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Problem Solving in Chilean Higher Education: Figurations Prior in Interpretations of Cartesian Graphs

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***Abstract*—**A Cartesian graph, as a mathematical object, becomes a tool for configuration of change. Its best comprehension is done through everyday life problem-solving associated with its representation. Despite this, the current educational framework favors general graphs, without consideration of their argumentation. Students are required to find the mathematical function without associating it to the development of graphical language. This research describes the use made by students of configurations made prior to Cartesian graphs with regards to an everyday life problem related to a time and distance variation phenomenon. The theoretical framework describes the function conditions of study and their modeling. This is a qualitative, descriptive study involving six undergraduate case studies that were carried out during the first term in 2016 at University of Los Lagos. The research problem concerned the graphic modeling of a real person's movement phenomenon, and two levels of analysis were identified. The first level aims to identify local and global graph interpretations; a second level describes the iconicity and referentiality degree of an image. According to the results, students were able to draw no figures before the Cartesian graph, highlighting the need for students to represent the context and the movement of which causes the phenomenon change. From this, they managed Cartesian graphs representing changes in position, therefore, achieved an overall view of the graph. However, the local view only indicates specific events in the problem situation, using graphic and verbal expressions to represent movement. This view does not enable us to identify what happens on the graph when the movement characteristics change based on possible paths in the person's walking speed.

***Keywords*—**Cartesian graphs, higher education, movement modeling, problem solving.

I. Introduction

ONE of the implied aspects that is accepted in the current Chilean curriculum at all levels is the effect of the teaching and learning of mathematics problem-solving. Solving a problem is both a means and an end of good mathematics education [25]. The current challenge is to make students develop mathematics competences, which are considered essential in the curricula through their school years. These integrate abilities, knowledge, and capacities that imply a positive attitude towards the comprehension of the structure of a problem and the capacity to implement solving processes [13].

According to [14], problem solving corresponds to a competence that is not acquired or developed in an abstract way. Instead, it requires concrete situations, spaces, and people, as well as concrete activities that are part of the daily activities of the learner. For this reason, the activities are pragmatic and reflective, and they are closely linked to the acquisition of different kinds of knowledge [9].

The demand of solving problems in mathematical or real-life contexts that are not likely to be experienced by students is explained in TIMSS 2003 [15]. Additionally, the PISA results [31]-[33] have shown the importance of problem solving in mathematics compulsory education. Collaborative problem solving and global competency are the two major areas that the OECD nominated in 2015 and 2018 for major development in the Programme for International Student Assessment (PISA), in addition to scientific literacy, math, and reading literacy [30]. However, even though this ability is considered important, authors like [2] and [39] admit that the attempts of teaching general problem solving strategies to students have not been successful. Besides, it is important to mention that textbooks do not pay much attention to the heuristic learning of problem solving strategies [41], [34].

Regarding Cartesian graphs as mathematical objects, they are the main mathematical tool for the figuration of change. However, the frame of reference of the school system has preferred graphs in general. It does not allow them to be considered as a means of argumentation by themselves, but only as a representation of a function. The tasks students do are restricted to finding a function, without developing a graphic language.

How we account for the use and interpretation of graphs used by students naturally implies taking an epistemological position that allows the interpretation of what the student is doing. Currently, the pairing of modeling and use of graphs is considered by social epistemology [43], [42]. This represents the core for implementing actions in the didactic system through the design of situations of movement modeling, thus offering the possibility of creating a learning situation that allows studying change through graphs.

According to [26], there is evidence of the students' difficulties in associating the origin of the coordinates with the place from which the movement is described and the way in which these difficulties are overcome [28], [29], [36], [37], [40].

In this framework, the purpose of the research is to characterize the role of students' figurations, prior to a Cartesian graph, when faced to a time and space variation phenomenon.

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II. Theoretical Framework

The origin of the concept of function has always been linked to the study of phenomena that is subject to change. Some documents written by Babylonian astronomers are the earliest references to the concept of function. In the Middle Ages, the study of functions is linked to the concept of movement. One of the first who did this was Nicole Oresme (1323-1382) who represented in coordinate axes graphs related to the change of speed with respect to time. Three centuries later, in 1630, Galileo studied movement from a quantitative point of view, creating experimental justification and, from that, establishing relationships and laws between magnitudes.

