

# Positive Solutions for Systems of Nonlinear Third-Order Differential Equations with p-Laplacian

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**Abstract**—In this paper, by constructing a special set and utilizing fixed point theory, we study the existence and multiplicity of the positive solutions for systems of nonlinear third-order differential equations with p-laplacian, which improve and generalize the result of related paper.

**Keywords**—p-Laplacian, cone, fixed point theorem, positive solution.

## I. INTRODUCTION

THE boundary value problems of differential equation with p-Laplacian arises in a variety of problems related to applied mathematics, physics and engineering. However there is still a little research about it. In recent years, some results concerning the problems have been obtained. We refer the readers to [1]-[6] and the references cited therein. In the thesis [5], the author investigated the following elliptic systems:

$$\begin{cases} \Delta u + \lambda k_1(|x|)f(u, v) = 0, \\ \Delta v + \mu k_2(|x|)g(u, v) = 0, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases}$$

In the thesis [6], the author investigated the following coupled singular boundary value problems:

$$\begin{cases} (\phi_p(u''(t)))' + \omega_1(t)f_1(t, v(t)) = 0, & t \in (0, 1), \\ (\phi_p(v''(t)))' + \omega_2(t)f_2(t, u(t)) = 0, & t \in (0, 1), \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \gamma_1 u(1) + \delta_1 u'(1) = 0, u''(0) = 0, \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \gamma_2 v(1) + \delta_2 v'(1) = 0, v''(0) = 0. \end{cases}$$

Motivated by the thesis [5], [6], in this paper. We consider the following systems of third-order boundary value problems:

$$\begin{cases} (\phi_p(u''(t)))' + \omega_1(t)f_1(t, u(t), v(t)) = 0, & t \in (0, 1), \\ (\phi_p(v''(t)))' + \omega_2(t)f_2(t, u(t), v(t)) = 0, & t \in (0, 1), \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \gamma_1 u(1) + \delta_1 u'(1) = 0, u''(0) = 0, \\ \alpha_2 v(0) - \beta_2 v'(0) = 0, \gamma_2 v(1) + \delta_2 v'(1) = 0, v''(0) = 0. \end{cases} \quad (1)$$

where  $\phi_p(s) = |s|^{p-2}s$ ,  $p \geq 2$ ,  $\alpha_i, \gamma_i > 0$ ,  $\beta_i, \delta_i \geq 0$ ,  $f_i \in C((0, 1) \times [0, +\infty) \times [0, +\infty), [0, +\infty))$ ,  $\omega_i(t) \in C((0, 1), [0, +\infty))$  and  $f_i, \omega_i(t)$  may be singular at  $t = 0, 1$ . In thesis [5], the control functions need to be continuous and monotonic. In thesis [6],  $f_1$  and  $f_2$  are functions of one variable. Different from the works mentioned above, our purpose here is to deal with more general functions than that of thesis [6]. Moreover, the conditions that we used are weaker than that of thesis [5]. The organization of this paper is as

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follows, we shall first introduce some definitions and lemmas in the rest of this section. The main results will be stated and proved in Section II. In Section III, we present an example to check our result.

For the sake of convenience, we first give some conditions.

(H<sub>1</sub>)  $f_i(t, u(t), v(t)) \leq g_i(t)h_i(u(t), v(t))$ ,  $g_i(t) : (0, 1) \rightarrow [0, +\infty)$  may be singular at  $t = 0, 1$ ,  $h_i(u, v) : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$  is continuous,  $i=1, 2$ .

(H<sub>2</sub>)  $\omega_i \in C((0, 1), [0, +\infty))$ ,  $\omega_i$  may be singular at  $t = 0, 1$ , and

$$0 \leq \int_0^1 \omega_i(s)g_i(s)ds < +\infty.$$

(H<sub>3</sub>)

$$0 \leq \limsup_{(u,v) \rightarrow 0} \frac{h_1(u, v)}{(u+v)^{p-1}} < \eta_1^{p-1}$$

and

$$0 \leq \limsup_{(u,v) \rightarrow 0} \frac{h_2(u, v)}{(u+v)^{p-1}} < \eta_2^{p-1}.$$

