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Primary subgroups and p-nilpotency of finite groups

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Abstract—In this paper, we investigate the influence of S-semipermutable and weakly S-supplemented subgroups on the p-nilpotency of finite groups. Some recent results are generalized.

Keywords—S-semipermutable, weakly S-supplemented, p-nilpotent.

I. Introduction

All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [1]. G denotes always a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$. Two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of G is said to be S-permutable (or S-quasinormal, π-quasinormal) in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel in [2]. More recently, Q. Zhang and L. Wang generalized s-permutable subgroups to S-semipermutable subgroups. H is said to be S-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$ [3]. L. Wang and Y. Wang [4] showed the following theorem: Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are S-semipermutable in G, then G is p-nilpotent. As another generalization of s-permutable subgroups, Skiba [5] introduced the following concept: A subgroup H of a group G is called weakly S-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-quasinormal in G. In fact, this concept is also a generalization of c-supplemented subgroups given in [6]. Skiba proposed in [5] two open questions related to weakly S-supplemented subgroups. In this paper we are concerned with another problems in this context. There are examples to show that weakly S-supplemented subgroups are not S-semipermutable subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using S-semipermutable and weakly S-supplemented subgroups.

II. Preliminaries

Lemma 2.1. *Suppose that H is an S-semipermutable subgroup of a group G and N is a normal subgroup of G. Then*

(1) *H is S-semipermutable in K whenever $H \leq K \leq G$.*

(2) *If H is p-group for some prime $p \in \pi(G)$, then HN/N is S-semipermutable in G/N.*

(3) *If $H \leq O_p(G)$, then H is s-permutable in G.*

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Proof: (a) is [3, Property 1], (b) is [3, Property 2], and (c) is [3, Lemma 3].

Lemma 2.2. ([5], Lemma 2.10) *Let H be a weakly S-supplemented subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly S-supplemented in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly S-supplemented in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly S-supplemented in G/N.*

Lemma 2.3. ([7], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.4. ([8], Lemma 2.2.) *If P is an s-permutable p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.5. ([4], Theorem 3.3) *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime divising $|G|$. If every maximal subgroup of P is S-semipermutable in G, then G is p-nilpotent.*

Lemma 2.6. ([10], Lemma 3.4) *Let H be a normal subgroup of a group G such that G/H is p- nilpotent and let P be a Sylow p-subgroup of H, where p is the smallest prime divisor $|G|$. If $|P| \leq p^2$ and G is A_4-free, then G is p-nilpotent.*

Lemma 2.7. ([1], IV, 5.4) Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.8. ([1], III, 5.2) Suppose G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then

(a) G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(b) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(c) If P is non-abelian and $p > 2$, then the exponent of P is p; If P is non-abelian and $p = 2$, then the exponent of P is 4.

(d) If P is abelian, then the exponent of P is p.

(e) $Z(G) = \Phi(P) \times \Phi(Q)$.

III. Main results

Theorem 3.1. *Let p be the smallest prime divisor of $|G|$ and G_p be a Sylow p-subgroup of a group G. If every* *maximal subgroup of G_p is either weakly S-supplemented or S-semipermutable in G, then G is p-nilpotent.*

Proof: Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider G/N. we will show that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of G_pN/N. It is easy to see $M = G_1N$ for some maximal subgroup G_1 of G_p. It follows that $G_1 \cap N = G_p \cap N$ is a Sylow p-subgroup of N. If G_1 is S-semipermutable in G, then M/N is S-semipermutable in G/N by Lemma 2.1. If G_1 is weakly S-supplemented in G, then there is a subgroup T of G such that $G = G_1T$ and $G_1 \cap T \le (G_1)_{sG}$. So $G/N = M/N \cdot TN/N = G_1N/N \cdot TN/N$. Since

$$(|N : G_1 \cap N|, |N : T \cap N|) = 1,$$

we have

$$(G_1 \cap N)(T \cap N) = N = N \cap G = N \cap G_1T.$$

By Lemma 2.3, $(G_1N) \cap (TN) = (G_1 \cap T)N$. It follows that $(G_1N/N) \cap (TN/N) = (G_1N \cap TN)/N = (G_1 \cap T)N/N \le (G_1)_{sG}N/N \le (G_1N/N)_{sG}$. Hence M/N is weakly S-supplemented in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \ne 1$, then $N \le O_{p'}(G)$ by step (1). Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$.