In the current mathematics curriculum in Chile, the notion of function is introduced as linear change, and its representations are mainly used with the ability to model daily-life situations and situations from other subjects, considering open problems that could be solved through the function and its representations. The curricular guidelines and curricula from primary school to higher education, give relevance to mathematical modeling. The aim of developing this thinking skill is to make the student create a simplified and abstract version of a system that works in reality. They must also understand the key patterns and express them through mathematical symbols [24]. Simple modeling forms are used first. Then, in secondary and higher education linear models that represent the relationship between variables are used to finally model different situations through functions. Besides, graphs allow people to observe the behavior of functions, to study their properties, observing their domain, increase, decrease, their maximums and minimums, their continuity, and regularity. They are treated as the representation of the concept of function, not as a manifestation of the uses of knowledge. In the curriculum, graphs are included in the second-to-last level of primary school, in a unit called algebra and functions, but they are not explicitly associated to the concept of function [23].

In the international context, the study of graphs is being consolidated as a line of research, where the reference practices associated to the use of graphs in the mathematical school discourse are studied [42]. The tasks of the mathematics teacher, in the framework given to the Cartesian graphs by the educational system, are focused on achieving the correct articulation of their semiotic elements, foster the analysis of a graphic representation, and achieving a proper interpretation [3]. In this scenario, what is required - also from the teacher - is an instrumental use of mathematical symbols used without understanding the concepts that are supposed to be represented. The role of the teacher then, is to propose tasks to promote what [21] calls direct conversions between registers of representations.

The research oriented towards teaching and learning of mathematics in the IT environments of [44], of [42], and [7] have contributed with information about the type of graphs used currently in primary and secondary education. This has provided evidence that their use could support the building of mathematical knowledge. In these studies of the use of graphs [10] there is an attempt to characterize the use of graphs as a kind of knowledge that has its own structure and development.

The use of graphs in school mathematical discourse has been studied under the social epistemological perspective [11], [4], [18], [5], [38], [22]. In these research texts, two principles are present. First, functional mathematics is the knowledge that should be integrated to life to transform it by a permanent reconstruction of meaning. Second, the volume and nature of the knowledge acquired by human beings is determined by their social practices, i.e. their control over the world around them. These authors have linked practices of use of graphs, modeling, and prediction. By doing this the behavior of curves is anticipated by the local and global behavior, focusing the attention in the applications and development of the use of graphs, therefore they have approached to functional mathematics [17].

Specifically [7], in a social epistemological study, suggests that creating graphs allows the student to use argumentation, i.e. mathematical knowledge can be constructed and explained through the use of graphs. Similarly, the use of graphs could be incorporated to institutional practices in the model of knowledge, accounting for the mathematical knowledge and the real causes of that knowledge. Reference [44] suggests that meaning and symbolic systems are found directly in graphs. These meanings could be detected through qualitative analysis of position and velocity graphs. Meanings are reflected in relationships that students manage to establish, i.e. position and velocity graphs show intervals that indicate when movement is slower, faster, or when the body stops, and when velocity is positive or negative.

Describing the way students from different educational levels represent the movement of objects through Cartesian graphs or drawings is a task that several mathematics education researchers have carried out [8], [16], [29], [28], [40], and [19]. From these bodies of research, it could be stated that for a beginning student, the study of phenomena related to movement is not an easy task. Besides, the use of Cartesian graphs and algebra formulae in the research of movement implies the comprehension of how does a cultural form of graphic-visual description that highlights both qualitative and quantitative aspects of movement using complex semiotics that is far from clear for students work [27].

According to [26], mathematical objects are generated by individuals through their cultural historic development; specifically, these objects are not substantial entities. The objects are understood as culturally coded forms of movement and, based on the theory of objectification; Cartesian graphs where linear movement of objects is represented are signs of an activity of reflection about movement. This reflection is part of the western culture since the first half of the XIV century.