(H<sub>4</sub>)

$$(M_1^{-1}\xi_1)^{p-1} < \liminf_{(u,v) \rightarrow \infty} \frac{f_1(s, u, v)}{(u+v)^{p-1}} \leq \infty,$$

or

$$(M_2^{-1}\xi_2)^{p-1} < \liminf_{(u,v) \rightarrow \infty} \frac{f_2(s, u, v)}{(u+v)^{p-1}} \leq \infty.$$

(H<sub>5</sub>)

$$0 \leq \limsup_{(u,v) \rightarrow \infty} \frac{h_1(u, v)}{(u+v)^{p-1}} < \eta_1^{p-1}$$

and

$$0 \leq \limsup_{(u,v) \rightarrow \infty} \frac{h_2(u, v)}{(u+v)^{p-1}} < \eta_2^{p-1}.$$

(H<sub>6</sub>)

$$(M_1^{-1}\xi_1)^{p-1} < \liminf_{(u,v) \rightarrow 0} \frac{f_1(s, u, v)}{(u+v)^{p-1}} \leq \infty$$

or

$$(M_2^{-1}\xi_2)^{p-1} < \liminf_{(u,v) \rightarrow 0} \frac{f_2(s, u, v)}{(u+v)^{p-1}} \leq \infty.$$

where  $\eta_i$  and  $\xi_i$  ( $i = 1, 2$ ) are constants such that

$$0 < \eta_i \left( \max_{0 \leq t \leq 1} \int_0^1 G_i(t, \tau) \phi_q \left( \int_0^\tau \omega_i(s)g_i(s)ds \right) d\tau \right) \leq \frac{1}{2}$$

and

$$\xi_i \int_\theta^{1-\theta} G_i\left(\frac{1}{2}, \tau\right) \phi_q \left( \int_0^\tau \omega_i(s)g_i(s)ds \right) d\tau \geq 1.$$

It is easy to show the systems (1) are equivalent to the following integral equations

$$\begin{cases} u(t) = \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau, \\ v(t) = \int_0^1 G_2(t, \tau) \phi_q \left( \int_0^\tau \omega_2(s) f_2(s, u(s), v(s)) ds \right) d\tau, \end{cases} \quad (2)$$

where  $q$  is a constant such that  $\frac{1}{p} + \frac{1}{q} = 1$ , and

$$G_i(t, s) = \begin{cases} \frac{\varphi_i(t)\psi_i(s)}{\rho_i}, & 0 \leq s \leq t \leq 1, \\ \frac{\varphi_i(s)\psi_i(t)}{\rho_i}, & 0 \leq t \leq s \leq 1, \end{cases}$$

where

$$\begin{aligned} \rho_i &= \alpha_i \gamma_i + \alpha_i \delta_i + \beta_i \gamma_i, \\ \varphi_i(t) &= \gamma_i + \delta_i - \gamma_i t, \\ \psi_i(t) &= \beta_i + \alpha_i t, \quad 0 \leq t \leq 1. \end{aligned}$$

Let

$$A(u, v)(t) = \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau,$$

$$B(u, v)(t) = \int_0^1 G_2(t, \tau) \phi_q \left( \int_0^\tau \omega_2(s) f_2(s, u(s), v(s)) ds \right) d\tau,$$

$$F(u, v)(t) = (A(u, v)(t), B(u, v)(t)).$$

Then systems (2) are equivalent to the fixed point equation

$$F(u, v) = (u, v)$$

in the Banach space  $E = X \times X$ , where

$$X = \{u : u, \phi_p(u'') \in C^1[0, 1]\}.$$

The following fixed -point theorem of cone expansion and compression type is crucial in the following argument.

