# Postbuckling Analysis of End Supported Rods under Self-Weight Using Intrinsic Coordinate Finite Elements

C. Juntarasaid, T. Pulngern, S. Chucheepsakul

*Abstract*—A formulation of postbuckling analysis of end supported rods under self-weight has been presented by the variational method. The variational formulation involving the strain energy due to bending and the potential energy of the self-weight, are expressed in terms of the intrinsic coordinates. The variational formulation is accomplished by introducing the Lagrange multiplier technique to impose the boundary conditions. The finite element method is used to derive a system of nonlinear equations resulting from the stationary of the total potential energy and then Newton-Raphson iterative procedure is applied to solve this system of equations. The numerical results demonstrate the postbluckled configurations of end supported rods under self-weight. This finite element method based on variational formulation expressed in term of intrinsic coordinate is highly recommended for postbuckling analysis of end-supported rods under self-weight.

*Keywords*—Variational method, postbuckling, finite element method, intrinsic coordinate.

## I. Introduction

POSTBUCKLING analysis of rods is a fundamental problem in elastic theory of structures. The classical theory of elastic rods is presented widely in texts books, such as those by Love [1], Timmoshenko and Goodier [2], Antman [3], Wang [4] and Bigoni [5] which the buckling load has been given in texts books [1]-[5]. For the past several years ago, almost all research dealt with buckling loads of columns/beams has attracted the attention of many investigators such as cantilever columns loaded laterally by a tip force [6], heavy columns with various support conditions subjected to an axial load and restrained by internal supports [7], [8], elastic columns of constant and variable cross-sections [9], non-prismatic columns under self-weight and tip force [10], the axially non-uniform elastically restrained beams [11] and the buckling load of columns under self-weight with various boundary conditions [12].

Nowadays, postbuckling analysis of rods is considered to be a fundamental topic in elastica theory of structures, and an interesting practical problem in structural stability. In case of methods for solving the postbuckled configuration of rods/columns, the most research focused on applying numerical solution of shooting method to solve postbuckling behavior of beam and rods/columns such as a standing

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sandwich beam under terminal force and self-weight [13], a clamped-simply supported rod under axial force [14], slender elastic rod subjected to axial terminal forces restrained by hinged with a rotational spring [15], a cantilever column due to self-weight [16], slender rods with double-hinged boundary condition subjected to axial terminal forces and self-weight [17], hinged-fixed beam under uniformly distributed follower forces [18] and slender rod with two hinged ends under self-weight [19]. In addition, the postbuckling behavior of rods/ columns under various loading and supporting conditions are also analyzed by using the Runge-Kutta and Regula-Falsi methods [20], Butcher's fifth-order Runge-Kutta method [21], Sturm-Liouville boundary value problem [22], Differential quadrature method [23], Matched asymptotic perturbation method [24], A canonical dual finite element method [25].

From the literature in the field of offshore engineering, the analysis of buckling and postbuckling of vertical rods have been investigated in several studies for example Huang and Dareing [26], [27], Lubinski [28], Kokkinis and Bernitsas [29], as well as Vaz and Patel [30], [31]. The previous research related to offshore engineering application also mentioned that self-weight of long submersed rod is main factor cause to its buckle.

Although the model formulation of postbuckling rods/ columns has been presented by several scholars, their model formulations have not yet used the intrinsic coordinates finite elements. Therefore, the purpose of this study is to present intrinsic coordinates finite elements based on variational method for postbuckling analysis of rods under self-weight with various end conditions. The variational formulation developed in the present study involves strain energy due to bending, and the potential energy of the self-weight. To accomplish the formulation, the constraint of the boundary condition in a Lagrange multiplier technique is employed. The results obtained in this study are compared with those results reported by Liu et al. [19]. Moreover, the obtained buckled configurations of end supported rods under self-weight can be used to predict the postbuckling behavior of rods and other engineering structure.

