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Optimization of Lakes Aeration Process

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Abstract—The aeration process via injectors is used to combat the lack of oxygen in lakes due to eutrophication. A 3D numerical simulation of the resulting flow using a simplified model is presented. In order to generate the best dynamic in the fluid with respect to the aeration purpose, the optimization of the injectors location is considered. We propose to adapt to this problem the topological sensitivity analysis method which gives the variation of a criterion with respect to the creation of a small hole in the domain. The main idea is to derive the topological sensitivity analysis of the physical model with respect to the insertion of an injector in the fluid flow domain. We propose in this work a topological optimization algorithm based on the studied asymptotic expansion. Finally we present some numerical results, showing the efficiency of our approach

Keywords—Quasi Stokes equations, Numerical simulation, topological optimization, sensitivity analysis.

I. Introduction

THE mechanical aeration process in water reservoirs is one of the most used techniques to combat eutrophication. It consists on pomping a source of compressed air in the reservoir bottom via injectors in order to create a dynamic and aerate the water by bringing it in contact with the surface air. We focus in this work in the first hand to the direct problem. It concerns the numerical simulation of the resulting two phase water air-bubbles flow. Different models can be used to describe this problem [2], [4], [5], [9]. Using the fact that the water phase is dominant. We used a simplified model in which the water phase is governed by the Navier-Stokes equations and the aeration effects are taken into account through a local boundary condition for the velocity on the injector holes. Our discretization method is based on three dimensional mixed finite element method P^1+ bubble$/P^1$ [3]. The Uzawa algorithm is used to solve the obtained matrix system. In the other hand, we look at the inverse problem: find the optimal injectors location generating the best motion in the fluid with respect to the aeration purpose. The optimal injectors location is characterized as the solution to a topological optimization problem. The topological sensitivity analysis is used to solve this problem [6], [7], [8], [10]. The main idea is to compute the asymptotic topological expansion with respect to the insertion of an injector.

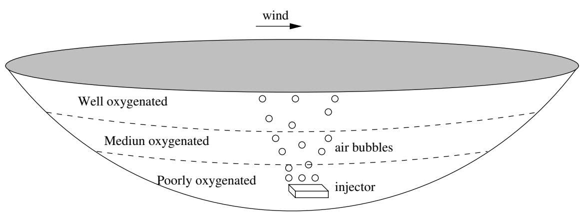

Fig. 1. Aeration process

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The paper is organized as follows. The used model, its numerical analysis and a direct numerical simulation is presented in section 2. Section 3 is devoted to a topological sensitivity analysis for the Quasi-Stokes equations. The obtained results are valid for a large class of cost functions. Finally, we illustrate the efficiency of the proposed method by a numerical test.

II. Direct simulation

Let Ω be a three dimensional flow domain representing the eutrophized lake. The used model is based on three dimensional Navier-Stokes equations for water flow in which we integrate the effect of momentum released by the injected bubbles by adding a local boundary condition for the velocity on the injector holes. In the presence of an injector $\omega_{inj} \subset \Omega$, the velocity $u(x,t)$ and the pressure $p(x,t)$ solve the following system

$$\begin{cases} \text{Find } u \text{ and } p \text{ solutions of} & \\ \dfrac{\partial u}{\partial t} + u.\nabla u - \nu\,\Delta u + \nabla p = \mathcal{G} & \text{in } \Omega_i \times [0,T] \\ \operatorname{div} u = 0 & \text{in } \Omega_i \times]0,T] \\ u^0 = u_0 & \text{in } \Omega_i \\ u = u_d & \text{on } \Gamma_i \times]0,T] \end{cases} \tag{1}$$

where $\Omega_i = \Omega \backslash \overline{\omega_{inj}}$ is the lake domain in the presence of the injector ω_{inj}, ν is the water viscosity, $\mathcal{G}$ is the gravitational force, T is the final time of simulation, u_0 is the initial velocity field, $\Gamma_i = \Gamma_s \cup \Gamma_w \cup \partial\omega_{inj}$ the boundary of Ω_i and

$$u_d = \begin{cases} u_{wind} & \text{on } \Gamma_s: \text{ the surface lake boundary,} \\ 0 & \text{on } \Gamma_w: \text{ the bottom lake boundary,} \\ u_{inj} & \text{on } \partial\omega_{inj}: \text{ the injector boundary.} \end{cases} \tag{2}$$

Using characteristics method in (1), we obtain

$$\begin{cases} \text{Find } u^{n+1} \text{ and } p^{n+1} \text{ solutions of} & \\ \alpha u^{n+1} + \nu\,\Delta u^{n+1} + \nabla p^{n+1} = F^{n+1} & \text{in } \Omega_i \\ \operatorname{div} u^{n+1} = 0 & \text{in } \Omega_i \\ u^{n+1} = u_d & \text{on } \Gamma_i, \end{cases} \tag{3}$$

where $\alpha = \dfrac{1}{\Delta t}$, $F^{n+1} = \dfrac{1}{\Delta t} u^n o \chi^n + \mathcal{G}$, u^{n+1} and p^{n+1} are the approximations of u and p on time $t^{n+1} = (n+1)\Delta t$ and $\chi^n(x) = X^n(t^{n+1}; x)$ represents the position at time t^{n+1} of the particle of fluid which is at point x at time t^n.

