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OWA Operators in Generalized Distances

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***Abstract*—**Different types of aggregation operators such as the ordered weighted quasi-arithmetic mean (Quasi-OWA) operator and the normalized Hamming distance are studied. We introduce the use of the OWA operator in generalized distances such as the quasi-arithmetic distance. We will call these new distance aggregation the ordered weighted quasi-arithmetic distance (Quasi-OWAD) operator. We develop a general overview of this type of generalization and study some of their main properties such as the distinction between descending and ascending orders. We also consider different families of Quasi-OWAD operators such as the Minkowski ordered weighted averaging distance (MOWAD) operator, the ordered weighted averaging distance (OWAD) operator, the Euclidean ordered weighted averaging distance (EOWAD) operator, the normalized quasi-arithmetic distance, etc.

***Keywords*—**Aggregation operators, Distance measures, Quasi-OWA operator.

I. Introduction

THE quasi-arithmetic distances are very useful techniques that generalize a wide range of distance measures such as the Hamming distance, the Euclidean distance and the Minkowski distance. These particular cases of the quasi-arithmetic distance are very useful techniques that have been used in a lot of applications such as fuzzy set theory, multicriteria decision making, business decisions, etc.

Often, when calculating distances, we want an average result of all the individual distances. We call this the normalization process. In the literature, we find basically, two types of normalized distances. The first type is the case when we normalize the distance giving the same importance to all the individual distances. This case is known for the quasi-arithmetic distance, the normalized quasi-arithmetic distance. The second type is the case when we normalize the distance giving different degrees of importance to the individual distances. Then, we get the weighted quasi-arithmetic distance.

Sometimes, when calculating the normalized distance, it would be interesting to consider the attitudinal character of the decision maker in order to modify the results of the aggregation with optimistic or pessimistic attitudes. A very useful technique for the aggregation of the information considering the attitudinal character of the decision maker is the ordered weighted averaging (OWA) operator introduced by Yager in [1]. The OWA operator provides a parameterized family of aggregation operators that include, among others, the maximum, the minimum and the average criteria. Since its appearance, it has been used in a wide range of applications such as [2]–[25].

In this paper, we suggest a new type of distance measure consisting in normalize the quasi-arithmetic distance with the OWA operator. Then, the normalization developed will be able to modify the results of the aggregation by using different degrees of pessimism or optimism and it will provide a parameterized family of distance operators that include the maximum distance, the minimum distance and the average distance. Note that from a mathematical perspective, the attitudinal character of the decision maker in the aggregation is seen as the orness or the andness of the aggregation [1]. We will call this generalization as the ordered weighted quasi-arithmetic distance operator or the Quasi-OWAD operator, for short. We will also study a wide range of particular cases of Quasi-OWAD operators such as the Minkowski ordered weighted averaging distance (MOWAD) operator, the Hamming ordered weighted averaging distance (HOWAD) operator, the Euclidean ordered weighted averaging distance (EOWAD) operator, the ordered weighted geometric averaging distance (OWGAD) operator and the ordered weighted harmonic averaging distance (OWHAD) operator. We should note that some considerations about using OWA operators in distance measures have been studied in [21].

In order to do so, the remainder of the paper is organized as follows. In Section II, we briefly describe some basic aggregation operators such as the Hamming distance and the Quasi-OWA operator. Section III, develops the Quasi-OWAD operator. In Section IV, we study different families of Quasi-OWAD operators. Finally, in Section V, we summarize the main conclusions of the paper.

II. Preliminaries

A. Normalized Hamming Distance

The normalized Hamming distance is a distance measure used for calculating the differences between two elements, two sets, etc. In fuzzy set theory, it is very useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. For two sets *A* and *B*, it can be defined as follows.

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Definition 1. A normalized Hamming distance of dimension n is a mapping d_H: $R^n \times R^n \rightarrow R$ such that:

$$d_H(A,B) = \frac{1}{n}\sum_{i=1}^{n}|a_i - b_i| \tag{1}$$

where a_i and b_i are the ith arguments of the sets A and B respectively.

