}-1}\exp\left[\frac{-0.5}{B\alpha_{sx}^2 Q_{Ax}}\left(t_{x2}Q_{Ax}-j\alpha_{sx}^2 Q_{Vx}\right)^2\right]$$
$$\left.\times\left[\frac{0.5}{B\alpha_{sx}^2 Q_{Ax}}\left(t_{x2}Q_{Ax}-j\alpha_{sx}^2 Q_{Vx}\right)^2\right]^{0.5(k_{x1}-1)-k_{x2}}\right\}$$
$$-\pi^{0.5}\frac{\left(k_{x1}-1\right)!!}{2^{0.5k_{x1}}}\left|\operatorname{rem}\left(k_{x1}-1,2\right)\right|$$
$$\times\left\{\operatorname{erf}\left[\left(\frac{0.5}{B\alpha_{sx}^2 Q_{Ax}}\right)^{0.5}\left(t_{x1}Q_{Ax}-j\alpha_{sx}^2 Q_{Vx}\right)\right]\right.$$
$$\left.\left.-\operatorname{erf}\left[\left(\frac{0.5}{B\alpha_{sx}^2 Q_{Ax}}\right)^{0.5}\left(t_{x2}Q_{Ax}-j\alpha_{sx}^2 Q_{Vx}\right)\right]\right\}\right) \quad (6)$$

From (6), S_{Vy} is obtained by changing all x subscripts to y and n indices to m. S_{Yx} is the same as S_{Vx} except that all V_x s' are to be replaced by Y_x s'. Finally S_{Yy} is the same as S_{Vy} except that all V_y s' are to be replaced by Y_y s'. The definitions for the rest of the terms appearing in (5) and (6) are,

$$Q_{Vx} = BV_x + kp_x,\; Q_{Vy} = BV_y + kp_y,\; Q_{Yx} = BY_x + kp_x,$$
$$Q_{Yy} = BY_y + kp_y,\; Q_{Ax} = B + jkA\alpha_{sx}^2,\; Q_{Ay} = B + jkA\alpha_{sy}^2,$$

g_{x1} and g_{x2} take on the values +1 and –1 determined by the following conditions,

$$g_{x1} = +1,\; g_{x2} = +1,\; \text{when real}\left(j\alpha_{sx}^2 Q_{Vx}/Q_{Ax}\right) \le t_{x1} < t_{x2}$$
$$g_{x1} = -1,\; g_{x2} = +1,\; \text{when } t_{x1} < \text{real}\left(j\alpha_{sx}^2 Q_{Vx}/Q_{Ax}\right) < t_{x2}$$
$$g_{x1} = -1,\; g_{x2} = -1,\; \text{when } t_{x1} < t_{x2} \le \text{real}\left(j\alpha_{sx}^2 Q_{Vx}/Q_{Ax}\right)$$

real(x) refers to the real part of x, ! means the factorial notation, $(k)!! = 1\text{x}3\text{x} \ldots(k\text{-}1)$ or $(k)!! = 2\text{x}4\text{x} \ldots(k\text{-}1)$ depending on whether k is odd or even. The square brackets appearing as the upper limit of some summations indicate that the integral part of the enclosing expression is to be taken. The operator, rem(k,2), generates the remainder of k when divided by 2. The sign | x | implements the absolute value operation on x. All forms of $\binom{C_1}{C_2}$ correspond to Binomial coefficients, such that, $\binom{C_1}{C_2} = C_1!/\left[\left(C_1-C_2\right)!\;C_2!\right]$,

$T_{lx1} = 1\times 3\times\cdots\times\left(2l_{x1}-1\right)$ for $l_{x1}\neq 0$, **erf** is the error function.

For the intensity at receiver plane, we need to multiply the receiver field with its conjugate, hence

$$I_r(\boldsymbol{p}, z=L) = u_r(\boldsymbol{p}, z=L)\,u_r^*(\boldsymbol{p}, z=L) \quad (7)$$

where, * stands for the conjugate. Notice that in the case of evaluating the receiver intensity, the exponential on the first line of (5) resolves to unity.