**Lemma 1** [7] Let  $K$  be a cone in Banach space  $E$ . Assume  $\Omega_1, \Omega_2$  are open subsets of  $E$  with  $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ , and let  $F : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$  be a completely continuous operator such that either

$$\|Fu\| \leq \|u\|, u \in K \cap \partial\Omega_1, \|Fu\| \geq \|u\|, u \in K \cap \partial\Omega_2;$$

or

$$\|Fu\| \geq \|u\|, u \in K \cap \partial\Omega_1, \|Fu\| \leq \|u\|, u \in K \cap \partial\Omega_2,$$

then  $F$  has a fixed point in  $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

In what follows we set

$$\|(u, v)\| = \|u\| + \|v\|,$$

where

$$\|u\| = \max_{t \in [0, 1]} |u(t)|.$$

In order to apply Lemma 1, we let  $K$  be the cone defined by

$$K = \{(u, v) : (u, v) \in E : u, v \geq 0, \min_{t \in [\theta, 1-\theta]} (u(t) + v(t)) \geq M(\|u\| + \|v\|)\},$$

where  $\theta \in (0, \frac{1}{2})$ ,  $M = \min\{M_1, M_2\}$ , and

$$M_i = \min\left\{\frac{\delta_i + \theta\gamma_i}{\gamma_i + \delta_i}, \frac{\theta\alpha_i + \beta_i}{\alpha_i + \beta_i}\right\}.$$

**Lemma 2** [8] If  $p \geq 2, \frac{1}{p} + \frac{1}{q} = 1$ , then

$$|\phi_q(x) - \phi_q(y)| \leq \phi_q(x - y).$$

**Lemma 3** Suppose that conditions  $(H_1) - (H_2)$  hold, then  $F : K \rightarrow K$  is completely continuous.

**Proof:** First we show that  $F(K) \subset K$ .

$\forall (u, v) \in K, t \in [0, 1]$ , we have

$$\begin{aligned} A(u, v)(t) &= \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau \\ &\leq \int_0^1 G_1(\tau, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau. \end{aligned}$$

Hence

$$\begin{aligned} \|A(u, v)\| &\leq \int_0^1 G_1(\tau, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau. \end{aligned}$$

On the other hand, for  $t \in [\theta, 1-\theta], \frac{G_i(t, \tau)}{G_i(\tau, \tau)} \geq M_i, \tau \in [0, 1]$ , we have

$$\begin{aligned} \min_{\theta \leq t \leq 1-\theta} A(u, v)(t) &= \min_{\theta \leq t \leq 1-\theta} \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau \\ &\geq M_1 \int_0^1 G_1(\tau, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau \\ &\geq M_1 \|A(u, v)\|. \end{aligned}$$

Similarly

$$\min_{\theta \leq t \leq 1-\theta} B(u, v)(t) \geq M_2 \|B(u, v)\|.$$

Thus

$$\begin{aligned} \min_{\theta \leq t \leq 1-\theta} (A(u, v)(t) + B(u, v)(t)) &\geq \min_{\theta \leq t \leq 1-\theta} A(u, v)(t) + \min_{\theta \leq t \leq 1-\theta} B(u, v)(t) \\ &\geq M_1 \|A(u, v)\| + M_2 \|B(u, v)\| \\ &\geq M \|(A(u, v), B(u, v))\|. \end{aligned}$$

We conclude that  $F(K) \subset K$ .

Next we show that  $F : K \rightarrow K$  is completely continuous.

Let  $\forall D \in K$  be a bounded set, i.e.  $\exists M > 0$  such that  $\forall (u, v) \in K, \|(u, v)\| \leq M$ , we have

$$\begin{aligned} A(u, v)(t) &= \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau \\ &\leq \int_0^1 G_1(\tau, \tau) \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) h_1(u(s), v(s)) ds \right) d\tau \\ &\leq \{ \max \phi_q(h_1(u(s), v(s)) : 0 \leq s \leq 1) \} \\ &\quad \times \int_0^1 G_1(\tau, \tau) \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) ds \right) d\tau \\ &\leq \{ \max \phi_q(h_1(u(s), v(s)) : 0 \leq s \leq 1) \} \\ &\quad \times \int_0^1 G_1(\tau, \tau) d\tau \phi_q \left( \int_0^1 \omega_1(s) g_1(s) ds \right) \\ &= N_1 < \infty. \end{aligned}$$

So  $\|A(u, v)\| \leq N_1$ .

