Periodic oscillations in a delay population model

Changjin Xu, Peiluan Li

Abstract—In this paper, a nonlinear delay population model is investigated. Choosing the delay as a bifurcation parameter, we demonstrate that Hopf bifurcation will occur when the delay exceeds a critical value. Global existence of bifurcating periodic solutions is established. Numerical simulations supporting the theoretical findings are included.

Keywords—Population model; Stability; Hopf bifurcation; Delay; Global Hopf bifurcation.

I. INTRODUCTION

IN 1976, in order to investigate the control of sigle populations of cells, Nazarenko [1] proposed the nonlinear delay differential equation

$$\frac{dx(t)}{dt} = -px(t) + \frac{qx(t)}{r + x^m(t-\tau)}, t \geq 0, \qquad (1)$$

where $x(t)$ is the size of the population, p is the death rate and the feedback is given by the function $f(u, u(t-\tau)) = \frac{qu(t)}{r+u^m(t-\tau)}$ and τ denotes the generation time. $p, q, r, \tau \in (0, +\infty)$ and $m \in N = \{1, 2, 3, \cdots\}$. Nazarenko[1] proved that every positive nonoscillatory solution converges to the unique positive equilibrium $(q/p - r)^{-m}$ and established a sufficient condition for oscillation of all positive solution about $(q/p - r)^{-m}$. Kubiaczyk and Saker [2] considered the convergence and the oscillation of system (1). Considering that the periodic changes in the environment, Saker and Agarwal [3] modified (1) as the following form

$$\frac{dx(t)}{dt} = -p(t)x(t) + \frac{q(t)x(t)}{r + x^m(t-\tau)}, \qquad (2)$$

where $p(t)$ and $q(t)$ are positive periodic functions of period ω and m is a positive integer. Saker and Agarwal [3] obtained that system (2) has a positive periodic solution $\bar{x}(t)$ with period ω and established some sufficient conditions for oscillation and global attractivity of positive solutions by using the Brouwer's fixed point theorem. Saker [4] discussed the existence and global attractivity of periodic solution of the discrete version of system (2) which takes the form

$$x(n+1) = x(n)\exp\left(-p(n) + \frac{q(n)}{r + x^m(n-\omega)}\right), \qquad (3)$$

where $n = 0, 1, 2, \cdots$, $xp(n)$ and $q(n)$ are positive periodic sequences of period ω, m and ω are positive integer and $\omega > 1$. Song and Peng [5] further investigated the periodic

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solution of the more general non-autonomous periodic models of population with continuous and discrete time as follows

$$\frac{dx(t)}{dt} = -p(t)x(t) + \frac{q(t)x(t)}{r + x^m(t-\tau(t))} \qquad (4)$$

and

$$x(k+1) = x(k)\exp\left\{-p(k) + \frac{q(k)}{r + x^n(k)}\right\}, \qquad (5)$$

where $k = 0, 1, 2, 3, \cdots$. In order to unify continuous and discrete analysis, Zhang and Wen [6] investigated the periodic solutions the periodic solution of dynamical equation on the scales of the form

$$x^\Delta(t) = -p(t) + \frac{q(t)}{r + e^{nx(t-\tau(t))}}, \qquad (6)$$

where $p(t)$ and $q(t)$ are positive ω-periodic function on time scale T. With the aid of coincidence degree theory, Zhang and Wen [6] obtained the sufficient conditions for the existence of periodic solutions of system (6).

Based on former work [1-6], we further devote to explore the dynamical behaviors of system (1), i.e., we will investigate the natures of Hopf bifurcation of system (1). For simplification, we assume that $m = 3$, then system becomes

$$\frac{dx(t)}{dt} = -px(t) + \frac{qx(t)}{r + x^3(t-\tau)}, \qquad (7)$$

where p, q, r positive constants and $\tau \geq 0$ is a delay. The initial value is $x(t) = \varphi(t), -\tau \leq t \leq 0, \varphi(t) \in ([-\tau, 0], R^+), \varphi(0) > 0$.

