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Performance Comparison and Analysis of Different Schemes and Limiters

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Abstract**—** Eight difference schemes and five limiters are applied to numerical computation of Riemann problem. The resolution of discontinuities of each scheme produced is compared. Numerical dissipation and its estimation are discussed. The result shows that the numerical dissipation of each scheme is vital to improve scheme's accuracy and stability. MUSCL methodology is an effective approach to increase computational efficiency and resolution. Limiter should be selected appropriately by balancing compressive and diffusive performance.

Keywords**—**Scheme; Limiter; Numerical simulation; Riemann problem

I. Introduction

AS an new discipline, the Computational Fluid Dynamics formed from sixties of 20th century . Since 80s, because of the great progress of numerical methods and unprecedented development of computer technology, CFD has been one of the important tools in Fluid Dynamics research, just as wind tunnel and theory analysis. Furthermore, CFD is widely applied in oceanography, meteorology, aerospace, automotive, energy sources and so on.

Difference schemes are the nucleus and the most active factor in CFD. All through the development of CFD, every progress in difference schemes always make great contribution to it.

In this paper, we chose several classical difference schemes in the development history of CFD, such as MacCormack scheme, Jameson scheme, FVS-family schemes, FDS-family schemes, AUSM-family schemes and so on; moreover, applied some limiters to one-dimensional shock-tube problem; at last, we carried on numerical simulation, and compared the performance of difference schemes and limiters mentioned.

II. Riemann Problem and Computational Model

A. Riemann problem

The physical analogue of the Riemann problem is the shock-tube problem. Shock-tube problems have played, over a period of more than 100 years, a fundamental role in fluid dynamics research. The structure of the solution of the Riemann problem is described as a set of elementary waves such as shock waves, contacts, rarefactions. The Riemann problem is always the fundamental and crucial problem of CFD, The exact Riemann problem solution is an invaluable reference solution that is useful in assessing the performance of numerical methods and to check the correctness of programs in the early stages of development. In respect that, most of the difference schemes based on the approximate Riemann solver, from which developed to multi-dimensional [1] .

B. One-dimensional time-dependent Euler equations

Here we study the classical Riemann problem for the one-dimensional time-dependent Euler equations; it's an Initial Value Problem (IVP), for the non-linear hyperbolic conservation laws

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}}{\partial x}=0, \\ &\boldsymbol{U}=\begin{Bmatrix} \rho \\ \rho u \\ \rho E \end{Bmatrix}, \boldsymbol{F}=\begin{Bmatrix} \rho u \\ \rho u^2+p \\ (\rho E+p)u \end{Bmatrix} \end{aligned}\right\} \tag{1}$$

From the assumption of hyperbolicity the Jacobian matrix

$$A(U)=\frac{\partial F}{\partial U} \tag{2}$$

May be expressed as

$$\boldsymbol{A}(\boldsymbol{U})=\boldsymbol{R}(\boldsymbol{U})\cdot\boldsymbol{\Lambda}(\boldsymbol{U})\cdot\boldsymbol{L}(\boldsymbol{U}) \tag{3}$$

Where $\boldsymbol{\Lambda}$ is the diagonal matrix formed by the eigenvalues of A, namely

$$\boldsymbol{\Lambda}=\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}=\begin{bmatrix} u & & \\ & u-c & \\ & & u+c \end{bmatrix} \tag{4}$$

The $\boldsymbol{R}(\boldsymbol{U}),\boldsymbol{L}(\boldsymbol{U})$ is the right eigenvector and left eigenvector of A, respectively [1]; the A can be expressed as

$$\boldsymbol{A}=\begin{pmatrix} 0 & 1 & 0 \\ (\gamma-3)\dfrac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3-\gamma E u & -3(\gamma-1)\dfrac{u^2}{2}+\gamma E & \gamma u \end{pmatrix} \tag{5}$$

Initial conditions (IC)

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$$\mathrm{U}(x,0)=\mathrm{U}^{(0)}(x)=\begin{cases}U_L, x\in[-1,0)\\ U_R, x\in[0,1]\end{cases} \tag{6}$$

$$\begin{aligned}&U_L:(\rho=1.0, u=0.0, p=1.0)\\ &U_R:(\rho=0.125, u=0.0, p=0.1)\end{aligned} \tag{7}$$

This text we will get the results at t=0.4s.