If $O_p(G) \ne 1$, Step (1) yields $N \le O_p(G)$ and $\Phi(O_p(G)) \le \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $G_p = N(G_p \cap M)$. Furthermore $G_p \cap M < G_p$, thus there exists a maximal subgroup G_1 of G_p such that $G_p \cap M \le G_1$. Hence $G_p = NG_1$. By the hypothesis, G_1 is either S-semipermutable or weakly s-permutable in G. If we assume G_1 is S-semipermutable in G, then G_1M_q is a group for $q \ne p$. Hence

$$G_1 < M_p, M_q | q \in \pi(M), q \ne p >= G_1M$$

is a group. Then $G_1M = M$ or G by maximality of M. If $G_1M = G$, then $G_p = G_p \cap G_1M = G_1(G_p \cap M) = G_1$, a contradiction. If $G_1M = M$, then $G_1 \le M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore we may assume G_1 is weakly S-supplemented in G. Then there is a subgroup T of G such that $G = G_1T$ and $G_1 \cap T \le (G_1)_{sG}$. From Lemma 2.4 we have $O^p(G) \le N_G((G_1)_{sG})$. Since $(G_1)_{sG}$ is subnormal in G, we have

$$G_1 \cap T \le (G_1)_{sG} \le O_p(G) = N.$$

Thus $(G_1)_{sG} \le G_1 \cap N$ and $(G_1)_{sG} \le ((G_1)_{sG})^G = ((G_1)_{sG})^{O^p(G)P} = ((G_1)_{sG})^{G_p} \le (G_1 \cap N)^{G_p} = G_1 \cap N \le N$. It follows that $((G_1)_{sG})^G = 1$ or $((G_1)_{sG})^G = G_1 \cap N = N$. If $((G_1)_{sG})^G = G_1 \cap N = N$, then $N \le G_1$ and $G_p = NG_1 = G_1$, a contradiction. If $((G_1)_{sG})^G = 1$, then $G_1 \cap T = 1$ and so $|T|_p = p$. Hence T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \le N_G(M_{p'}) \le G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \trianglelefteq G$, and $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we may assume $M = N_G(M_{p'})$. By applying a deep result of Gross([9], main Theorem) and Feit-Thompson's theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence $T^g \le N_G(T_{p'}^g) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of G_1. Thus $G = G_1T^g = G_1M$ and $G_p = G_1(G_p \cap M) = G_1$, a contradiction.

(4) The final contradiction.

If every maximal subgroup of G_p is S-semipermutable in G, then G is p-nilpotent by Lemma 2.5, a contradiction. Thus there is a maximal subgroup G_1 of G_p such that G_1 is weakly S-supplemented in G. Then there exists a subgroup T of G such that $G = G_1T$ and

$$G_1 \cap T \le (G_1)_{sG} \le O_p(G) = 1.$$

By [11, Theorem 2.2], G is not simple and G has a Hall p'-subgroup. Suppose $NG_p < G$, then NG_p satisfies the hypothesis of the theorem. The choice of G yields that N is p-nilpotent, a contradiction with steps (2) and (3). Therefore we may assume $G = NG_p$. Then we may suppose that N has a Hall p'-subgroup $N_{p'}$. By Frattini's argument, $G = NN_G(N_{p'}) = (G_p \cap N)N_{p'}N_G(N_{p'}) = (G_p \cap N)N_G(N_{p'})$ and so $G_p = G_p \cap G = G_p \cap (G_p \cap N)N_G(N_{p'}) = (G_p \cap N)(G_p \cap N_G(N_{p'}))$. Since $N_G(N_{p'}) < G$, it follows that $G_p \cap N_G(N_{p'}) < G_p$. Consider a maximal subgroup G_1 of G_p such that $G_p \cap N_G(N_{p'}) \le G_1$. Then $G_p = (G_p \cap N)G_1$. By the hypothesis, G_1 is either S-semipermutable or weakly S-supplemented in G. If G_1 is S-semipermutable in G, then $G_1N_G(N_{p'}) = G_1N_{p'}$ forms a group. Since $|G : G_1N_{p'}| = p$ and p is the smallest prime divisor of $|G|$, we have $G_1N_{p'} \trianglelefteq G$. By Frattini's argument again, $G = G_1N_{p'}N_G(N_{p'}) = G_1N_G(N_{p'}) < G$, a contradiction. Now assume that G_1 is weakly S-supplemented in G. Then there is a subgroup T of G such that $G = G_1T$ and