Sometimes, when normalizing the Hamming distance it is better to give different weights to each individual distance. Then, the distance is known as the weighted Hamming distance. It can be defined as follows.

Definition 2. A weighted Hamming distance of dimension n is a mapping d_{WH}: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$. Then:

$$d_{WH}(A,B) = \sum_{i=1}^{n} w_i|a_i - b_i| \tag{2}$$

where a_i and b_i are the ith arguments of the sets A and B respectively. Note that Definitions 1 and 2 are the general expressions. For the formulation used in fuzzy set theory see for example [27]–[29].

B. Quasi-OWA operator

The Quasi-OWA operator [5] is a generalization of the OWA operator by using quasi-arithmetic means. The quasi-arithmetic mean was introduced in [30]–[32] and it represents a generalization to a wide range of mean aggregations such as the generalized mean, the arithmetic mean, the geometric mean, the harmonic mean or the quadratic mean. It can be defined as follows.

Definition 3. A quasi-arithmetic mean of dimension n is a mapping QM: $R^n \rightarrow R$ such that:

$$QM(a_1, a_2,\ldots, a_n) = g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g(a_i)\right) \tag{3}$$

where a_i is the argument variable and g is a continuous strictly monotonic function. Note that depending on the form of g, we obtain different types of means. When $g(a_i) = a_i$, we obtain the arithmetic mean. When $g(a_i) = a_i^2$, the quadratic mean. When $g(a_i) = a_i^{-1}$, the harmonic mean. When $g(a_i) = a_i^0$, the geometric mean. More generally, when $g(a_i) = a_i^{\lambda}$, we get the generalized mean.

Note that if the arguments have different weights, then, the quasi-arithmetic mean is transformed in the weighted quasi-arithmetic mean. With this information, we can define the Quasi-OWA operator as follows.

Definition 4. A Quasi-OWA operator of dimension n is a mapping $QOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$QOWA(a_1, a_2,\ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_{(j)})\right) \tag{4}$$

where b_j is the jth largest of the a_i, and g is a continuous strictly monotonic function.

From a generalized perspective of the reordering step, we can distinguish between the descending Quasi-OWA (Quasi-DOWA) operator and the ascending Quasi-OWA (Quasi-AOWA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the Quasi-DOWA and w^*_{n-j+1} the jth weight of the Quasi-AOWA operator.

It can be demonstrated that the Quasi-OWA operator generalizes a wide range of aggregation operators [3], [5] such as the maximum, the minimum, the generalized OWA operator [2], [17], the arithmetic mean, the geometric mean, the quadratic mean, the harmonic mean, the weighted average, the weighted geometric mean, the OWA operator [1], the ordered weighted quadratic averaging (OWQA) operator [17], the ordered weighted harmonic averaging (OWHA) operator [17], etc.

III. The Quasi-Ordered Weighted Averaging Distance Operator

The Quasi-OWAD operator represents an extension of the traditional normalized quasi-arithmetic distance [33] by using OWA operators. The difference is that the reordering of the arguments is developed according to the values of the individual distances. Then, it is possible to calculate the distance between two elements, two sets, two fuzzy sets, etc., modifying the results according to the attitudinal character of the decision maker. For example, this type of distance is very useful when a decision maker wants to compare two fuzzy subsets but he wants to give more importance to the highest individual distance because he believes that it will be more significant in the analysis. Note that this type of normalized distance operator can be constructed by mixing the quasi-arithmetic distance with OWA operators, by mixing the Hamming distance with quasi-OWA operators or by mixing the Hamming OWAD operator with quasi-arithmetic means. It can be defined as follows.

Definition 5. A Quasi-OWAD operator of dimension n is a mapping $QOWAD$: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$. Then, the distance between two sets A and B is:

$$QOWAD(d_1, d_2,\ldots, d_n) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\left(D_{(j)}\right)\right) \quad (5)$$

where $D_{(j)}$ is the jth largest of the d_i and d_i is the individual distance between A and B. That is, $d_i = |a_i - b_i|$. Note that g is a continuous strictly monotonic function. As we can see, we adapt the characteristics of the quasi-arithmetic mean to the characteristics of the OWAD operator.