III. Results and Discussion

In this section, we present our results in the form of graphs. Initially, to ensure the reliability of the currently offered formulation, we test it against the well proven cases. By letting $V_x = V_y = -j50,\ Y_x = Y_y = j50,\ l_1 = l_2 = 0.5,$ $t_{x1} \to -\infty,\ t_{x2} \to +\infty$, that is the Hermite cosh-Gaussian beam without any aperture confinements, the receiver intensity including the source intensity are plotted in Fig. 2, at propagation distances of L = 2, 5 and 20 km. The numeric values of other source parameters used in this plot are shown in the box inset of Fig. 2, with the undisplayed parameters of the y axis being the same as those of the x axis. Here we note that Fig. 2 of this paper would be equivalent, both in shape and magnitude terms, to Fig. 6 of [5], if in the latter, the turbulence is eliminated, which means $C_n^2 \to 0$, where C_n^2 is the structure constant. From Fig. 2, it is further noted that, due to the absence of turbulence and the mode indices (n, m) employed, the intermediate stage of Hermite sine-Gaussian beam is observed at L = 20, instead of the eventual Gaussian profile as in Fig. 6 of [5]. At this stage, we would like to point out that the conversion of a hyperbolic source beam to a sinosiodal beam after propagation and vice versa is an event which we have named **reciprocity**. This subject was treated in details in our earlier publications [1] – [5]. In the light of explanations provided in [3] – [5] Hermite sine-Gaussian, rather than Hermite cos-Gaussian, tranformation is to be expected under these circumstances, since the sum of mode indices is odd, i.e., $n + m = 1$.

In Fig. 2 and in the subsequent plots, we have adopted the following normalization procedure for source and receiver intensities respectively.

$$I_{sN}\left(s_x,s_y\right) = I_s\left(s_x,s_y\right) / Max\left[I_s\left(s_x,s_y\right)\right]$$
$$I_{rN}\left(s_x,s_y\right) = I_r\left(s_x,s_y\right) / Max\left[I_s\left(s_x,s_y\right)\right] \tag{8}$$

Here, Max selects the peak value in $I_s\left(s_x,s_y\right)$, where, $I_s\left(s_x,s_y\right)$, is the intensity at source plane, thus

$$I_s\left(s_x,s_y\right) = u_s\left(s_x,s_y\right) u_s^*\left(s_x,s_y\right) \tag{9}$$

Next, leaving the examination of cascaded optical elements on the propagation path to future studies, we investigate the effects of finite aperture sizes in free space conditions. To this end, Fig. 3 shows the progress of the same beam of Fig. 2 along the propagation axis, after setting the ABCD matrix as in (2). Here we observe that, the existence of the aperture naturally lowers the magnitude, but more importantly, the conversion towards Hermite sine-Gaussian shape is accelerated. Hence the transformation to Hermite sine-Gaussian beam occurs at earlier distances. It is further observed that at the base of Hermite sine-Gaussian profile, there is an annular construction, despite the fact that our aperture is a square one.

Finally, we illustrate in Fig. 4, the contour plot of the receiver beam of Fig. 3 at L = 5 km, where the borders of the aperture are also marked. Fig. 4 shows that Hermite sine-Gaussian shape is slightly deformed, also the symmetry of the whole beam, with respect to the slanted axis, is radially oriented due to unequal mode indices [4].

IV. Conclusion

In this study, using Collins integral, we have examined the propagation properties of a generalized Hermite sinosiodal or hyperbolic Gaussian beam passing through an ABCD system with a finite aperture. In addition to adopting a general source beam, our formulation also incorporates an arbitrary aperture shape and freely selectable matrix elements of the ABCD system. A narrow focus on the subject has been offered presently, but due to the generality of our formulation, it is envisaged that studies on other aspects can easily be undertaken in future work.

Appendix

In this appendix, the derivations steps from (4) to (5) are somewhat elaborated.

After writing the series expansion for Hermite polynomials, the integral in (4) is converted into a simplified form of

$$I = \int_a^b x^n \exp\left(-px^2 + 2q\right) dx \tag{A1}$$

For finite values of a and b, no readily available solution exists in the literature for (A1), therefore we have had to evaluate it via the method of integration by parts. Solving (A1) as an indefinite integral, and then inserting the limits, an analytic result is obtained, with various terms exhibiting conditional dependency on the parameters a, b, q and p as stated below,

$$\begin{aligned}
I = {} & \exp\left(q^2/p\right)\left(q/p\right)^n \frac{n!}{2p^{0.5}} \sum_{k_1}^{n} \frac{q^{-k_1} p^{0.5k_1}}{k_1!\left(n-k_1\right)!} \\
& \times \left(\sum_{k_2=0}^{\left[0.5\left(k_1-1\right)\right]} \frac{\left(k_1-1\right)!!}{\left(k_1-1-2k_2\right)!!\,2^{k_2}} \left\{ g_1^{k_1-1} \exp\left[-p\left(a-q/p\right)^2\right] \right.\right. \\
& \times \left[p\left(a-q/p\right)^2\right]^{0.5\left(k_1-1\right)-k_2} g_2^{k_1-1} \exp\left[-p\left(b-q/p\right)^2\right] \\
& \left. \times \left[p\left(b-q/p\right)^2\right]^{0.5\left(k_1-1\right)-k_2} \right\} \\
& -\pi^{0.5} \frac{\left(k_1-1\right)!!}{2^{0.5k_1}} \left|\operatorname{rem}\left(k_1-1,2\right)\right| \\
& \left. \times \left\{ \operatorname{erf}\left[p^{0.5}\left(a-q/p\right)\right] - \operatorname{erf}\left[p^{0.5}\left(b-q/p\right)\right] \right\} \right)
\end{aligned} \tag{A2}$$