Reference [35] considers Cartesian graphs as objects of semiotic mediation of a particular historic-cultural way of conceiving movement. This implies that the way we conceive the learning of these graphs must be rearranged. The epistemological support of the theory of objectification describes the general outline of that rearrangement through a socio-cultural point of view of learning. For this author, thinking is not produced exclusively in the mind, but also through a complex coordination among language, body, gestures, symbols, and tools. Moreover, most of the abstract entities in mathematics are created through cognitive devices that expand the structure of corporal experiences [45]. In this way, gestures have a more complex role than just giving emphasis to communication and they become part of the mathematics learning process and an important pedagogical resource in the classroom [45].

When [12] study figuration practices, they understand them as practices of construction and interpretation of a flat figure of entities that are distinguished in a variation phenomenon. They define the notion of epistemic space of figuration that is the act of constructing and interpreting figures of variation phenomena when students are in the complex interrelationship of the environment, a figure and a phenomenon. The act of knowing what happens with the interpretation of graphs is a perceptive, operational and experience space, in which interaction happens between those who interpret a graph and in the mathematical space of gestures.

To address the research question of this study, the objective that was formulated was to characterize student's use of figurations prior to Cartesian graphs of a time and space variation phenomenon. For this purpose, the theoretical framework that was considered was the social epistemological approach [6] that suggests that the construction of knowledge must correspond to modeling and the use of mathematics, i.e. with the language of tools resulting from human activities. Regarding graphs, the epistemological dimension is the one that is directly related to the mathematical learning content, which must be studied from the perspectives of its origin and operation. In other words, what are the methods used when teaching how to create graphs and what the beliefs students have when studying their local and global aspects.

For the analysis of student's work, a problem related to the graphic modeling of movement was chosen. Two levels of analysis were considered. The first one is based on the model [44], where local and global interpretations of graphs are identified. The second one describes each figuration from the degree of iconicity and the associated textuality.

III. Research Methodology

This is a qualitative research, descriptive, and exploratory, with case study, carried out during the first semester of 2016. A qualitative study was chosen because the identification of figurations performed by students happened in the classroom, and the purpose the research was to account for its sense and the effects of the proposed task. The subjects were six students in the 3rd year of higher education, belonging to the teaching career in mathematics and computer science at the University of Los Lagos.

A situation that was previously validated by [22] was used with the aim of achieving a better characterization of student's understanding prior to the use of graphs and for the construction of models that allow them to describe the variation of position and velocity in a situation of movement. This situation was used in a regular mathematics class teaching mathematics at university. The original author used calculators that could create graphs and a speed sensor (CBR) in the original movement problem. In this research, no technology was used.

The content of the problem situation that was selected for this research in correspondence with the curriculum of mathematics of secondary education and students must know to practice as future teachers of this level of study. This established that they must acquire knowledge about the meaning of the slope of a straight line (for example, in the context of movement of objects), as well as the formula $v = d/t$. Cartesian graphs that represent linear movement of objects with constant velocity are generally used by mathematics teachers when teaching graphic representations of functions in the form $f(x) = mx+b$. Frequently, for the analysis of this kind of graphs students are asked to determine the distance an object has covered in a specific period of time.

The following problem situation was given to students in the first session:

> *Virnia arrived to her mathematics lesson early. She was about to sit down when she realized she had forgotten her notebook in the library. She could not miss the start of her lesson, so she rushed to the library, got her notebook and returned to her seat in time to begin her mathematics lesson. On the way, however, she had met her beloved Antonio and stopped to exchange a few words with him; this took her 4 minutes, and forced her to speed up in order to catch up with time.*
>
> *The library is located opposite the classroom within the school circular yard, which is 500 m in diameter. Virnia's trip took 9 minutes. [12].*

The story of the problem considers the different positions in which a person might be, as well as the way she moves (fast, slow, faster, stopped), and the type of velocity related to each change in direction. In this activity, students must understand the problem and create a graph that represents the position change regarding time, going through a series of inquiries before creating it. Basically, at the moment of doing this task, students must make decisions over the intervening variables, the graph scale, and the distances covered at different times.

Inquiries and reflections allow putting the meanings generated by students together, so they will be able to build a quantitative appraisal of velocity during the distance from the graph of position in relation to time. By doing this, the learning activity allows building knowledge from simulation and modeling.

IV. Analysis of Information and Results

The time for the session was planned according to the structure of activities of modeling and use of graphs, and considering enough time for the students to read and solve the problem using pen and paper.