A fundamental aspect of the Quasi-OWAD operator is the reordering of the arguments based upon their values. That is, the weights rather than being associated with a specific argument, as in the case with the usual quasi-arithmetic mean, are associated with a particular position in the ordering. This reordering introduces nonlinearity into an otherwise linear process. Note that the Quasi-OWAD operator follows a similar methodology than the Quasi-OWA operator [3], [5], [8].

If D is a vector corresponding to the ordered arguments $g(D_{(j)})$, we shall call this the ordered argument vector, and W^T is the transpose of the weighting vector, then the Quasi-OWAD aggregation can be expressed as:

$$QOWAD(d_1, d_2,\ldots, d_n) = g^{-1}\left(W^T D\right) \quad (6)$$

From a generalized perspective of the reordering step, we have to distinguish between the descending Quasi-OWAD (Quasi-DOWAD) and the ascending Quasi-OWAD (Quasi-AOWAD) operators. The weights of the Quasi-DOWAD are related to those of the Quasi-AOWAD by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the Quasi-DOWAD and w^*_{n-j+1} the jth weight of the Quasi-AOWAD operator.

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the Quasi-OWAD operator can be expressed as:

$$QOWAD(d_1, d_2\ldots, d_n) = g^{-1}\left(\frac{1}{W}\sum_{j=1}^{n} w_j\, g\left(D_{(j)}\right)\right) \quad (7)$$

The Quasi-OWAD operator is a mean or averaging operator. This is a reflection of the fact that the operator is monotonic, bounded, commutative, and idempotent.

Theorem 1 (Commutativity). *Assume f is the Quasi-OWAD operator, then*:

$$f(d_1, d_2\ldots, d_n) = f(e_1, e_2\ldots, e_n) \quad (8)$$

where $(d_1, d_2\ldots, d_n)$ *is any permutation of the arguments* $(e_1, e_2\ldots, e_n)$.

Proof. Let

$$f(d_1, d_2\ldots, d_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \quad (9)$$

$$f(e_1, e_2\ldots, e_n) = \left(\sum_{j=1}^{n} w_j d_j^{\lambda}\right)^{1/\lambda} \quad (10)$$

Since $(d_1, d_2\ldots, d_n)$ is a permutation of $(e_1, e_2\ldots, e_n)$, we have $d_j = e_j$, for all j, and then

$$f(d_1, d_2\ldots, d_n) = f(e_1, e_2\ldots, e_n) \quad \blacksquare$$

Theorem 2 (Monotonicity). *Assume f is the Quasi-OWAD operator, if* $d_i \geq e_i$, *for all i, then*:

$$f(d_1, d_2\ldots, d_n) \geq f(e_1, e_2\ldots, e_n) \quad (11)$$

Proof. Let

$$f(d_1, d_2\ldots, d_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \quad (12)$$

$$f(e_1, e_2\ldots, e_n) = \left(\sum_{j=1}^{n} w_j c_j^{\lambda}\right)^{1/\lambda} \quad (13)$$

Since $d_i \geq e_i$, for all d_i, it follows that, $d_i \geq e_i$, and then

$$f(d_1, d_2\ldots, d_n) \geq f(e_1, e_2\ldots, e_n) \quad \blacksquare$$

Theorem 3 (Bounded). *Assume f is the Quasi-OWAD operator, then*:

$$\text{Min}\{d_i\} \leq f(d_1, d_2\ldots, d_n) \leq \text{Max}\{d_i\} \quad (14)$$