$g_1 = +1,\ g_2 = +1,$ when $q/p \le a < b$

$g_1 = -1,\ g_2 = +1,$ when $a < q/p < b$

$g_1 = -1,\ g_2 = -1,$ when $a < b \le q/p$

Upon inspection, it is seen that, the conditions listed above are essentially the same as those stated in the main text. We note that in the limit of $a \to -\infty,\ b \to \infty$, (A2) will reduce to (3.462.2) of [15].

Since, there is no coupling between s_x and s_y, applying (A2) to (4) twice individually, we finally arrive at (5).

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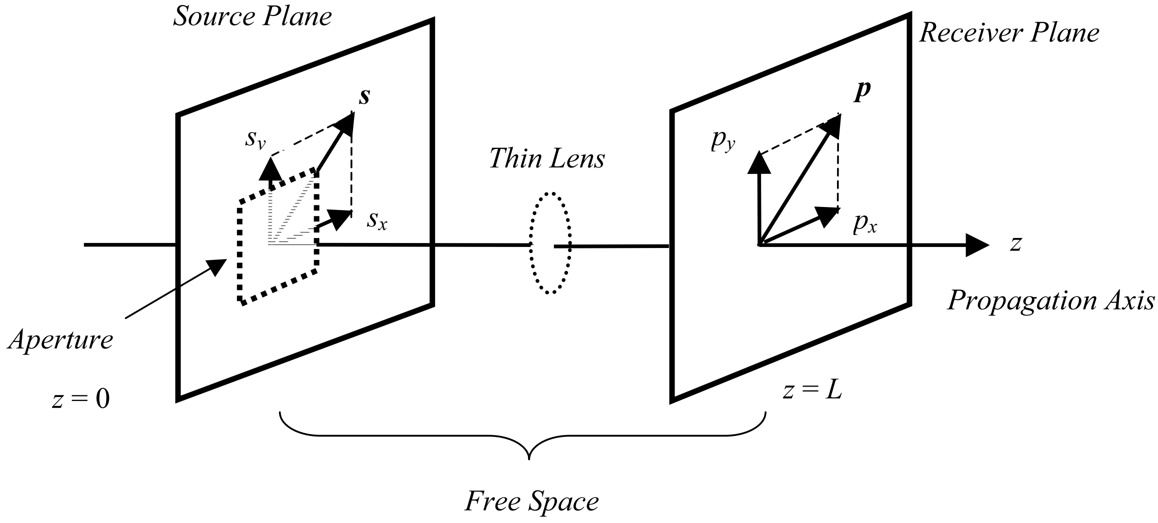