C. Structure of the solution

The wave patterns under given IC above is shown in Fig.1. When t=0, two stationary gases in a tube are separated by a diaphragm. On the left of the diaphragm is higher pressure p_L; on the right is lower pressure p_R. The rupture of the diaphragm generates a time-dependent right shock wave, a left rarefaction and a contact discontinuity. At the time of t (t>0), the gas in the tube was divided into four domains, A and D are the regions which are not disturbed by waves remain the initial values; B is the region the left rarefaction passed, it's pressure denoted p_B; C is the region after the right shock, it's pressure denoted p_C, moreover, $p_B = p_C$; but the density and temperature of the two regions are different, so B and C was divided by a contact.

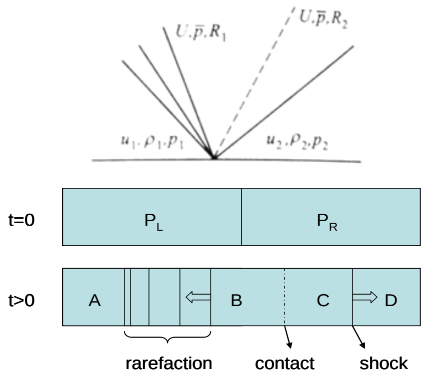

Fig. 1 Wave pattern in the solution of the given Riemann problem and the corresponding distribution in the shock-tube.

III. Difference Schemes and Limiters

The construction and choice of difference schemes is the key to computation, when utilizing numerical methods to solve the Euler equations above. Here we will follow the clue of the development of difference schemes, introduce six classical difference schemes in different period, and then take the computation case of one-dimensional shock-tube problem for example to carry on numerical simulation.

A. Difference schemes

At first, we will give a brief introduction to the difference schemes this paper utilized; limited to the section, we haven't list the specific expressions, for details the reader can see the related reference.

(1) MacCormack scheme

In 1969, MacCormack scheme a evolution from Lax-wendroff scheme was presented. It is a Centred scheme with second-order accuracy and two steps. It was the protagonist of the computation of two-dimensional steady flow problem and widely applied from 60s to 70s in 20th century [2].

(2) Jameson scheme

Jameson, Schmidt, and Turkel presented the famous JST centered finite volume scheme in 80s of the 20th century [3]. It employed Centred difference and constructed second-order, fourth-order artificial viscous term in order to restrain fluctuate. It is based on the gradient of the pressure to confirm the weight of second-order term and fourth-order term. In the case of pressure gradient is great (just as shock region), the second-order term played a main role on restrain the fluctuate of numerical solution, which may be appear in the vicinity of the high gradients; on the other hand, when pressure gradient is little (like the smooth region), it introduced fourth-order dissipation term to restrain the fluctuate caused by odd-even decoupling, therefore improved the stability of centered scheme.

Jameson scheme is widely applied in the computation of subsonic, transonic, supersonic flow and flow through turbine; particularly, this scheme got great success in aerofoil computation and transonic detour flow.

(3) FVS (Flux Vector Splitting) scheme

The former two schemes are belong to Centred scheme, which needs add extra artificial viscous term to restrain fluctuate. So it requires the experienced user, and the addition of the artificial viscous term is easy to pollute the physical solution, influence the accuracy. Analyzed from the essence of its construction, Centred schemes have not consider the direction of propagation of information. Therefore, there is a nature conflict between the unidirectional propagation of the waves and the nondirectional Centred scheme [4].

Since 80s of the 20th century, the upwind schemes have got remarkable development. Different from the Centred scheme, upwind scheme considered the propagation direction of the waves. So it has the natural predominance to express the characteristic of flow. Up to today, kinds of upwind schemes have been the mainstream of CFD.

Flux Vector Splitting (FVS) scheme is one of the upwind schemes; it is convenient, efficient and has a strong ability in capturing shock wave. The flux vector splitting based on the direction of propagation of correlative information; then, the splitting components is required to perform upwind difference i.e. applied backward difference when the direction is positive and forward difference when the direction is reverse. There are various FVS schemes because of the difference of splitting approaches. The Van leer [5] and Steger-Warming scheme [6] are highly recommended. The former scheme does splitting based on local Mach, while the latter is on the sigh of eigenvalues of the flux. See for details the paper [7].

(4) FDS (Flux Difference Splitting) scheme

Corresponding to FVS scheme, the other essential approach for identifying upwind directions is Flux Difference Splitting Method, which derived from Riemann approach, also refer to Godunov approach. The representative scheme of FDS is Roe scheme [8], which has high resolution of shock wave and contact discontinuity. However, when the eigenvalues of Jacobin matrix of flux is little, it will against the entropy condition and generate unphysical solution, therefore must introduce entropy amend.