## II. Variational Formulation

According to the elastica theory, the exact curvature is used to obtain postbuckling behavior of end supported rods. The model formulation is developed by elastica theory based on the assumption that the deflection of rods can be large, but strain is small. Then, torsional and shear rigidities are

neglected. An energy functional of rods expressed in the intrinsic coordinates ($\theta$, $s$) can be written as:

$$\pi = \int_0^L \frac{1}{2} EI\theta'^2 ds - w_s \int_0^L \int_0^s [1-\cos\theta] ds\,ds \qquad (1)$$

The first term in (1) is the strain energy due to bending and the second term is the potential energy of the self-weight ($w_s$). In the constrained boundary condition of the end supported rods, a Lagrange multiplier technique [32] is introduced; thus, the constrained boundary condition can be given as:

$$g = y(L) - \int_0^L \sin\theta\,ds = 0 \qquad (2)$$

where $y(L)$ is the lateral displacement of the end supported rods at position $L$. The multiplier $\lambda$ is added to the system in accordance with the Lagrange multiplier technique. The modified total potential energy function of the system can be expressed as:

$$\pi^* = \int_0^L \frac{1}{2} EI\theta'^2 ds - w_s \int_0^L \int_0^s [1-\cos\theta] ds\,ds + \int_0^L \lambda(y' - \sin\theta)]\,ds \quad (3)$$

The rotation angle ($\theta$) and the Lagrange multipliers ($\lambda$) in the variational formulation in (3) are determined simultaneously using finite element method and Newton-Raphson iteration. The boundary conditions in this study (Fig. 1) are as follows:

Simply supported;

$$y(0) = 0, \ y(L) = 0, \ M(0) = 0, \ M(L) = 0 \qquad (4)$$

Clamped-pinned;

$$y(0) = 0, \ y(L) = 0, \ \theta(0) = 0, \ M(L) = 0 \qquad (5)$$

Clamped-clamped;

$$y(0) = 0, \ y(L) = 0, \ \theta(0) = 0, \ \theta(L) = 0 \qquad (6)$$

### III. Finite Element Method

The total arc-length of rods is divided into a number of $k^{th}$ elements in the finite element procedure. The rotation function $\theta(s)$ within the $k^{th}$ element can be approximated by using Lagrange interpolation functions as:

$$\theta(s) = [\mathbf{N}]\{q\} \qquad (7)$$

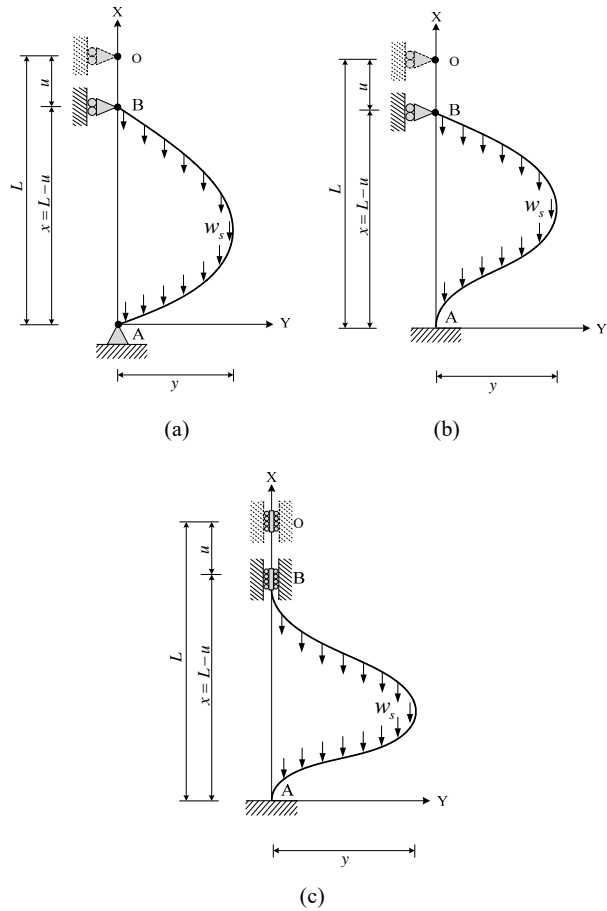where $[\mathbf{N}]$ is the shape function components consisting of:



Fig. 1 Deformed configurations of end supported rods under self-weight: (a) simply supported, (b) clamped-pinned, (c) clamped-clamped

$$[\mathbf{N}] = \left[ 1 - \frac{11s}{2l} + \frac{9s^2}{l^2} - \frac{9s^3}{2l^3} \quad \frac{9s}{l} - \frac{45s^2}{2l^2} + \frac{27s^3}{2l^3} \right.$$
$$\left. -\frac{9s}{2l} + \frac{18s^2}{l^2} - \frac{27s^3}{2l^3} \quad \frac{s}{l} - \frac{9s^2}{2l^2} + \frac{9s^3}{2l^3} \right] \qquad (8)$$

The vector $\{q\}$ denotes the local degrees of freedom representing the value of $\theta(s)$ and $l$ is the element length of a rod for each discretized $k^{th}$ element.

Gaussian quadrature integration for a line element is performed for single integration, $\int_0^L f(s)\,ds$, in (3). This integral can be transformed to an integration having limits $\zeta = -1$ and $\zeta = 1$ by substitution $s = \frac{L}{2}(1+\zeta)$, becomes:

$$\int_0^L f(s)\,ds = \int_{-1}^1 f(s)\left(\frac{ds}{d\zeta}\right) d\zeta = \frac{L}{2}\sum_{i=1}^n W_i f\left(\frac{L}{2}(1+\zeta_i)\right) \qquad (9)$$

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The double integration, $\int_0^L \int_0^s f(s)\,ds\,ds$, in (3), can be changed in the form of $\int_0^L \int_0^s f(t)\,dt\,ds$. The Gaussian quadrature integration formula can be applied to any arbitrary interval [0, $s$] with transformation $t = \frac{s}{2}(1+\xi)$, yields:

$$\int_0^s f(t)\,dt = \int_{-1}^1 f(t)\left(\frac{dt}{d\xi}\right)d\xi = \frac{s}{2}\sum_{i=1}^n W_i f\left(\frac{s}{2}(1+\xi_i)\right) \quad (10)$$

Then, the double integration $\int_0^L \int_0^s f(t)\,dt\,ds$ can be given as:

$$\int_0^L \int_0^s f(t)\,dt\,ds = \int_{-1}^1 \left(\int_{-1}^1 f(t)\frac{dt}{d\xi}d\xi\right)\frac{ds}{d\zeta}d\zeta$$

$$= \frac{L}{2}\sum_{j=1}^n W_j \left[\frac{1}{2}\left(\frac{L}{2}(1+\zeta_j)\right)\sum_{i=1}^n W_i f\left(\frac{1}{2}\left(\frac{L}{2}(1+\zeta_j)\right)(1+\xi_i)\right)\right] \quad (11)$$

where the Gauss points $\xi_i$ and $\zeta_j$ are the value of coordinates at a specific point in the line element, and the $W_i$ and $W_j$ are the weight appropriate to $\xi_i$ and $\zeta_j$, respectively.