Proof. Let $\max\{d_i\} = c$, and $\min\{d_i\} = d$, then

$$f(d_1, d_2\ldots, d_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \leq \left(\sum_{j=1}^{n} w_j c^{\lambda}\right)^{1/\lambda} = \left(c^{\lambda}\sum_{j=1}^{n} w_j\right)^{1/\lambda} \quad (15)$$

$$f(d_1, d_2\ldots, d_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \geq \left(\sum_{j=1}^{n} w_j d^{\lambda}\right)^{1/\lambda} =$$

$$\left(d^{\lambda} \sum_{j=1}^{n} w_j \right)^{1/\lambda} \tag{16}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f\,(d_1, d_2\ldots, d_n) \leq c \tag{17}$$

$$f\,(d_1, d_2\ldots, d_n) \geq d \tag{18}$$

Therefore,

$$\text{Min}\{d_i\} \leq f\,(d_1, d_2\ldots, d_n) \leq \text{Max}\{d_i\} \qquad \blacksquare$$

Theorem 4 (Idempotency). *Assume f is the Quasi-OWAD operator, if $d_i = d$, for all d_i, then*:

$$f\,(d_1, d_2\ldots, d_n) = d \tag{19}$$

Proof. Since $d_i = d$, for all d_i, we have

$$f\,(d_1, d_2\ldots, d_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} =$$

$$\left(\sum_{j=1}^{n} w_j d^{\lambda} \right)^{1/\lambda} = \left(d^{\lambda} \sum_{j=1}^{n} w_j \right)^{1/\lambda} \tag{20}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f\,(d_1, d_2\ldots, d_n) = d \qquad \blacksquare$$

Another interesting issue to analyze is the attitudinal character of the Quasi-OWAD operator. Based on the measure developed for the Quasi-OWA operators in [2], it can be formulated in two different forms depending on the type of ordering used. For the first type we get the following:

$$\alpha(W) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\left(\frac{n-j}{n-1} \right) \right) \tag{21}$$

A further issue to consider is the measure of dispersion of the weights W. It is a measure that provides the type of information being used. Using the same methodology as in [1], it can be defined as follows.

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \tag{22}$$

For example, if $w_j = 1$ for some j, then $H(W) = 0$, and the least amount of information is used. If $w_j = 1/\text{n}$ for all j, then, the amount of information used is maximum.

A third measure that could be studied in the aggregation is the divergence of the weights W. It can be defined as follows.

$$Div(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W) \right)^2 \tag{23}$$

Note that the divergence can also be formulated with an ascending order in a similar way as it has been shown in the attitudinal character.

IV. Families of Quasi-OWAD Operators

A. Analysing the Weighting Vector W

By using a different manifestation of the weighting vector in the Quasi-OWAD operator, we are able to obtain different types of distance aggregation operators. For example, we can obtain the maximum distance, the minimum distance, the normalized quasi-arithmetic distance and the weighted quasi-arithmetic distance.

Remark 1: The maximum distance is obtained when $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. And the minimum distance is found when $w_n = 1$ and $w_j = 0$, for all $j \neq n$. As we can see, the maximum and the minimum distances are obtained independently of the value of g.

Remark 2: It should also be noted that the median can also be used as Quasi-OWAD operator. We will call it the Quasi-OWAD median and it is possible to distinguish between two situations. If n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the $[(n + 1)/2]$th largest argument d_i. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest d_i.

Remark 3: More generally than the maximum, the minimum and the median, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any g, $QOWAD(d_1, d_2,\ldots, d_n) = D_k$, where D_k is the kth largest or lowest of the arguments d_i. This type of Quasi-OWAD operator is known as the step-Quasi-OWAD operator. Note that if $k = 1$, the step-Quasi-OWAD is transformed in the maximum and if $k = n$, the step-Quasi-OWAD becomes the minimum.

Remark 4: For the weighted-Quasi-OWAD median, we select the argument that has the kth largest d_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Remark 5: The normalized quasi-arithmetic distance and the weighted quasi-arithmetic distance are also particular cases of the Quasi-OWAD operator. The normalized quasi-arithmetic distance is obtained when $w_j = 1/n$, for all j. The weighted quasi-arithmetic distance is obtained when $j = i$, for all i and j, where j is the jth argument of D_j and i is the ith argument of d_i.