In 1983, Harten [9] presented the Total Variable Diminishing (TVD) scheme by analyzing the reason why traditional difference schemes always produce spurious oscillations in the vicinity of shock waves. At the same time, he constructed a second-order TVD scheme which has a high resolution and excellent in capture shock wave with no oscillations. Therefore, it is popular as soon as been proposed; later, the approach of ENO, WENO and so on emerged.

Note that, the Harten TVD scheme essentially is Roe scheme added TVD limiter, so made the scheme possessed TVD properties. In this paper, we utilize the Harten-Yee scheme [4].

(5) NND (Non Oscillatory Non Free Parameter Dissipation Scheme) scheme.

In China, academician Zhang Han-xin [10] constructed the Non Oscillatory Non Free Parameter Dissipation (NND) Scheme by analyzed the reason of oscillations of numerical solution in the vicinity of the shock and made its third-order dissipation term confined to certain relationship by applied different difference schemes at fore-and-aft of the shock wave. The research indicates that the NND scheme has the TVD properties also belong to TVD schemes family.

(6) AUSM (Advection Upwind Splitting Method) scheme

At present, one of the developing trends of upwind schemes is to combine the advantages to construct schemes with higher quality; namely, the Hybrid Upwind Splitting (HUS) scheme. The most famous HUS scheme is the AUSM scheme proposed by Liou and Steffen [11]. In theory, the AUSM scheme distinguishes the flow field into the linear field related with characteristic velocity u and non-linear field related with characteristic velocity $u \pm a$, moreover, splitting the pressure term from the advection flux, respectively. As far as the construction method concerned, AUSM scheme is the improvement of Van Leer scheme; likewise it is the combination of FVS and FDS analyzed from the dissipation term [4].

The characteristics of AUSM scheme are excellent. Such as: easy to construction, no matrices operation, high shock resolution, stable, possess the accuracy of FDS scheme in the boundary and the robust of FVS scheme when captured the strong discontinuities at the same time. After ten years developing, it has been applied to various computation models from slow flow to supersonic flow, formed a series of AUSM schemes.

There are two embranchments after the AUSM scheme proposed. One is leaded by Liou to modify the Mach splitting function and pressure splitting function, developed the AUSMDV, AUSM+ schemes, furthermore, introduce the pressure dissipation, deduced to AUSM+up [12]–[13] scheme ; the other one is leaded by Kim to introduce the pressure weight function amendment technology, developed AUSMPW, AUSMPW+[14], M-AUSMPW+, MLP (Multi-dimensional Limiting Process) scheme [15].

B. Limiters

Nowadays, the computation procedure of CFD is commonly consist of two steps: the first step is called data reconstruction, which is to reconstruction the value at computing cell interface by utilizing the value of successive computing cells; the second step is to utilize the value obtained by the data reconstruction, apply kinds of difference schemes no matter upwind schemes or centred schemes to construct the flux to numerically solve the equation, it is called the resolve step.

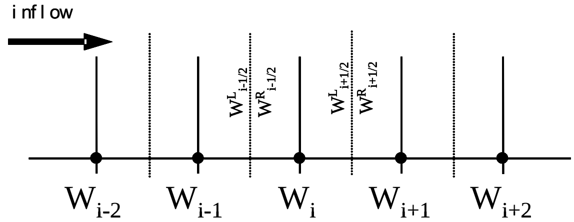

Fig. 2 Reconstruction of data in computing cells

The MUSCL (Monotone Upstream-Centred Scheme for Conservation Laws) approach is routinely used in practice to increase the high-order accuracy. The reconstruction is constrained appropriately so as to avoid spurious oscillations. Generally speaking, the reconstruction step includes two parts: interpolation and limit. Interpolation is to obtain the value at the interface of computing cells by utilizing the value of successive computing cells; limit is to avoid spurious oscillations by constraining the gradient of the value. It is introduced in the form of limiters. Here we chose 5 limiters as follows [15].

SUPERBEE is given by

$$\varphi(r) = \max[\min(2r,1), \min(r,2)] \tag{8}$$

Double MINMOD is given by

$$\varphi(r) = \begin{cases} \min[2r, 2, (1+r)/2] & r > 0 \\ 0 & r \le 0 \end{cases} \tag{9}$$

VANLEER is given by

$$\varphi(r) = \frac{r + |r|}{1+r} \tag{10}$$

VANALBADA is given by

$$\varphi(r) = \frac{r^2 + r}{1 + r^2} \tag{11}$$

MINMOD is given by

$$\varphi(r) = \min\operatorname{mod}(r,1) \tag{12}$$

The Fig.3 illustrates five of these flux limiters constructed from TVD region; the upper borderline of the shadow is the SUPERBEE, the lower borderline is MINMOD, and the rest is Double MINMOD, VANLEER, and VANALBADA from the top down.