Following the virtual work-energy principle, the equilibrium equations are obtained by taking the variation of the modified total potential energy in (3). The energy functional of the $k^{th}$ element can be expressed in terms of its local degrees of freedom as:

$$\delta\pi_k^* = \left(\frac{\partial\pi_k^*}{\partial q_i}\right)\delta q_i + \left(\frac{\partial\pi_k^*}{\partial\lambda}\right)\delta\lambda = 0 \quad (12)$$

Since $\delta q_i$ and $\delta\lambda$ are arbitrary, $\dfrac{\partial\pi_k^*}{\partial q_i}$ and $\dfrac{\partial\pi_k^*}{\partial\lambda}$ in (12) are approximation to the solution. Taylor's series expansion about $\dfrac{\partial\pi_k^*}{\partial q_i}$ and $\dfrac{\partial\pi_k^*}{\partial\lambda}$ by truncating the second-order terms, yields

$$\left\{\frac{\partial\pi_k^*}{\partial q_i}\right\}_{n+1} \cong \left\{\frac{\partial\pi_k^*}{\partial q_i}\right\}_n + \left\{\frac{\partial}{\partial q_j}\left(\frac{\partial\pi_k^*}{\partial q_i}\right)\right\}_n \Delta q_n$$
$$+ \left\{\frac{\partial}{\partial\lambda}\left(\frac{\partial\pi_k^*}{\partial q_i}\right)\right\}_n \Delta\lambda_n = 0 \quad (13)$$

$$\left\{\frac{\partial\pi_k^*}{\partial\lambda}\right\}_{n+1} \cong \left\{\frac{\partial\pi_k^*}{\partial\lambda}\right\}_n + \left\{\frac{\partial}{\partial q_i}\left(\frac{\partial\pi_k^*}{\partial\lambda}\right)\right\}_n \Delta q_n$$
$$+ \left\{\frac{\partial}{\partial\lambda}\left(\frac{\partial\pi_k^*}{\partial\lambda}\right)\right\}_n \Delta\lambda_n = 0 \quad (14)$$

where $\{\Delta q\}_n = \{q_i\}_{n+1} - \{q_i\}_n$, $\Delta\lambda_n = \lambda_{n+1} - \lambda_n$, and $n$ = number of iterations. Then, (13) and (14) can be arranged into a matrix form as:

$$\begin{bmatrix} [\mathbf{K}_{NL}]_{N\times N} & \{\mathbf{K}_\lambda\}_{N\times 1} \\ \{\mathbf{K}_\lambda\}_{1\times N}^{\mathrm{T}} & 0 \end{bmatrix}\begin{Bmatrix} \{\Delta q_i\} \\ \Delta\lambda \end{Bmatrix} = -\begin{Bmatrix} \{\mathbf{R}_i\} \\ \mathbf{R}_\lambda \end{Bmatrix} \quad (15)$$

The integer value $N$ is the number of nodal rotations of the rod system.

The matrix $[\mathbf{K}_{NL}]$ is obtained as an assemblage of the local stiffness matrices $\left[\dfrac{\partial^2\pi_k^*}{\partial q_i\partial q_j}\right]$ from all elements, and the vector $\{\mathbf{K}_\lambda\}$ represents the assembling of the element vectors $\left\{\dfrac{\partial^2\pi_k^*}{\partial\lambda\,\partial q_i}\right\}$. The vector $\{\mathbf{R}_i\}$ is the assemblage of element vectors $\dfrac{\partial\pi_k^*}{\partial q_i}$ and the parameter $\mathbf{R}_\lambda$ is the value of $\dfrac{\partial\pi_k^*}{\partial\lambda}$, respectively.

In order to find the solution of system in (15), the Newton-Raphson iterative procedure is employed to obtain the numerical solutions. The numerical procedure can be summarized in the following as:

1. Specify the initial values $\{q\} = 0$ and $\lambda = 0$.
2. Determine the matrix $[\mathbf{K}_{NL}]$, the vector $\{\mathbf{K}_\lambda\}$, the vector $\{\mathbf{R}_i\}$ and the parameter $\mathbf{R}_\lambda$.
3. Apply the boundary conditions and solve for $\{\Delta q_i\}$ and $\Delta\lambda$.
4. Add the $\{\Delta q_i\}$ to $\{q_i\}$ and $\Delta\lambda$ to $\lambda$, which give the new values of $\{q_i\}$ and $\lambda$.
5. Repeat steps 2 through 5 until $\{\Delta q_i\}$ and $\Delta\lambda$ are converged.