Fig. 1 Propagation geometry

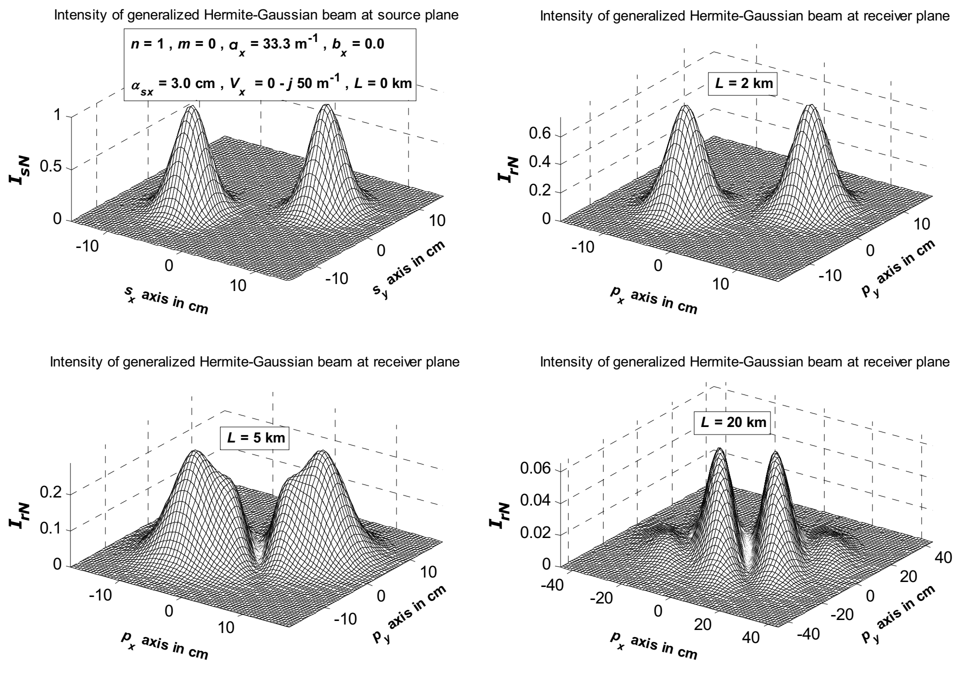

Fig. 2 Combined 3-D plots of source and receiver plane intensities at L = 2, 5, 20 km with no aperture

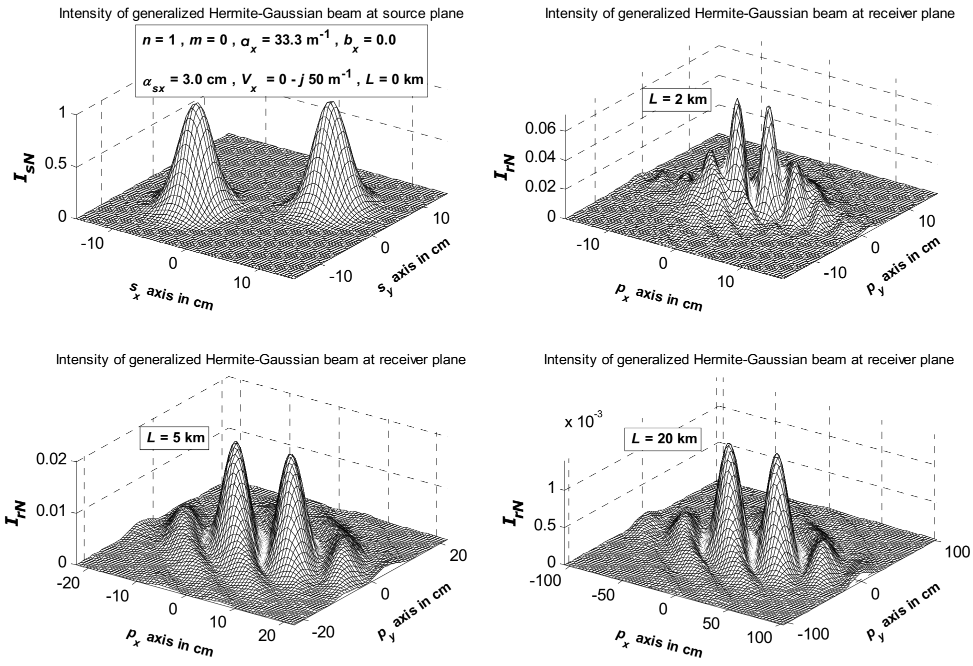

Fig. 3 Combined 3-D plots of source and receiver plane intensities at L = 2, 5, 20 km with aperture

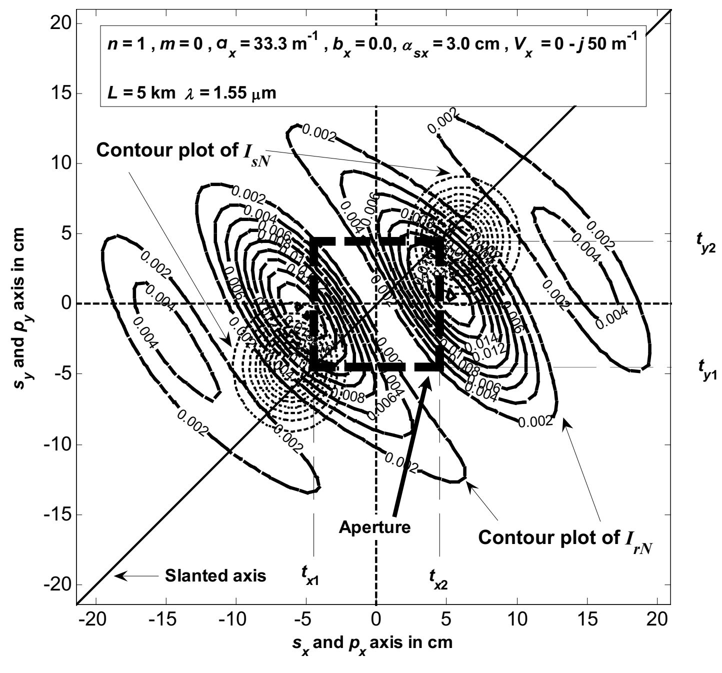

Fig. 4 Combined contour plots of source and receiver plane intensities at L = 5 km with aperture