A. First Level of Analysis

For the first level of analysis of students' work, the model of

[44] was used, where local and global interpretations of graphs are identified.

Students are expected to identify local and global aspects of graphs, both in position and velocity, when straight lines are used. Also when straight lines and parabolas are used, for example, in the form of graphs; when velocity could be positive, negative, or zero; when it could be slow, fast, slower, faster, or when it stops.

B. Second Level of Analysis

For the second level of analysis of students' work, every figuration is described from their degree of iconicity, as the degree of referentiality of an image, i.e., the relationship between the appearance of the image and its referent.

The description is made from the components of the image and its syntax, distinctions of perception form Gestalt (perception laws) articulated through the image theory proposed by [1], who claims that images are polysemic and the person who observes may choose some and discard others. The objective is to recognize the message associated to the figuration, which is the link between a particular meaning and an image.

C. Students' Interpretations

Previous figurations used by four students to visualize a problem can be observed in Fig. 1. It is worth mentioning that these are the real samples, thus the original Spanish language was kept; nevertheless, remarks within figures are translated into English in text boxes under each figure.

First, in Fig. 1 it can be observed that all of them started drawing a circle. Only in one of them, arrows were drawn as vectors between two iconic representations of the school. The second interpretation of the problem of these four students is presented in the Fig. 2.

Figurations in Fig. 2 represent Cartesian graphs close to the phenomenon. However, to obtain them, non-Cartesian figurations were produced first, as it can be observed in Fig. 1.

D. Analysis of the Graphic Description of Movement Situations

Regarding the global vision of the phenomenon, all students gave a global vision of the position change. However, only one of them is able to draw curve lines that account for velocity change. It is a figuration that seems to account only for the global vision of movement.

From a local point of view, when creating a Cartesian graph, student's use two axes (time/distance). There are differences between those who scale the axes, and those who indicate only the points where the graph changes. All students, but one complement, the graph with text, indicating positive, negative or zero velocity. As it can be observed in Fig. 2, only two students are able to describe the phenomenon of movement through straight lines and parabolas. In the other tasks, straight lines were used; therefore, they do not indicate changes in velocity.

The change of direction can be observed in the graphs, because students marked all the points of change of the curve, and identified there zero movement for some minutes. This was distributed in different ways, depending on how students understood the problem and the actions through the period of time. They use descriptions of space and movement trace, leaving some aspects implicit. Two students represent the movement that the action implies. The section of reality that students draw responds to the setting where movement happened. They did a graphic representation of the description. The elements that are present are: the circular playground, its diameter and dimensions, and text to indicate where the classroom and the library are located and to give some context.

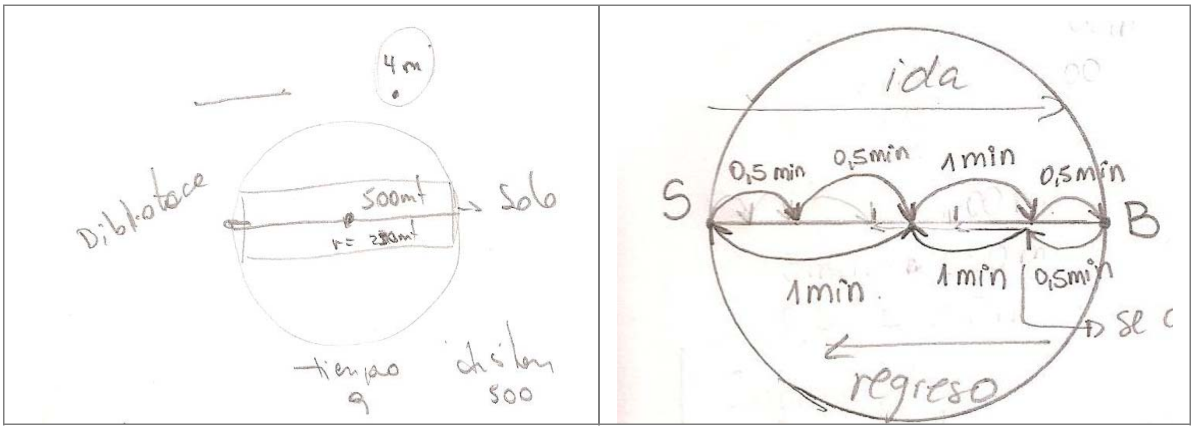
E. Representation Sequences of Students

Two different compositions, from students Loreto and Alex, can be observed in Fig. 3. Fig. 3 corresponds to Loreto, and it can be observed that he draws a one-panel comic strip where he represents the different situations in Virnia's story. In all of them, a person in walking position can be observed. This iconicity expresses movement. He represents the problem situation through a circle, where he adds distances and the location of the music classroom and the library. It can be observed that in his graph he uses two variables: time in minutes, and distance in meters. In the axis of time, he only indicates the point where direction changes. He does not use exact minutes and uses only straight lines.