$$G_1 \cap T \le (G_1)_{sG} \le O_p(G) = 1.$$

Since $|T|_p = p$, we have T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a Hall p'-subgroup of G. A application of the result of Gross ([9], Main Theorem) and Feit-Thompson's theorem yields $T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in G_1$ such that $T_{p'}^g = N_{p'}$. Hence

$$G = (G_1T)^g = G_1T^g = G_1N_G(T_{p'}^g) = G_1N_G(N_{p'})$$

and

$$G_p = G_p \cap G = G_p \cap G_1N_G(N_{p'}) = G_1(G_p \cap N_G(N_{p'})) \leq G_1,$$

a contradiction.

Theorem 3.2. *Let p be the smallest prime dividing the order of a group $|G|$ and G_p a Sylow p-subgroup of G. Suppose that G is A_4-free and every 2-maximal subgroup of G_p is either weakly S-supplemented or S-semipermutable in G. Then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) By Lemma 2.6, $|G_p| \geqslant p^3$ and so every 2-maximal subgroups G_2 of G_p is non-identity.

(2) G has a unique minimal normal subgroup N such that G/N is p-nilpotent, Moreover $\Phi(G) = 1$.

(3) $O_{p'}(G) = 1$.

(4) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (3) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, hence by G, the uniqueness of N yields $N = O_p(G)$. Clearly, $G_p = N(G_p \cap M)$. Furthermore $G_p \cap M < G_p$. If $G_p \cap M$ is a maximal subgroup of G_p, then N is a subgroup of order p. By applying [7, Lemma 2.8], we obtain that $N \leq Z(G)$. Since G/N is p-nilpotent, it follows that G is p-nilpotent, a contradiction. Therefore $G_p \cap M$ is contained in a 2-maximal subgroup G_2 of G_p. By the hypothesis, G_2 is either S-semipermutable or weakly S-supplemented in G. If we assume G_2 is S-semipermutable in G, then G_2M_q is a group for $q \neq p$. Hence

$$G_2 < M_p, M_q | q \in \pi(M), q \neq p >= G_2M$$

is a group. Then $G_2M = M$ or G by maximality of M. If $G_2M = G$, then $G_p = G_p \cap G_2M = G_2(G_p \cap M) = G_2$, a contradiction. If $G_2M = M$, then $G_2 \leq M$. Therefore, $P_2 \cap N = 1$. Since $G_p = NP_2$, we have $|N| = p^2$. Then the p-nilpotency of G/N implies the p-nilpotency of G by Lemma 2.6, a contradiction. Now we suppose G_2 is weakly S-supplemented in G. Then there is a subgroup T of G such that $G = G_2T$ and $G_2 \cap T \leq (G_2)_{sG}$. From Lemma 2.4 we have $O^p(G) \leq N_G((G_2)_{sG})$. Since $(G_2)_{sG}$ is subnormal in G,

$$G_2 \cap T \leq (G_2)_{sG} \leq O_p(G) = N.$$

Thus, $(G_2)_{sG} \leq G_1 \cap N$, where p_1 is a maximal subgroup of G_p which contains G_2. Then $(G_2)_{sG} \leq ((G_2)_{sG})^G = ((G_2)_{sG})^{O^p(G)G_p} = ((G_2)_{sG})^{G_p} \leq (G_1 \cap N)^{G_p} = G_1 \cap N \leq N$. It follows that $((G_2)_{sG})^G = 1$ or $((G_2)_{sG})^G = G_1 \cap N = N$. If $((G_2)_{sG})^G = G_1 \cap N = N$, then $N \leq G_1$ and $G_p = NG_1 = G_1$, a contradiction. If $((G_2)_{sG})^G = 1$, then $G_2 \cap T = 1$ and so $|T|_p = p^2$. Hence T is p-nilpotent by Lemma 2.6. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \lhd G$, $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we must have $M = N_G(M_{p'})$. By applying a deep result of Gross ([9],main Theorem) and Feit-Thompson's theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence $T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of G_2. Thus $G = G_2T^g = G_2M$ and $G_p = G_2(G_p \cap M) = G_1$, a contradiction.

(5) The final contradiction.