Limiter function $\Phi(r)$ must be included in the shadow region, so as to make the difference schemes satisfy the TVD condition and achieve second-order accuracy. The closer to the

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top of the shadow, the less of the dissipation caused by the limiter; its resolution is higher, however, its stability, convergence getting worse at one time; on the contrary, when close to the bottom of the shadow, the dissipation caused by limiter increased, its resolution reduced, the stability and convergence of the limiter got enhanced.

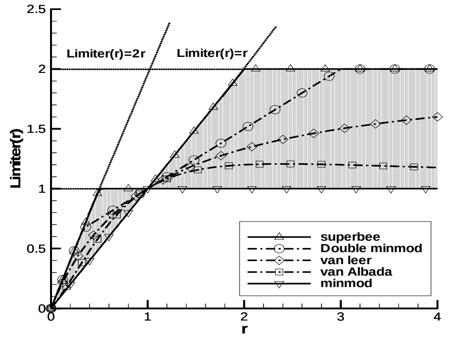


Fig.3 Comparison of five flux limiter functions

IV. NUMERICAL RESULTS AND DISCUSSION

A. The comparison and analysis of the numerical results of different difference schemes

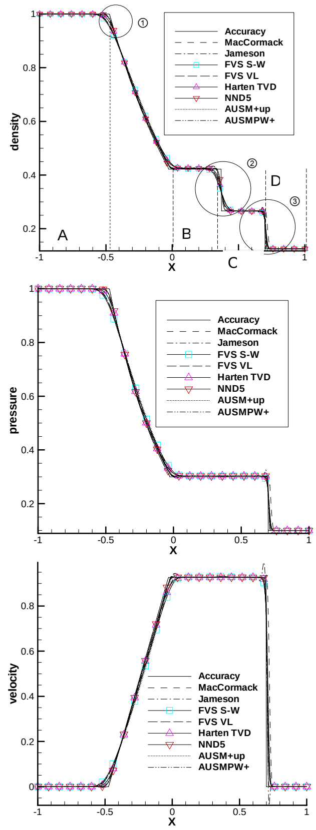


Fig.4 Comparison the numerical (symbol) solution of 8 schemes with the exact (line) solution

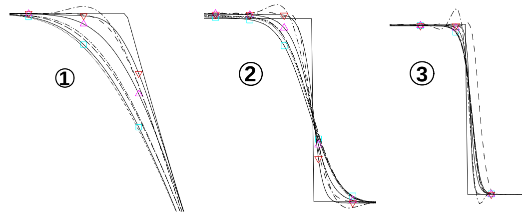


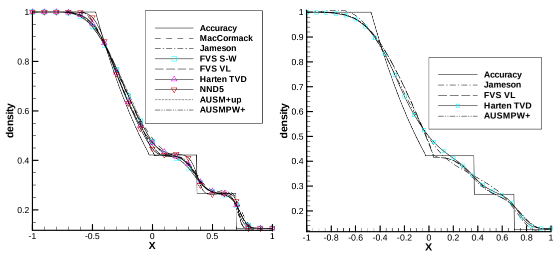
Fig.5 Microscope of 3 concerned regions of density

The given text is a modified version of the popular Sod's test; the solution consists of a right shock wave, a right traveling contact wave and a left sonic rarefaction wave; this test is very useful in assessing the entropy satisfaction property of numerical methods. Fig.4 show comparisons between exact solutions and numerical solutions at output time (t=0.4) obtained by these 8 schemes listed; the quantities shown are density, pressure, and particle velocity. The spatial domain is the interval [-1, 1] which is discretized with M=500 computing cells; boundary conditions are transmissive. We can find that all results obtained by applied these 8 schemes to the text are approximate to the exact solution. In A, B, C, D four regions, the numerical results are virtually identical with the exact solution. In the smooth regions, the results of Centred schemes almost accord with the results of the upwind schemes; but the differences emerged in the vicinity of high gradients. We can see from the Fig.5 that, ① represent the region near the left rarefaction wave, ②represent the region near the right contact wave, ③ represent the region near the shock wave, the results of two Centred schemes (MacCormack scheme and Jameson scheme) emerged spurious oscillations to a certain extent, especially in Fig.5.③, the oscillations are distinct; Otherwise, the upwind schemes are stable, but the curves are smoothed because of the implicit dissipation of upwind schemes.