In Fig. 4, two previous drawings are displayed. This was drawn by Alex and it is relevant to understand the situation. He draws his graph only considering the variable of distance; therefore, he does not present a coordinate system. For Alex, the situation begins in the library (Biblioteca), and he associates the "back" (Ida) and then ends up in the room (Sala de Clases), which was associated with the "return" (Vuelta). For this reason, his graph has one step more than Loreto's. To draw the graph, he uses only straight lines in the axis of distance. It can be observed that the intervals are based only on this variable. The changes in position coincide roughly with minutes. He identifies only one of the changes in velocity, but he identifies changes in direction clearly.

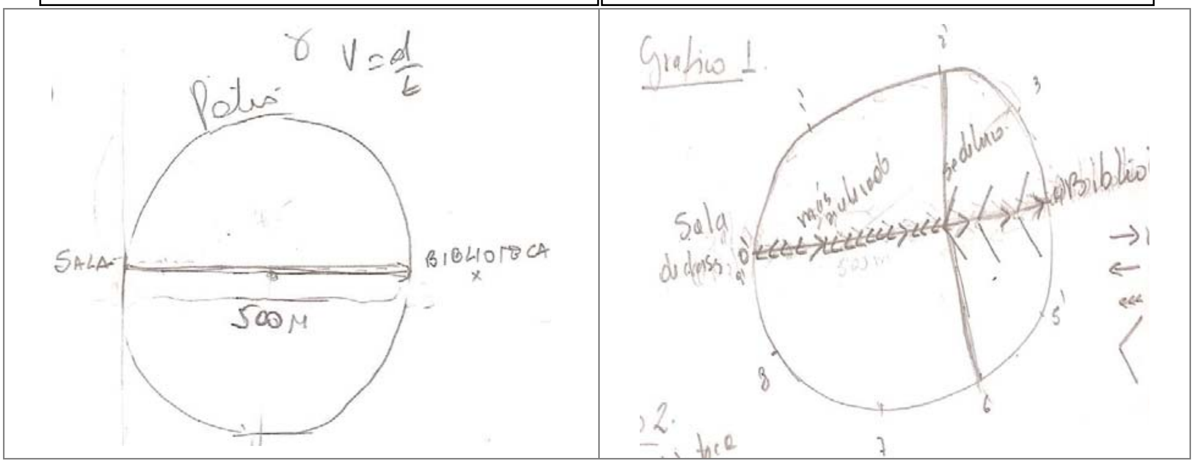
In summary, observing both figures, it can be observed that figuration is constructed in sequence, using moment of reference, indicating changes in position and connotations that are part of moments that happen later, compared to the moments of reference. The level of iconicity of the figurations is high, and it is possible to identify space relationships of the phenomenon, since this figuration is complemented with the representation of the setting where movement happens.

In a general analysis, we can state that the representation of the circular playground and the location of the music classroom and the library provide a frame and a space to the area where Virnia's movements are drawn. They are the background and frame of the image, and allow the student to put the proposed movements in a visual context. They tag the points where movement changes with numbers present in the statement, thus indicating the distance to be walked. Movement is implicit in lines. These allow to visually follow the path in the image. They mostly use lines to express Virnia's movements, but they never formulate the function by sections resulting from the whole path, neither do they formulate the velocity function by sections. It is important to mention that there is no evidence of the construction of sufficient knowledge about the concept of slope, since they do not account for the relationship between velocity and slope in a graph of position.