The purpose of this paper is to investigate the existence of local and global Hopf bifurcation of model (7). This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, the existence of global Hopf bifurcation is established. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

II. STABILITY OF THE EQUILIBRIUM AND LOCAL HOPF BIFURCATIONS

Considering the biological interpretations of population, in this paper, we only investigate the positive equilibrium point of system (7). If the condition $q > pr$ holds, then system (7) has a unique positive equilibrium

$$x^* = \sqrt[3]{\frac{q - pr}{p}}.$$

Let $\bar{x}(t) = x(t) - x^*$, Substituting this into (7) and still denote $\bar{x}(t)$ by $x(t)$, then (7) takes the form

$$
\begin{aligned}
\frac{dx(t)}{dt} &= a_1 x(t-\tau) + a_2 x(t) x(t-\tau) \\
&\quad + a_3 x(t) x^2(t-\tau) + a_4 x^3(t-\tau),
\end{aligned} \tag{8}
$$

where

$$
a_1 = -\frac{3q(x_1^*)^3}{[r+(x^*)^3]^2}, a_2 = -\frac{3q(x_1^*)^2}{[r+(x^*)^3]^2},
$$

$$
a_3 = -\frac{3q(x_1^*)^2[r+(x^*)^3] - 3(x*)^2}{[r+(x^*)^3]^4},
$$

$$
\begin{aligned}
a_4 &= \frac{12(x^*)^2[(12qr-6)(x^*)^4 + 6r(q-1)x^* + 6q(x^*)^7]}{[r+(x^*)^3]^7} \\
&\quad - \frac{6q[r^2 + 2r(x^*)^3 + (x^*)^6]}{[r+(x^*)^3]^7} \\
&\quad - \frac{[6r(x^*)^2 + 6x^*][6qx^* - 24(x^*)^3 - 6r]}{[r+(x^*)^3]^7}.
\end{aligned}
$$

Then the linearization of system (8) around the equilibrium (0,0) is

$$
\frac{dx(t)}{dt} = a_1 x(t-\tau). \tag{9}
$$

The associated characteristic equation of (9) is given by

$$
\lambda - a_1 e^{-\lambda\tau} = 0, \tag{10}
$$

Let $\lambda = i\omega_0, \tau = \tau_0$, and substituting this into (9). Separating the real and imaginary parts, we have

$$
a_1 \cos\omega_0\tau = 0, a_1 \sin\omega_0\tau = -\omega_0. \tag{11}
$$

Since $a_1 < 0$, then we can obtain

$$
\omega_0 = -a_1, \tau = \tau_k = \frac{1}{\omega_0}\left[k\pi + \frac{\pi}{2}\right], k = 0, 1, 2, \cdots. \tag{12}
$$

When $\tau = 0$, (10) becomes

$$
\lambda = a_1 < 0. \tag{13}
$$

In view of above analysis, we have

Lemma 2.1. *If $q > pr$ holds, then system (7) admits a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_k, k = 0, 1, 2, \cdots$.*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq.(2.3) near $\tau = \tau_k$ satisfying $\alpha(\tau_k) = 0$, $\omega(\tau_k) = \omega_0$. Due to functional differential equation theory, for every $\tau_k, k = 0, 1, 2, \cdots$, there exists a $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (10) and taking the derivative of λ with respect to τ, we get

$$
\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{1}{a_1\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.
$$

It follows together with (2.4) that

$$
\begin{aligned}
\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}\Big|_{\tau=\tau_k} &= -\mathrm{Re}\left\{\frac{1}{a_1\lambda e^{-\lambda\tau}}\right\}_{\tau=\tau_k} \\
&= -\frac{\sin\omega_0\tau_k}{\omega_0} = 1 > 0.
\end{aligned}
$$