Because of there is no dissipation term in the discrete equations of Centred schemes, so it needs to add extra artificial viscous term to restrain fluctuating near the high gradients. The artificial viscous term influenced the accuracy of solution, as a matter of fact, as to the computed case of this paper; the centred schemes results could achieve the same effects as the upwind schemes by adjusting the coefficients of the artificial viscous term to constrain the oscillations. Analyzed from the true physical situation, Centred schemes fit for the subsonic flow, conflicted with the physical characteristics of the propagation of information in both the supersonic and hypersonic flow.

Upwind schemes considered the direction of propagation of the information. When applied to discrete equations, it has the inherent implicit dissipation term differs from the artificial viscous term; accordingly, there are no free coefficients need to be adjusted. The governing equations of CFD referred to are mainly hyperbolic, the characteristic curves exist and the information or disturbances will propagation along the characteristic curves at characteristic speed. Upwind schemes are just designed to meet the direction of propagation to difference, that's why it has the natural predominance to express the physical characteristics of flow.Neither the ways of construction nor the implicit dissipation terms are same between different upwind schemes; so the performance of constraining oscillation is various. Just as the centred schemes the magnitude of the dissipation term will leads to exaggerated fluctuate or smooth influenced accuracy. In fact, difference schemes commonly contain numerical dissipation term, no matter the centred or upwind schemes; it's the byproduct of the numerical computation. Based on the Lax entropy condition, we concluded that suitable numerical dissipation is an insurance to got the true physical solution. Due to the diversity of the numerical dissipations, there are great deference in accuracy and reliability between schemes [4].

It is worth to pay attention to the AUSM-family schemes, which are the combination of FVS and FDS; they possess the high resolution of FDS and the excellent computation efficiency of FVS. A satisfactory feature of the numerical shock wave of AUSM scheme in Fig.4 is that it is monotone, there are no spurious oscillations in the vicinity of the shock, it has the identical excellent performance as TVD and NND schemes.



(a) 100 cells (b) 50 cells

Fig.6 The results in different numbers of computing cells

B. The comparison of different numbers of computing cells

Here we decrease the amount of computing cells discretized the spatial domain, did the same text case again with 100 cells and 50 cells respectively, Fig.6 shows the results. Compared the Fig.4 with Fig. 6 we can conclude that along with the reduction of the computing cells number, the computational accuracy of the results of all 8 schemes is fall. This phenomenon is caused by the diminishment of the numbers of points which described the information of the flow field. At the same time, the distinction of results between different schemes is more obvious. From Fig.6 (b) we can clearly see that the dissipation of Jameson centred scheme is the most severe, almost can not distinguish the four regions of the flow field. The next is Harten-Yee TVD, Van Leer and AUSM schemes have stable sensitivity of the mesh, exhibited a better performance.

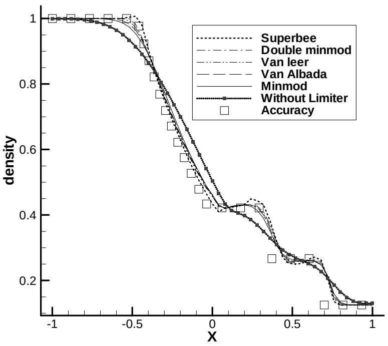


Fig.7 The result of interpolation of MUSCL on 50 cells

C. The comparison of limiters

This paper applied the MUSCL interpolation method to different schemes to achieve high-order accuracy. Take the AUSM+up scheme for example, the results of Fig.7 show the application of five limiters and first-order accuracy solution; the number of the computing cells is 50. Compared Fig.7 with Fig.6, it is obvious that the accuracy has been improved greatly after applied the limiters.

Limiters are essentially a function of limiting the gradient of solution. It plays an important role on the accuracy, stability and convergence of the computation. As shown in Fig.7, as to the SUPERBEE limiter, whose compressibility stronger than any other else emerged obvious fluctuation in the vicinity of discontinuities, because of the insufficiency of the dissipation. Therefore, to estimate a limiter in practical application need to combine with the difference schemes, balance the contradiction between the resolution and dissipation, find the limiters with high resolution, stability, and satisfactory accuracy.

V. CONCLUSION

Various difference schemes and limiters are applied in numerical computation of one-dimensional shock-tube problem; Resolution of discontinuities of each scheme is compared, and discussed the numerical dissipation of difference schemes. We compared the magnitude of numerical dissipation of centred schemes with that of upwind schemes by computation; and conclude that the numerical dissipation of each scheme is the key to improve the scheme's accuracy and assess its quality. It is effective to increase the computational efficiency and resolution of difference schemes by Using MUSCL method. Limiter should be selected appropriately by balancing compressive and diffusive performance.

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