Similarly,

$$\|B(u, v)\| \leq N_2 < \infty.$$

Hence,

$$\begin{aligned} \|F(u, v)\| &= \|A(u, v)\| + \|B(u, v)\| \\ &\leq N_1 + N_2 < +\infty. \end{aligned}$$

Correspondingly,  $F : K \rightarrow K$  is bounded uniformly.

Now, we show that  $F$  is equicontinuous. The continuity of  $G_i(t, s)$  on  $[0, 1] \times [0, 1]$  implies that  $G_i(t, s)$  is continuous uniformly. i.e.

$\forall \varepsilon > 0, \exists \delta > 0$ , such that  $\forall \xi \in [0, 1], |t_1 - t_2| < \delta$ , we have

$$\begin{aligned} &|G_i(t_1, \xi) - G_i(t_2, \xi)| \\ &< \frac{\varepsilon}{2} \{ \max \phi_q(h_i(u(s), v(s)) : 0 \leq s \leq 1) \} \\ &\quad \times \phi_q \left( \int_0^1 \omega_i(s) g_i(s) ds \right)^{-1}. \end{aligned}$$

Hence,  $\forall D \in K$ , we have

$$\begin{aligned} &|F(u, v)(t_1) - F(u, v)(t_2)| \\ &= |(A(u, v)(t_1), B(u, v)(t_1)) - (A(u, v)(t_2), B(u, v)(t_2))| \\ &= |(A(u, v)(t_1) - A(u, v)(t_2), (B(u, v)(t_1) - B(u, v)(t_2)))| \\ &\leq |(A(u, v)(t_1) - A(u, v)(t_2))| \\ &\quad + |(B(u, v)(t_1) - B(u, v)(t_2))|. \end{aligned}$$

$$\begin{aligned} &= \left| \int_0^1 (G_1(t_1, \tau) - G_1(t_2, \tau)) \right. \\ &\quad \times \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) h_1(u(s), v(s)) ds \right) d\tau \left. + \int_0^1 (G_2(t_1, \tau) - G_2(t_2, \tau)) \right. \\ &\quad \times \phi_q \left( \int_0^\tau \omega_2(s) g_2(s) h_2(u(s), v(s)) ds \right) d\tau \left. \right| \\ &\leq \left| \int_0^1 (G_1(t_1, \tau) - G_1(t_2, \tau)) \right. \\ &\quad \times \max \{ \phi_q(h_1(u(s), v(s)) : 0 \leq s \leq 1) \} \\ &\quad \times \phi_q \left( \int_0^1 \omega_1(s) g_1(s) ds \right) d\tau \left. + \int_0^1 (G_2(t_1, \tau) - G_2(t_2, \tau)) \right. \\ &\quad \times \max \{ \phi_q(h_2(u(s), v(s)) : 0 \leq s \leq 1) \} \\ &\quad \times \phi_q \left( \int_0^1 \omega_2(s) g_2(s) ds \right) d\tau \left. \right| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This means that  $F(K)$  is equicontinuous, so  $F(K)$  is relatively compact in  $K$ .

Finally we show  $F : K \rightarrow K$  is continuous.

Let  $\{(u_n, v_n)\} \subset K$  be sequence such that  $(u_n, v_n) \rightarrow (u_0, v_0), n \rightarrow \infty$ , we have

$$\begin{aligned} &|F(u_n, v_n)(t) - F(u_0, v_0)(t)| \\ &\leq \int_0^1 G_1(\tau, \tau) \left| \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u_n(s), v_n(s)) ds \right) \right. \\ &\quad \left. - \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u_0(s), v_0(s)) ds \right) \right| d\tau \\ &\quad + \int_0^1 G_2(\tau, \tau) \left| \phi_q \left( \int_0^\tau \omega_2(s) f_2(s, u_n(s), v_n(s)) ds \right) \right. \\ &\quad \left. - \phi_q \left( \int_0^\tau \omega_2(s) f_2(s, u_0(s), v_0(s)) ds \right) \right| d\tau \\ &\leq \int_0^1 G_1(\tau, \tau) d\tau \left| \phi_q \left( \int_0^1 \omega_1(s) (f_1(s, u_n(s), v_n(s)) \right. \right. \right. \\ &\quad \left. \left. - f_1(s, u_0(s), v_0(s)) ds \right) \right| \\ &\quad + \int_0^1 G_2(\tau, \tau) d\tau \left| \phi_q \left( \int_0^1 \omega_2(s) (f_2(s, u_n(s), v_n(s)) \right. \right. \right. \\ &\quad \left. \left. - f_2(s, u_0(s), v_0(s)) ds \right) \right|. \end{aligned}$$