Thus

$$
\mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|_{\tau=\tau_k}\right\} = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}\Big|_{\tau=\tau_k}\right\} > 0.
$$

According to the results of Kuang [7] and Hale [8], we have

Theorem 2.1. *If $q > pr$ holds, the positive equilibrium x^* of system (7) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau \geq \tau_0$. System (7) underdoes a Hopf bifurcation at the positive equilibrium x^* when $\tau = \tau_k, k = 0, 1, 2, \cdots$.*

III. GLOBAL EXISTENCE OF THE HOPF BIFURCATIONS

In this section, we mainly focus on the global existence of periodic solutions bifurcating from the positive equilibrium. Throughout this section, we closely use the notation as those in Wu [9] and define
$X = C([-\tau, 0], R)$,
$\Sigma = Cl\{(x, \tau, p) : x \text{ is } p-\text{periodic solution of system (7)}\} \in X \times R_+ \times R_+$,
$N = \{(\hat{x}, \tau, p) : p(r + \hat{x}^3) - q = 0\}$,
$\Delta = \lambda - a_1 e^{-\lambda\tau}$.
Let $C(N_*, \tau_k, \frac{2\pi}{\omega_0})$ denote the connected component of $(N_*, \tau_k, \frac{2\pi}{\omega_0})$ in Σ.

Lemma 3.1. *If $p > q$, then all the periodic solution of system (7) are uniformly bounded and system (7) has no nontrivial τ-periodic solutions.*

Proof. It follows from (7) that

$$
\frac{dx(t)}{dt} \leq x(t)(-p+q) < 0
$$

which leads to $x(t) < \varphi(0)$. Thus all the periodic solution of system (7) are uniformly bounded. The nontrivial τ-periodic solutions of system (7) is also the nonconstant periodic solution of the following ordinary differential equation

$$
\frac{dx(t)}{dt} = -px(t) + \frac{qx(t)}{r + x^3(t)}, \tag{14}
$$

It follows from (3.1) that

$$
\frac{dx(t)}{dt} = x(t)\left(-p + \frac{q}{r + x^3(t)}\right) \leq x(t)(-p+q) < 0.
$$

Then we can conclude that system (14) has no nonconstant periodic solution. This complete the proof.

Theorem 3.1. *If $q > pr$ and $p > q$, then periodic solutions of system (7) still exist when $\tau > \tau_k$.*

Proof. First of all, define

$$
F(x_t, \tau, p) = -px(t) + \frac{qx(t)}{r + x^3(t-\tau)}.
$$

It is easy to check that $(A_1) - (A_3)$ in Wu [9] are satisfied. Let $(\hat{x}_0, \alpha_0, p) = (N_*, \tau_k, \frac{2\pi}{\omega_0})$ and it is easy to verify that $(N_*, \tau_k, \frac{2\pi}{\omega_0})$ is the only isolated center. There exist $\varepsilon > 0, \delta > 0$ and a smooth function $\lambda : (\tau_k - \delta, \tau_k + \delta) \to C$ such that

$$
\Delta(\lambda(\tau)) = 0, |\lambda(\tau) - i\omega_0| < \varepsilon
$$

for any $\tau \in [\tau_k - \delta, \tau_k + \delta]$, and

$$\lambda(\tau_k) = i\omega_0, \quad \frac{d\text{Re}(\lambda(\tau))}{d\tau} > 0.$$