If $NG_p < G$, then NG_p satisfies the hypothesis of the theorem. The choice of G yields that N is p-nilpotent, a contradiction with steps (4) and (5). Therefore we must have $G = NG_p$. Since G/N is a p-subgroup, we may assume G has a normal subgroup M such that $|G : M| = p$ and $N \leq M$. Hence the maximal subgroups of Sylow p-subgroup $G_p \cap M$ of M are the 2-maximal subgroups of Sylow p-subgroup G_p of G. By Lemmas 2.1 and 2.2, every maximal subgroup of Sylow p-subgroup $G_p \cap M$ is either S-semipermutable or weakly S-supplemented in M. Now applying Theorem 3.1, we get M is p-nilpotent, and so G is p-nilpotent, a contradiction.

Theorem 3.3. *Suppose N is a normal subgroup of a group G such that G/N is p-nilpotent, where p is a fixed prime number. Suppose every subgroup of order p of N is contained in the hypercenter $Z_\infty(G)$ of G. If $p = 2$, in addition, suppose every cyclic subgroup of order 4 of N is either weakly S-supplemented or S-semipermutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false, and let G be a counterexample of minimal order.

(1) The hypotheses are inherited by all proper subgroups, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

In fact, $\forall K < G$, since G/N is p-nilpotent, $K/K \cap N \cong KN/N$ is also p-nilpotent. The cyclic subgroup of order p of $K \cap N$ is contained in $Z_\infty(G) \cap K \leq Z_\infty(K)$, the cyclic subgroup of order 4 of $K \cap N$ is either weakly S-supplemented or S-semipermutable in G, then is either weakly S-supplemented or S-semipermutable in K by Lemmas 2.1 and 2.2. Thus $K, K \cap N$ satisfy the hypotheses of the theorem in any case, so K is p-nilpotent, therefore G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. By Lemmas 2.7 and 2.8, $G = PQ$, $P \trianglelefteq G$ and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(2) $G/P \cap N$ is p-nilpotent.

Since $G/P \cong Q$ is nilpotent, G/N is p-nilpotent and $G/P \cap N \lesssim G/P \times G/N$, therefore $G/P \cap N$ is p-nilpotent.

(3) $P \leq N$.

If $P \nleq N$, then $P \cap N < P$. So $Q(P \cap N) < QP = G$. Thus $Q(P \cap N)$ is nilpotent by (1), $Q(P \cap N) = Q \times (P \cap N)$. Since

$$G/P \cap N = P/P \cap N \cdot Q(P \cap N)/P \cap N,$$

it follows that

$$Q(P \cap N)/P \cap N \trianglelefteq G/P \cap N$$

by Step (2). So Q char $Q(P \cap N) \trianglelefteq G$. Therefore, $G = P \times Q$, a contradiction.

(4) $p = 2$.

If $p > 2$, then $\exp(P) = p$ by (a) and Lemma 2.9. Thus $P = P \cap N \leq Z_\infty(G)$. It follows that $G/Z_\infty(G)$ is nilpotent, and so G is nilpotent, a contradiction.

(5) For every $x \in P \backslash \Phi(P)$, we have $\circ(x) = 4$.

If not, there exists $x \in P \backslash \Phi(P)$ and $\circ(x) = 2$. Denote $M = < x^G > \leq P$. Then $M\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$, we have that $P = M\Phi(P) = M \leq Z_\infty(G)$ as $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ by Lemma 2.9, a contradiction.

(6) For every $x \in P \backslash \Phi(P)$, $< x >$ is weakly S-supplemented in G.

If $< x >$ is S-semipermutable in G, then $< x >$ is S-permutable in G by Lemma 2.1(4), and so weakly S-supplemented in G.

(7) Final contradiction.

For any $x \in P \backslash \Phi(P)$, we may assume that x is weakly S-supplemented in G by Step (6). Then there is a subgroup T of G such that $G = < x > T$ and $< x > \cap T \leq < x >_{sG}$. It follows that $P = P \cap G = P \cap < x > T = < x > (P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If $P \cap T \leq \Phi(P)$, then $< x > = P \trianglelefteq G$, a contraction. If $P = (P \cap T)\Phi(P) = P \cap T$, then $T = G$ and so $< x > = < x >_{sG}$ is s-permutable in G. We have $< x > Q$ is a proper subgroup of G and so $< x > Q = < x > \times Q$, i.e., $< x > \leq N_G(Q)$. By Lemma 2.8, $\Phi(P) \subseteq Z(G)$. Therefore we have $P \leq N_G(Q)$ and so $Q \trianglelefteq G$, a contradiction.

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