Remark 6: Other families of aggregation operators could be used in the weighting vector. For example, the Hurwicz Quasi-OWAD criteria is obtained when $w_1 = \alpha$, $w_n = 1 - \alpha$, $w_j = 0$, for all $j \neq 1,n$. Note that if $\alpha = 1$, the Hurwicz Quasi-OWAD criteria becomes the maximum distance and if $\alpha = 0$, it becomes the minimum distance.

Remark 7: When $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-Quasi-OWAD operator based on the window-OWA operator [13]. Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, then, the window-Quasi-OWAD is transformed in the maximum distance. If $m = 1$, $k = n$, then, the window-Quasi-OWAD becomes the minimum distance. And if $m = n$ and $k = 1$, the window-Quasi-OWAD is transformed in the normalized quasi-arithmetic distance.

Remark 8: If $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$, we are using the olympic-Quasi-OWAD operator that it is based on the olympic average [16]. Note that if $n = 3$ or $n = 4$, the olympic-Quasi-OWAD operator is transformed in the Quasi-OWAD median and if $m = n - 2$ and $k = 2$, the window-Quasi-OWAD is transformed in the olympic-MOWAD operator.

Remark 9: Another interesting family is the S-Quasi-OWAD operator based on the S-OWA operator [13], [15]. It can be divided in three classes, the "orlike", the "andlike" and the generalized S-Quasi-OWAD operator. The generalized S-Quasi-OWAD operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-Quasi-OWAD operator becomes the "andlike" S-Quasi-OWAD operator and if $\beta = 0$, it becomes the "orlike" S-Quasi-OWAD operator. Also note that if $\alpha + \beta = 1$, the generalized S-Quasi-OWAD operator becomes the Hurwicz quasi-arithmetic distance criteria.

Remark 10: A further useful approach for obtaining the weights is the functional method introduced by Yager [16] for the OWA operator. For the Quasi-OWAD operator, it can be summarized as follows. Let f be a function f: $[0, 1] \rightarrow [0, 1]$ such that $f(0) = f(1)$ and $f(x) \geq f(y)$ for $x > y$. We call this function a basic unit interval monotonic function (BUM). Using this BUM function we obtain the Quasi-OWAD weights w_j for $j = 1$ to n as

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \tag{24}$$

It can easily be shown that using this method, the w_j satisfy that the sum of the weights is 1 and $w_j \in [0,1]$.

Remark 11: A further type of aggregation that could be used is the E-Z Quasi-OWAD weights based on the E-Z OWA weights [18]. In this case, we should distinguish between two classes. In the first class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n, and in the second class we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$. Note that for the first class, the maximum distance is obtained if $k = 1$ and $b_1 = \text{Max}\{a_i\}$, and the normalized quasi-arithmetic distance if $k = n$. In the second class, the minimum distance is obtained if $k = 1$ and $b_n = \text{Min}\{a_i\}$, and the normalized quasi-arithmetic distance if $k = n$.

Remark 12: Another method for obtaining the weights is the one suggested by Filev and Yager in [4]. Following the same methodology we can distinguish between two possibilities for the Quasi-OWAD weights. For the first method, the weights can be expressed as $w_1 = \alpha$, $w_n = w_{n-1}(1 - w_1)/w_1$, and $w_j = w_{j-1}(1 - w_1)$ for $j = 2$ to $n - 1$. And for the second method, the weights are obtained as $w_n = 1 - \alpha$, $w_1 = w_2(1 - w_n)/w_n$, and $w_j = w_j(1 - w_n)$ for $j = 2$ to $n - 1$.

Remark 13: Other families of Quasi-OWAD operators could be obtained such as the weights that depend on the aggregated objects [13]. Note that in the Quasi-OWAD operator, the aggregated objects are individual distances. Then, the weights depend on the distances between the elements of the different sets. For example, we could develop the BADD-Quasi-OWAD operator based on the OWA version developed in [13].

$$w_j = \frac{b_j^\alpha}{\sum_{j=1}^n b_j^\alpha} \tag{25}$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments d_i, that is, the individual distances. Note that the sum of the weights is 1 and $w_j \in [0,1]$. Also note that if $\alpha = 0$, we get the normalized quasi-arithmetic distance and if $\alpha = \infty$, we get the maximum distance.