By  $(H_2)$ , Lebesgue dominated convergence theorem and the continuity of  $f_1, f_2$ , we have

$$\|F(u_n, v_n) - F(u_0, v_0)\| \rightarrow 0, n \rightarrow \infty,$$

thus  $F$  is continuous.

By above arguments, we claim  $F$  is completely continuous.

## II. CONCLUSIONS

**Theorem 1** Suppose that conditions  $(H_1) - (H_4)$  hold, then systems (1) has at least one positive solution.

**Proof:**

By  $(H_3)$ , we may choose  $r > 0$ , for any  $\|u\| + \|v\| \leq r$  we have

$$h_1(u, v) \leq \eta_1^{p-1}(u + v)^{p-1},$$

and

$$h_2(u, v) \leq \eta_2^{p-1}(u + v)^{p-1}.$$

Set

$$\Omega_1 = \{(u, v) : (u, v) \in K; \|(u, v)\| < r\}$$

If  $(u, v) \in K \cap \partial\Omega_1$  then  $\|u\| + \|v\| \leq r$ , we have

$$A(u, v)(t)$$

$$= \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau,$$

$$\leq \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) h_1(u(s), v(s)) ds \right) d\tau$$

$$\leq \eta_1 \|(u, v)\| \max_{0 \leq t \leq 1} \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) ds \right) d\tau$$

$$\leq \frac{1}{2} \|(u, v)\|.$$

Which implies

$$\|A(u, v)\| \leq \frac{1}{2} \|(u, v)\|.$$

Similarly,

$$B(u, v)(t) \leq \frac{1}{2} \|(u, v)\|.$$

Hence, for  $(u, v) \in K \cap \partial\Omega_1$ , we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \leq \|(u, v)\|. \quad (3)$$

By  $(H_4)$ , if we further assume

$$(M_1^{-1} \xi_1)^{p-1} < \liminf_{(u,v) \rightarrow \infty} \frac{f_1(s, u, v)}{(u + v)^{p-1}} \leq \infty,$$

then there is an  $\bar{R} > r$ , for any  $\|u\| + \|v\| > \bar{R}$ , we have

$$f_1(s, u, v) \geq (M^{-1} \xi_1 (u + v))^{p-1}.$$

Let  $R > \max\{\bar{R}, M_1^{-1} \bar{R}, M_2^{-1} \bar{R}\}$ , we set

$$\Omega_2 = \{(u, v) : (u, v) \in K; \|(u, v)\| < R\}.$$

If  $(u, v) \in K \cap \partial\Omega_2$ , then

$$A(u, v)\left(\frac{1}{2}\right)$$

$$= \int_0^1 G_1\left(\frac{1}{2}, \tau\right) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau$$

$$\geq M_1^{-1} \xi_1 \int_\theta^{1-\theta} G_1\left(\frac{1}{2}, \tau\right)$$

$$\times \phi_q \left( \int_\theta^\tau \omega_1(s) (u(s) + v(s))^{p-1} ds \right) d\tau$$

$$\geq \xi_1 (\|u\| + \|v\|) \int_\theta^{1-\theta} G_1\left(\frac{1}{2}, \tau\right) \phi_q \left( \int_\theta^\tau \omega_1(s) ds \right) d\tau$$

$$\geq \|(u, v)\|.$$

which implies

$$\|A(u, v)\| \geq \|(u, v)\|.$$

Similarly

$$\|B(u, v)\| \geq \|(u, v)\|.$$

An analogous estimate holds for  $f_2$  in condition  $(H_4)$ .

Hence, for  $(u, v) \in K \cap \partial\Omega_2$  we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \geq \|(u, v)\|. \quad (4)$$

By Lemma 1,  $F$  has at least one fixed point  $(u, v) \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ . So, systems (1) has at least one positive solution.