Define $p_j = \frac{2\pi}{\omega_0}$ and $\Omega_{\varepsilon, p_j} = \{0, p\} : 0 < u < \varepsilon, |p - p_j| < \varepsilon\}$. Clearly, if $|\tau - \tau_j| \le \delta$ and $(u, p) \in \partial\Omega_\varepsilon$, then $\Delta_{N_*, \tau, p}(u + 2\pi/p) = 0$ if and only if $\tau = \tau_k, u = 0, p = p_j$. Thus the assumptions (A_4) in Wu [9] holds. Let

$$H^\pm(N_*, \tau_k, 2\pi/\omega_0)(u, p) = \Delta_{(N_*, \tau_k \pm \delta, p)}(u + i2\pi/p),$$

then we can calculate the crossing number as follows

$$\gamma(N_*, \tau_k, 2\pi/\omega_0) = \deg_B(H^-(N_*, \tau_k, 2\pi/\omega_0), \Omega_{\varepsilon, p_j})$$

$$- \deg_B(H^+(N_*, \tau_k, 2\pi/\omega_0), \Omega_{\varepsilon, p_j}) = -1.$$

By Lemma 3.1, the projection of $C(N_*, \tau_k, 2\pi/\omega_0)$ onto the x-space is bounded. When $k > 0$, we have $0 < 2\pi/\omega_0 < \tau_k$. Thus the projection of $C(N_*, \tau_k, 2\pi/\omega_0)$ onto the p-space is bounded. The projection of $C(N_*, \tau_k, 2\pi/\omega_0)$ onto the τ-space must be positive and has no upperbound. As a result, system (7) still has nontrivial periodic solutions when $\tau > \tau_k$.

IV. NUMERICAL EXAMPLES

In this section, we use the formulae obtained in Section 2 and Section 3 to verify the existence of local and global Hopf bifurcation. We consider the following special case of system (7)

$$\frac{dx(t)}{dt} = -0.4x(t) + \frac{0.2x(t)}{0.2 + x^3(t - \tau)}. \quad (15)$$

It is easy to see that the conditions $q > pr$ and $p > q$ hold, then system (15) has a unique positive equilibrium $x^* \approx 0.6694$. By direct computation by means of Matlab 7.0, we get $\omega_0 \approx 0.7272, \tau_0 \approx 2.16$. Thus the positive equilibrium x^* is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figs.1-2). When τ passes through the critical value $\tau_0 \approx 2.16$, the positive equilibrium x^* loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium x^* which are depicted in Figs.3-4. When τ is sufficiently large, periodic solutions still exist as shown in Figs.5-6.

Fig.1

Fig.2
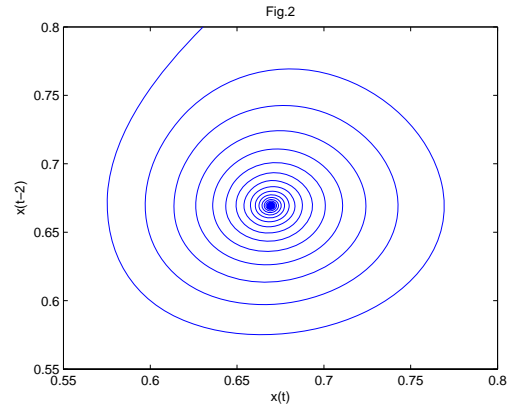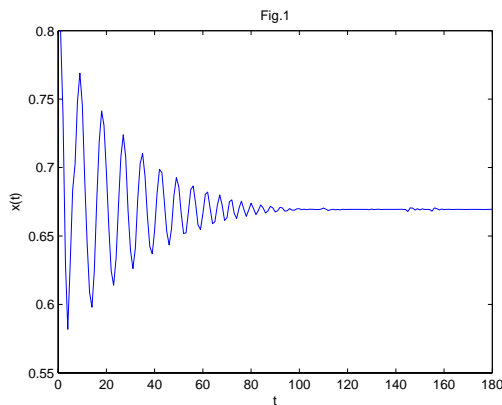
Figs.1-2 Dynamic behavior of system (15): times series of x. A Matlab simulation of the asymptotically stable positive equilibrium $x^* \approx 0.6694$ to system (15) with $\tau = 2 < \tau_0 \approx 2.16$. The initial value is 0.8.