Remark 14: A second family of Quasi-OWAD operator that depends on the aggregated objects is

$$w_j = \frac{(1/b_j)^\alpha}{\sum_{j=1}^n (1/b_j)^\alpha} \tag{26}$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments d_i. In this case, we also get the normalized Minkowski distance if $\alpha = 0$ and if $\alpha = \infty$, we get the minimum distance.

Remark 15: Another family of Quasi-OWAD operator that depends on the aggregated objects is

$$w_j = \frac{(1 - b_j)^\alpha}{\sum_{j=1}^n (1 - b_j)^\alpha} \tag{27}$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments d_i. Note that in this case if $\alpha = 0$, we also get the normalized quasi-arithmetic distance and if $\alpha = \infty$, we get the minimum distance. Note also that other families of dependent OWA operators could be developed in order to obtain the weighting vector.

Remark 16: A further type of aggregation operator that could be used in the Quasi-OWAD operator is the centered-

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OWA operator [19]. Following the same methodology, we could say that a Quasi-OWAD operator is a centered aggregation operator if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered-Quasi-OWAD operator. Note that the normalized quasi-arithmetic distance is an example of this particular case of centered-Quasi-OWAD operator. Another particular situation of the centered-Quasi-OWAD operator appears if we remove the third condition. We shall refer to it as a non-inclusive centered-Quasi-OWAD operator. For this situation, we find the median Quasi-OWAD as a particular case.

Remark 17: A special type of centered-Quasi-OWAD operator is the Gaussian Quasi-OWAD weights based on the Gaussian OWA weights [11]. In order to define it, we have to consider a Gaussian distribution $\eta(\mu, \sigma)$ where

$$\mu_n = \frac{1}{n}\sum_{j=1}^{n} j = \frac{n+1}{2} \tag{28}$$

$$\sigma_n = \sqrt{\frac{1}{n}\sum_{j=1}^{n}(j-\mu_n)^2} \tag{29}$$

Assuming that

$$\eta(j) = \frac{1}{\sqrt{2\Pi}\sigma_n} e^{-(j-\mu_n)^2/2\sigma_n^2} \tag{30}$$

we can define the Quasi-OWAD weights as

$$w_j = \frac{\eta_j}{\sum_{j=}^{n}\eta(j)} = \frac{e^{-(j-\mu_n)^2/2\sigma_n^2}}{\sum_{j=1}^{n} e^{-(j-\mu_n)^2/2\sigma_n^2}} \tag{31}$$

Note that the sum of the weights is 1 and $w_j \in [0,1]$.

Remark 18: By using the orness or attitudinal character and the dispersion measure it is also possible to obtain the weights of the Quasi-OWAD operator. For example, following [9] we could develop the maximal entropy Quasi-OWAD (MEQuasi-OWAD) as follows

$$\text{Maximize: } -\sum_{j=1}^{n} w_j \ln w_j \tag{32}$$

$$\text{Subject to: } g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\frac{n-j}{n-1}\right)\right) = \alpha(W) \tag{33}$$

where $\alpha \in [0, 1]$, $w_j \in [0,1]$, and the sum of the weights is 1. Note that other methods similar to the MEQuasi-OWAD could be developed for obtaining the Quasi-OWAD weights following the same methodologies than [6], [7], [10], [11].

B. Analysing the strictly continuous monotonic function g

If we analyze g, we obtain a wide range of particular cases that includes, among others, the Minkowski ordered weighted averaging distance (MOWAD) operator, the Hamming ordered weighted averaging distance (HOWAD) operator, the Euclidean ordered weighted averaging distance (EOWAD) operator, the ordered weighted geometric averaging distance (OWGAD) operator, the ordered weighted harmonic averaging distance (OWHAD) operator, etc.