**Theorem 2** Suppose that conditions  $(H_1)$ ,  $(H_2)$ ,  $(H_5)$ ,  $(H_6)$  hold, then systems (1) has at least one positive solution.

**Proof:** By  $(H_5)$ , there exist  $R_0, \varepsilon_i > 0$ , for  $\|u\| + \|v\| \geq R_0$ , we have

$$h_i(u, v) \leq (\eta_i - \varepsilon_i)^{p-1} (u + v)^{p-1}.$$

Let

$$a = \max_{i=1,2} \max\{\phi_q(h_i(u, v)) : u(t) \leq R_0, v(t) \leq R_0\},$$

$$R > \max\left\{\frac{a}{\varepsilon_1}, \frac{a}{\varepsilon_2}\right\}.$$

We set

$$\Omega_1 = \{(u, v) : (u, v) \in K; \|(u, v)\| < R\},$$

if  $(u, v) \in K \cap \partial\Omega_1$ , then

$$\begin{aligned} & A(u, v)(t) \\ &= \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) f_1(s, u(s), v(s)) ds \right) d\tau \\ &\leq \int_0^1 G_1(t, \tau) \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) h_1(u(s), v(s)) ds \right) d\tau \\ &= \int_0^1 G_1(t, \tau) \phi_q \left( \int_{\|u\|+\|v\| \geq R_0} \omega_1(s) g_1(s) \right. \\ &\quad \times h_1(u(s), v(s)) ds \left. \right) d\tau \\ &+ \int_0^1 G_1(t, \tau) \phi_q \left( \int_{\|u\|+\|v\| \leq R_0} \omega_1(s) g_1(s) \right. \\ &\quad \times h_1(u(s), v(s)) ds \left. \right) d\tau \\ &\leq (\eta_1 - \varepsilon_1)(\|u(t)\| + \|v(t)\|) \int_0^1 G_1(t, \tau) \\ &\quad \times \phi_q \left( \int_{\|u\|+\|v\| \geq R_0} \omega_1(s) g_1(s) ds \right) d\tau \\ &+ a \int_0^1 G_1(t, \tau) \phi_q \left( \int_{\|u\|+\|v\| \leq R_0} \omega_1(s) g_1(s) ds \right) d\tau \\ &\leq [(\eta_1 - \varepsilon_1)(\|u\| + \|v\|) + a] \max_{0 \leq t \leq 1} \int_0^1 G_1(t, \tau) \\ &\quad \times \phi_q \left( \int_0^\tau \omega_1(s) g_1(s) ds \right) d\tau \\ &\leq \frac{1}{2} \|(u, v)\|. \end{aligned}$$

Similarly,

$$B(u, v)(t) \leq \frac{1}{2} \|(u, v)\|.$$

Hence, for any  $(u, v) \in K \cap \partial\Omega_1$ , we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \leq \|(u, v)\|. \quad (5)$$

On the other hand, by  $(H_6)$ , if we further assume

$$(M_1^{-1} \xi_1)^{p-1} < \liminf_{(u,v) \rightarrow 0} \frac{f_1(s, u, v)}{(u+v)^{p-1}} \leq \infty,$$

then there is an  $0 < r < R$ , for any  $\|u\| + \|v\| \leq r$ , we have

$$f_1(s, u, v) \geq (M^{-1} \xi_1 (u+v))^{p-1}.$$

Set

$$\Omega_2 = \{(u, v) : (u, v) \in K; \|(u, v)\| < r\},$$

similar to the proof of Theorem 1, for  $(u, v) \in K \cap \partial\Omega_2$ , we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \geq \|(u, v)\|. \quad (6)$$

An analogous estimate holds for  $f_2$  in condition  $(H_6)$ .

By Lemma 1,  $F$  has at least one fixed point  $(u, v) \in K \cap (\bar{\Omega}_1 \setminus \Omega_2)$ , so system (1) has at least one positive solution.