Fig.3

Fig.4
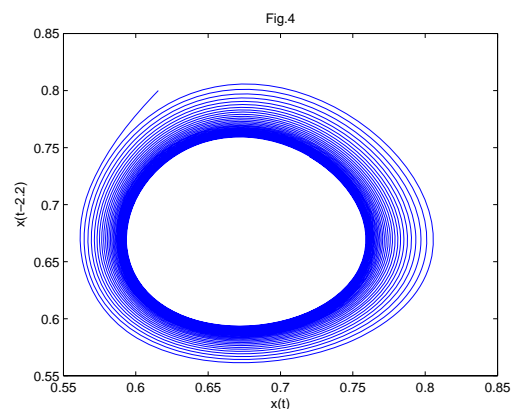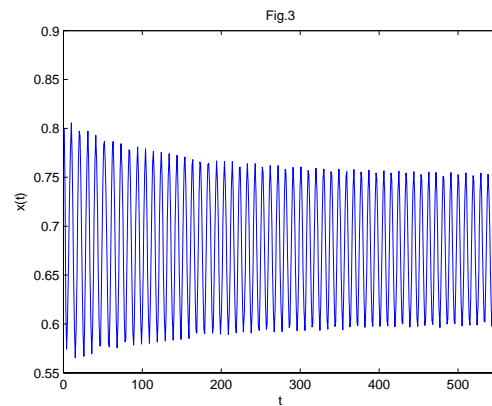
Figs.3-4 Dynamic behavior of system (15): times series of x. A Matlab simulation of a Hopf bifurcation from the positive equilibrium $x^* \approx 0.6694$ to system (15) with $\tau = 2.2 > \tau_0 \approx 2.16$. The initial value is 0.8.

Fig.5

Fig.6
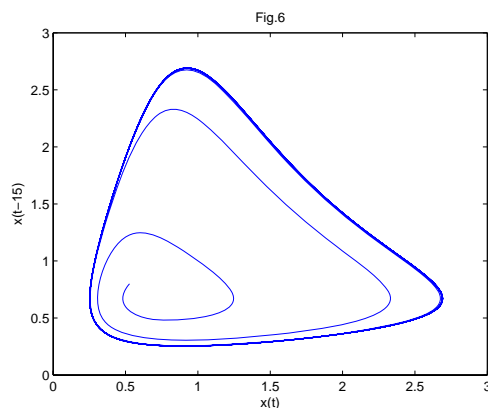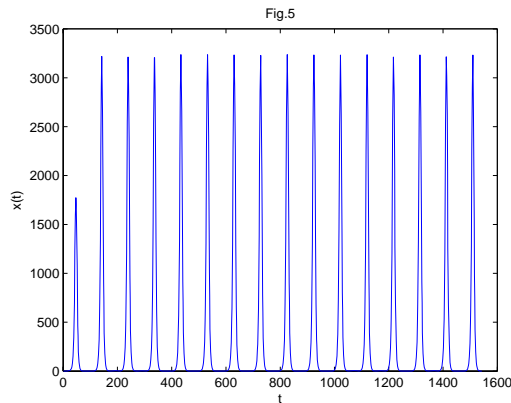
Figs.5-6 Dynamic behavior of system (15): times series of x. A Matlab simulation of a Hopf bifurcation from the positive equilibrium $x^* \approx 0.6694$ to system (15) with $\tau = 15 > \tau_0 \approx 2.16$. The initial value is 0.8.

V. CONCLUSIONS

In this paper, we have investigated the dynamical behaviors of a nonlinear delay population model. It is shown that under a certain condition, there exists a critical value τ_0 of the delay τ for the stability of the population system. If $\tau \in [0, \tau_0)$, the positive equilibrium of the population system is asymptotically stable which means that the size of the population will keep in a steady state. When the delay τ passes through some critical values $\tau = \tau_k, k = 0, 1, 2, \cdots$, the positive equilibrium of the population system loses its stability and a Hopf bifurcation will occur. Moreover, the existence of global Hopf bifurcation are established.

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