Remark 19: The MOWAD operator [21], [22] is found when $g(D_j) = D_j^{\lambda}$. Therefore, we can see that the Quasi-OWAD operator provides a further generalization to the MOWAD operator. It can be constructed as a particular case of the Quasi-OWAD operator, but it is also possible to construct it by mixing the OWA operator with the quasi-arithmetic distance or by mixing the Hamming distance with the Quasi-OWA operator. Note that $g^{-1}(D_j) = D_j^{-\lambda}$. Its formulation is as follows.

$$MOWAD(d_1, d_2, \ldots, d_n) = \left(\sum_{j=1}^{n} w_j D_j^{\lambda}\right)^{1/\lambda} \tag{34}$$

Note that from a generalized perspective of the reordering step it is possible to distinguish between descending (DMOWAD) and ascending (AMOWAD) orders. Note also that in this case we could also obtain a parameterized family of distance aggregation operators such as the maximum distance, the minimum distance, the normalized Minkowski distance, the weighted Minkowski distance, the HOWAD operator, the EOWAD operator, etc.

Remark 20: The Hamming OWAD operator or simply OWAD operator [23] is found when $g(D_j) = D_j$. Note that $g^{-1}(D_j) = D_j^{-1}$. Note also that it is also possible to obtain it as a particular case of the MOWAD operator when the parameter λ = 1. It can be formulated as follows.

$$HOWAD(d_1, d_2, \ldots, d_n) = \sum_{j=1}^{n} w_j D_j \tag{35}$$

In this case, we can also distinguish between the descending HOWAD (DHOWAD) and the ascending HOWAD (AHOWAD) operator.

Remark 21: The Euclidean OWAD operator [21], [24] or also the ordered weighted quadratic averaging distance (OWQAD) operator is found when $g(D_j) = D_j^2$. Note that in this case, $g^{-1}(D_j) = D_j^{-2}$. Its formulation is as follows.

$$EOWAD(d_1, d_2,\ldots, d_n) = \left(\sum_{j=1}^{n} w_j D_j^2\right)^{1/2} \quad (36)$$

As shown above for the other particular cases, it is possible to distinguish between descending and ascending orders.

Remark 22: Another particular case obtained with the Quasi-OWAD operator is the OWGAD operator [25]. This case is found when $g(D_j) = D_j^0$. Note that in this case we also get, $g^{-1}(D_j) = D_j^0$.

$$OWGAD(d_1, d_2,\ldots, d_n) = \sum_{j=1}^{n} D_j^{w_j} \quad (37)$$

Note that the geometric operators cannot aggregate negative numbers and the value zero. Therefore, this distance aggregation operator is only useful in some special situations. Note also that it is possible to transform this operator as suggested in [26], so it can deal with zero or negative numbers.

Remark 23: Another special case found in the Quasi-OWAD operator is the OWHAD operator. In this case, when $g(D_j) = D_j^{-1}$. Note that in this case, $g^{-1}(D_j) = D_j^{1}$. It can be formulated as follows.

$$OWHAD(d_1, d_2,\ldots, d_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{D_j}} \quad (38)$$

As shown above in the previous particular cases of the Quasi-OWAD operator, we can distinguish between descending (DOWHAD) and ascending (AOWHAD) orders.

V. Conclusion

In this paper, we have suggested a new generalization of the OWA operator by using distance measures. We have called it the ordered weighted quasi-arithmetic distance (Quasi-OWAD) operator. We have seen that it is a further generalization of the Minkowski distance by using quasi-arithmetic means. We have considered some of its main properties such as the distinction between descending and ascending orders and some basic measures to characterize the weighting vector. Next, we have developed a wide range of particular cases of the Quasi-OWAD operator that includes all the particular cases of the MOWAD operator. We have seen that these special cases also provide a parameterized family of aggregation operators with similar properties than the Quasi-OWAD operator. We have also considered the usual families found in the weighting vector such as the Quasi-OWAD median, the step-Quasi-OWAD, the window-Quasi-OWAD, the S-Quasi-OWAD, the olympic-Quasi-OWAD, the centered-Quasi-OWAD, etc.

This paper represents a first analysis about the possibility of using OWA operators in quasi-arithmetic distances. In future research, we will develop further analysis by using different extensions of the OWA operator.

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