**Theorem 3** Suppose that conditions  $(H_1) - (H_4)$  and  $(H_6)$  hold, then systems (1) has at least two positive solutions  $(u_1, v_1)$  and  $(u_2, v_2)$  with  $\|(u_1, v_1)\| < r < \|(u_2, v_2)\|$ .

**Proof** By  $(H_4)$  if we further assume that

$$(M_1^{-1} \xi_1)^{p-1} < \liminf_{(u,v) \rightarrow \infty} \frac{f_1(s, u, v)}{(u+v)^{p-1}} \leq \infty,$$

then we choose a  $R_1 > r$  large sufficiently, for any  $\|u\| + \|v\| > R_1$ , we have

$$f_1(s, u, v) \geq (M^{-1} \xi_1 (u+v))^{p-1}.$$

Set

$$\Omega_{R_1} = \{(u, v) : (u, v) \in K; \|(u, v)\| < R_1\},$$

similarly, for any  $(u, v) \in K \cap \partial\Omega_{R_1}$ , we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \geq \|(u, v)\|. \quad (7)$$

An analogous estimate holds for  $f_2$  in condition  $(H_4)$ .

On the other hand, by  $(H_6)$  we assume that

$$(M_1^{-1} \xi_1)^{p-1} < \liminf_{(u,v) \rightarrow 0} \frac{f_1(s, u, v)}{(u+v)^{p-1}} \leq \infty,$$

we choose  $R_2 < r$  small sufficiently, for any  $\|u(t)\| + \|v(t)\| < R_2$ , we have

$$f_1(s, u, v) \geq (M^{-1} \xi_1 (u+v))^{p-1}.$$

Set

$$\Omega_{R_2} = \{(u, v) : (u, v) \in K; \|(u, v)\| < R_2\},$$

similarly, for any  $(u, v) \in K \cap \partial\Omega_{R_2}$ , we have

$$\|F(u, v)\| = \|A(u, v)\| + \|B(u, v)\| \geq \|(u, v)\|. \quad (8)$$

An analogous estimate holds for  $f_2$  in condition  $(H_6)$ .

By Lemma 1,  $F$  has at least twice fixed points  $(u_1, v_1) \in \bar{\Omega}_{R_1} \setminus \Omega_1$  and  $(u_2, v_2) \in \bar{\Omega}_1 \setminus \Omega_{R_2}$ , so systems (1) has at least twice positive solutions with  $\|(u_1, v_1)\| < r < \|(u_2, v_2)\|$ .

**Theorem 4** Suppose that conditions  $(H_1) - (H_5)$  hold, then problem (1) has at least two positive solutions  $(u_1, v_1)$  and  $(u_2, v_2)$  with  $\|(u_1, v_1)\| < R < \|(u_2, v_2)\|$ .

The proof is similar to the Theorem 3, we omit it.

### III. EXAMPLE

As an application of Theorems 1 and 2, we give an example:

$$\begin{cases} (\phi_p(u''(t)))' + \frac{(u(t) + v(t))^{\alpha_1}}{6\sqrt{t(1-t)}} = 0, \\ (\phi_p(v''(t)))' + \frac{(u(t) + v(t))^{\alpha_2}}{6\sqrt{t(1-t)}} = 0, \\ u(0) = u(1) = 0, u'(0) = 0, \\ v(0) = v(1) = 0, v'(0) = 0. \end{cases}$$

where  $p \geq 2, \alpha_1 \geq \alpha_2 > p - 1$ .  $\omega_i(t) = \frac{1}{\sqrt[p]{t(1-t)}}$ ,

$$f_i(t, u, v) = \frac{(u+v)^{\alpha_i}}{6\sqrt[p]{t(1-t)}}.$$

Let

$$g_i(t) = \frac{1}{3\sqrt[p]{t(1-t)}}, h_i(u, v) = (u+v)^{\alpha_i},$$

$$\omega_i(t)g_i(t) = \frac{1}{3\sqrt[p]{t(1-t)}} \quad \text{and} \quad \int_0^1 \omega_i(s)g_i(s)ds = \frac{\pi}{3},$$

Theorem 1 holds.

If  $\alpha_1 \leq \alpha_2 < p - 1$ , then Theorem 2 holds.

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