Biblioteca= Library, Sala= Classroom, Tiempo= Time, Distancia= Distance

ida = departure, regreso = return, S = classroom, B=Library, min (minuto) = minute



Patio= Backyard, Sala= Classroom, Biblioteca=Library, V(velocidad) = speed

Grafico 1 = Graph 1, Sala de clases = Classroom, más acelerado = faster, se detuvo= stopped

Fig. 1 First interpretation of students' representations

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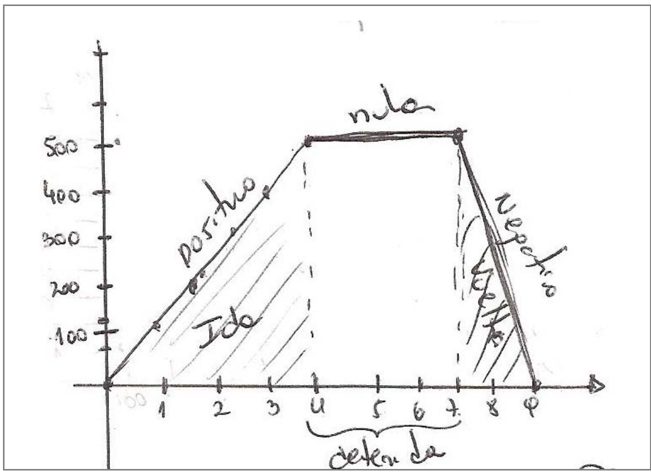
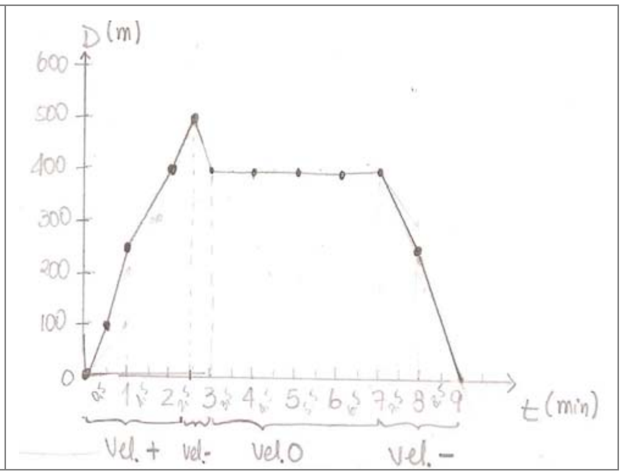
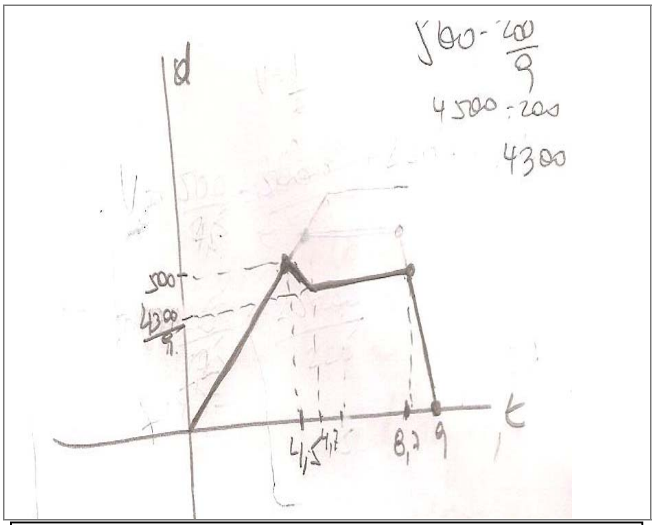
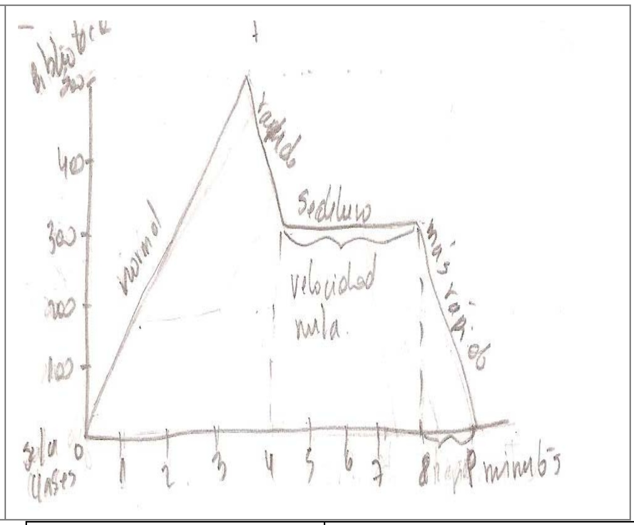