The resulting solutions are the rotation angle ($\theta$) at each node and $\lambda$.

## IV. RESULTS AND DISCUSSION

As found in the literature review, research studies on postbuckling behavior of a rod under self-weight are mostly concerned with simply supported rods. Therefore, the comparison numerical solution for postbuckling load ($\bar{w}_s$) of the simply supported rods under self-weight obtained in this work and previous researches are presented first, as shown in Table I. For Table I, the value of the shortening ($\bar{u} = u/L$) and

the maximum lateral displacement $(y_{max}/L)$ between the results of the finite element method (FEM) and other research studies undertaken by Liu et al. [19], in which the problem has been solved numerically by the shooting method (SM) are compared. The obtained numerical results show that FEM and SM from previous work [19] are in very good agreement for lower value of $\bar{w}_s$. However, small difference with a less than 5 percentage point were obtained in case of the shortening $(\bar{u}=u/L)$ for higher value of $\bar{w}_s$. The increased difference arises from the fact that the difference formulation technique is used. Moreover, the buckled configurations are obtained from FEM results are shown in Figs. 2-4. The buckled configurations of simply supported rod in Fig. 2 show the relationship between the axial displacements and lateral displacements which similar to the behavior of a double-hinged rod under self-weight, as presented in the work of Liu et al. [19]. In addition, Figs. 3 and 4 also display the postbuckle configurations of clamped-pinned and clamped-clamped rod under self-weight. In offshore engineering applications, the buckling of rods often occurs for long submersed rods used as drilling bars or oil sucker rods and marine risers in offshore oil/gas exploitation. The self-weight of the rod becomes the main factor to cause the buckling. The postbuckling behavior of rods is typically analyzed on the deformed rod after buckling and losing its stability. The presented analysis method can be used to predict the postbuckling behavior of rods and will be beneficial for the design of other engineering structures.
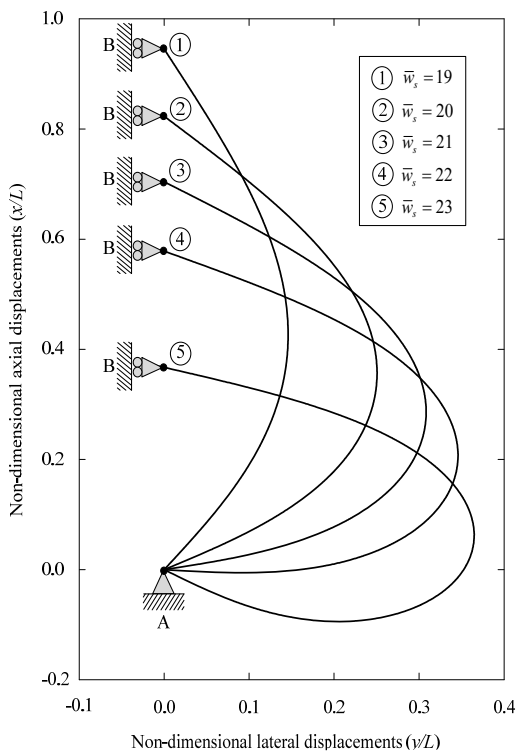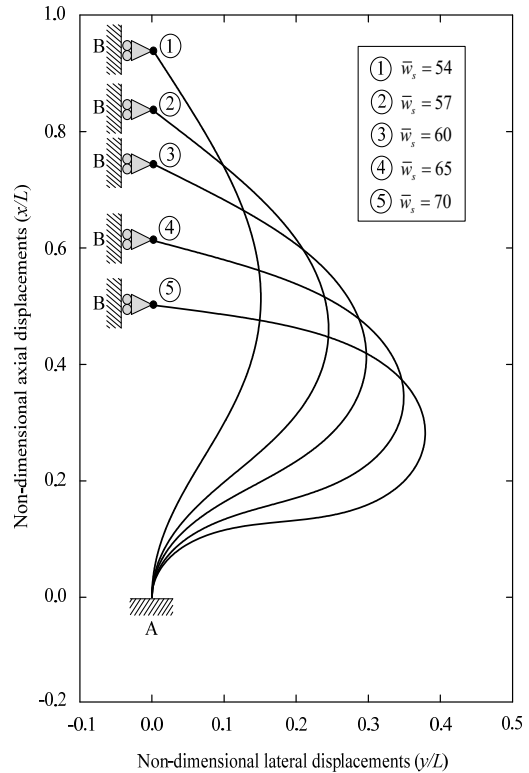