positivo= positive, nula= invalid, negativo= negative, Ida=departure, Vuelta = return, detenida= stopped

vel (velocidad) = speed, D (distancia) = distance, t(tiempo)=time



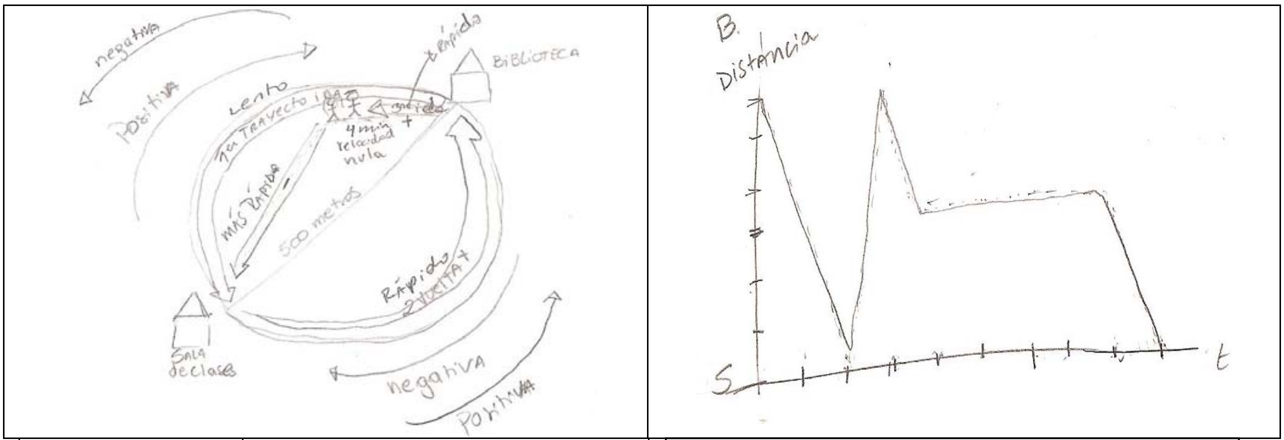


d (distancia) = distance

t (tiempo) = time

Biblioteca = Library	se detuvo= stopped
Sala de clases = Classroom	más rápido = faster
normal = normal	velocidad nula = invalid speed
rápido = fast	minutos = minutes

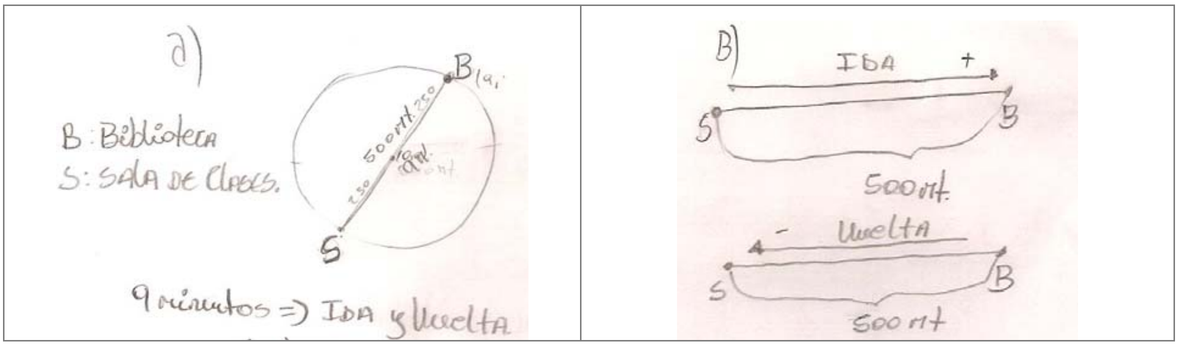
Fig. 2 Students' second interpretation graphic models of position and velocity



negativa = negative	BIBLIOTECA= Library
Positiva = Positive	Sala de clases = Classroom
normal = normal	1 er trayecto IDA = 1 st path out
rápido = fast	más Rápido = faster
500 metros = 500 meter	4 min + velocidad nula = 4 min + invalid speed

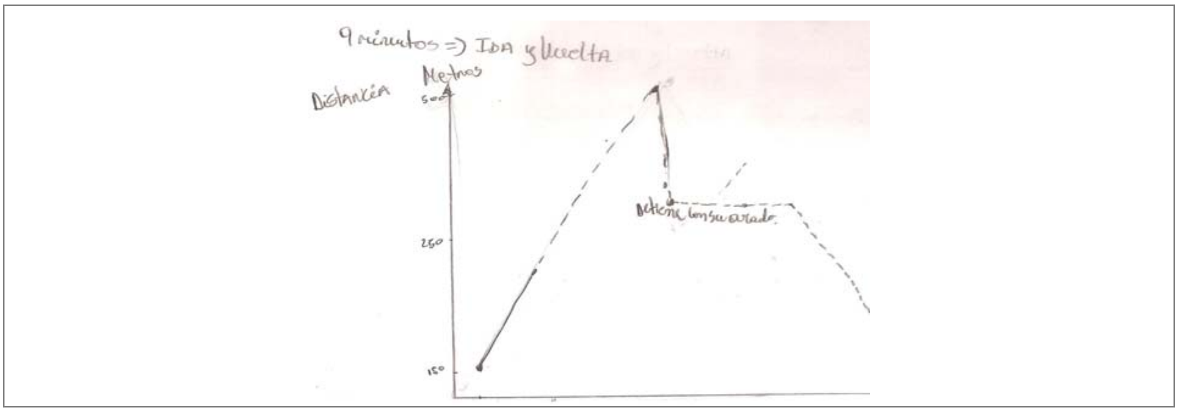
B (Biblioteca) = Library
Distancia = Distance
S (Sala) = Classroom
t (tiempo) = time

Fig. 3 Loreto's movement, context and graph sequence



B (Biblioteca)=Library, S (Sala)=Classroom, 9 minutos=9 minutes, IDA y vuelta = departure and return

IDA = departure, vuelta = return, S (Sala) = Classroom, B (Biblioteca) = Library



9 minutos IDA y vuelta = 9 minutes departure and return, distancia=distance, metros = meters, detiene con su amado = stopped to meet her love

Fig. 4 Alex's graph and context sequence

V. Conclusion and Discussion

A situation was chosen to study the use of graphs in a real movement phenomenon in the context of solving a problem. In this activity, six junior higher school students sought to understand the problem and elicit figuration practices - considering changes in position from and to a place in relation to time - so that they could identify what happens in the graph when the characteristics of movement change. By doing this, students analyze and represent a phenomenon at the same time.

Considering the research objective and the characteristics of the graphs drawn by students, it can be said that they were able to draw a graph that represents the changes in position, therefore, they achieved a global view of the graph.

Although all the students managed to create Cartesian graphs, in a local vision they only indicate specific events in the situation, not considering the scaled intervals in the axes, without making visible the possible changes in the velocity of the person walking, thus assuming a constant movement that is reflected in the high presence of straight lines used in the graphs. Even though only one of them managed to draw curve lines, it can be said that the nature of the task, i.e. creating a graph from a situation, makes students use everything they know to complete the requested graph.

The fact that they drew non-Cartesian figurations prior to the Cartesian graph of the task is evident of the need that students have of representing the context and movement of what makes the phenomenon change. This was observed in all the students' tasks.

Regarding the representations, we know that any person has mental representations, i.e. a group of images and assumptions that one may have about a situation and about what is associated to it. Semiotic representations, i.e. those constructed through signs (statements in natural language, algebra formulas, graphs, and geometric shapes) do not seem to be more than the mean that a person has to reflect their mental representations, and to make them observable and accessible to others. However, according to [20] this goes beyond that. Semiotic representations are not only essential for communications purposes, but they are necessary to do mathematics itself.

Specifically, regarding the graphic representation, students in general were able to represent the different times and positions of the problem. They used as variables the horizontal axis for time and the vertical axis for distance. A couple of students identified in the section of position the intervals when the movement was slow, fast, slower, faster or stopped. Regarding velocity, they identified intervals in which this became constant, zero, positive, and negative.

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