Fig. 3 Postbuckling configurations of a clamped-pinned rod under self-weight



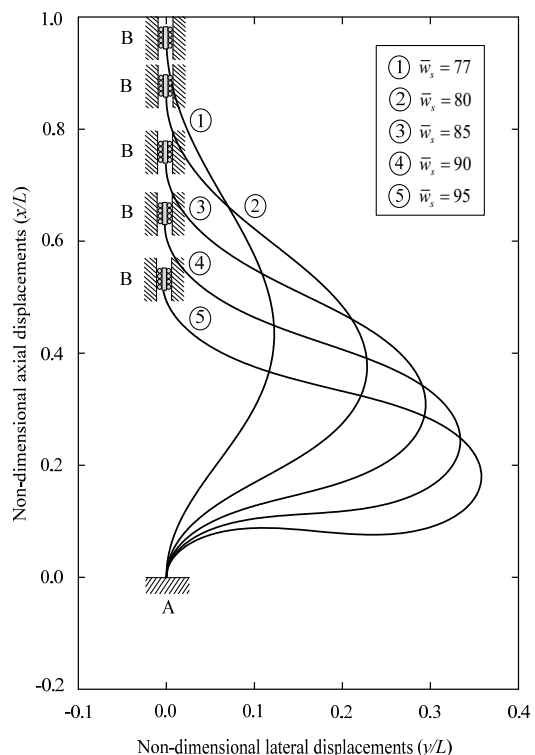Fig. 2 Postbuckling configurations of a simply supported rod under self-weight



Fig. 4 Postbuckling configurations of a clamped-clamped rod under self-weight

TABLE I
COMPARISON OF NUMERICAL RESULTS FOR POSTBUCKLING LOAD OF SIMPLY-
SUPPORTED RODS UNDER SELF-WEIGHT

| $\bar{w}_s$ | $\bar{u} = u/L$ | | | $v/L$ | | |
|---|---|---|---|---|---|---|
| | FEM | SM | Difference | FEM | SM | Difference |
| 19 | 0.0549 | 0.0550 | 0.1821 | 0.1460 | 0.1459 | 0.0685 |
| 20 | 0.1767 | 0.1770 | 0.1698 | 0.2505 | 0.2510 | 0.1996 |
| 21 | 0.2951 | 0.2980 | 0.9827 | 0.3082 | 0.3097 | 0.4867 |
| 22 | 0.4202 | 0.4320 | 2.8082 | 0.3456 | 0.3493 | 1.0706 |
| 23 | 0.6284 | 0.5990 | 4.6785 | 0.3648 | 0.3653 | 0.1370 |

Note: FEM = finite element method; SM = shooting method [19]

## V. CONCLUSIONS

A variational method for postbuckling analysis of end supported rods under self-weight is presented. The strain energy due to bending and the potential energy of the self-weight are the complement of a variational formulation. Lagrange multiplier technique is included by identified the Lagrange multiplier as the reaction at the end support. FEM with Newton-Raphson iteration procedure are used for numerical solution. The variational method presented here is highly recommended to analyze the postbuckling behavior of end supported rods under self-weight.

## ACKNOWLEDGMENT

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