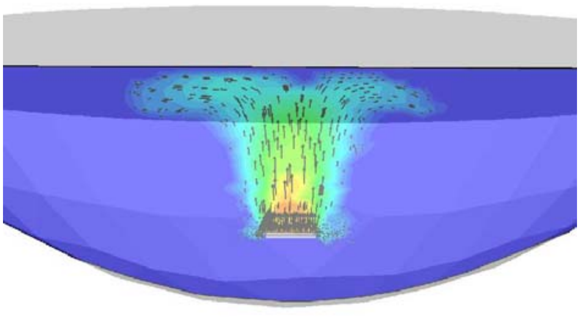
System (3) is solved iteratively for $n = 0, 1, \ldots$. At each time step, we have to solve a steady state problem of Quasi-Stokes type having the following generic form.

For $F \in L^2(\Omega_i)^3$, and $u_d \in H^{\frac{1}{2}}(\Gamma_i)^3$ such that $\displaystyle\int_{\Gamma_i} u_d.n\, ds = 0$, find u in $H^1(\Omega_i)^3$ and p in $L^2_0(\Omega_i)$ solutions of the problem

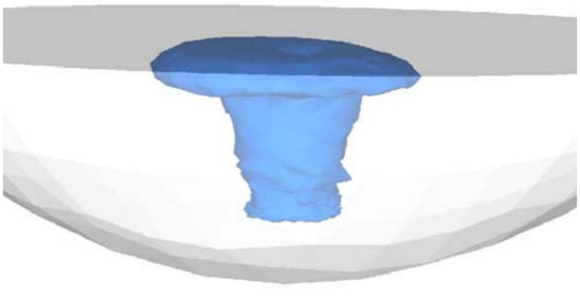
$$\begin{cases} \alpha\, u - \nu\, \Delta u + \nabla p = F & \text{fin } \Omega_i \\ \operatorname{div} u = 0 & \text{in } \Omega_i \\ u = u_d & \text{on } \Gamma_i. \end{cases} \tag{4}$$

Using 'P^1 +bubble/P^1' mixed finite element method (see [3]) for the space approximation, we derive a linear matrix system. The resolution is based on Uzawa method and conjugate gradient algorithm [1].

For the numerical simulation, we used the following boundary conditions: $u_{wind} = (0.01, 0, 0)m/s$ the wind velocity at the surface, no slip condition at the bottom and $u_{inj} = (0, 0, 0.1m/s)$ the injection velocity on the injector. We present in figure 2 the numerical simulation of the aeration effect on the water flow in a three dimensional domain containing one injector obtained for $T = 10mn$.



a) 2D cut of the water velocity isovalues.



b) created dynamic zone.

Fig. 2. Numerical simulation of the aeration process

This result shows that the aeration effect is located in the region between the injector and the top surface. Then we have to use more injectors in order to aerate all the lake domain. For this reason, we are interested in the following optimization problem: for a given injectors number, find their optimal location generating the best motion in the water.

III. Optimization problem

Our aim in this section is to design an efficient method to optimize the injectors location in order to generate the best motion of the fluid.

For the sake of simplicity, we shall assume that the injectors are well separated and have the geometry form $\omega_{z_k,\varepsilon} = z_k + \varepsilon\omega^k$, $1 \le k \le m$, where ε is the shared diameter and $\omega^k \subset \mathbb{R}^3$ are bounded and smooth domains containing the origin. The points $z_k \in \Omega$, $1 \le k \le m$ determine the location of the injectors. The domains ω^k describe the injectors geometries.

Here we limit ourselves to the steady state system described by the Quasi-Stokes equations (4). Then, in the presence of injectors, the velocity u_ε and the pressure p_ε satisfy the following equations

$$\begin{cases} \alpha u_\varepsilon - \nu\Delta u_\varepsilon + \nabla p_\varepsilon &= F \quad \text{in } \Omega\backslash\cup_{k=1}^m \overline{\omega_{z_k,\varepsilon}} \\ \nabla.u_\varepsilon &= 0 \quad \text{in } \Omega\backslash\cup_{k=1}^m \overline{\omega_{z_k,\varepsilon}} \\ u_\varepsilon &= u_d \quad \text{on } \Gamma \\ u_\varepsilon &= u^k_{inj} \quad \text{on } \partial\omega_{z_k,\varepsilon},\ 1 \le k \le m, \end{cases} \tag{5}$$

where u^k_{inj} is the injection velocity of the injector $\omega_{z_k,\varepsilon}$.

Consider now a design function j having the form

$$j(\Omega\backslash \cup_{k=1}^m \overline{\omega_{z_k,\varepsilon}}) = J_\varepsilon(u_\varepsilon), \tag{6}$$

where J_ε is a given cost function describing the optimization criteria and u_ε is the solution of (5).

Our identification problem can be formulated as a topological optimization problem: find the optimal location of the injectors $\omega_{z_k,\varepsilon} = z_k + \varepsilon\omega^k$, $1 \le k \le m$, inside the water reservoir domain Ω minimizing the function j:

$$(\mathcal{P}) \begin{cases} \text{Find } z^*_k \in \Omega,\ 1 \le k \le m, \text{ such that :} \\ j(\Omega\backslash \cup_{k=1}^m \overline{\omega_{z^*_k,\varepsilon}}) = \min\limits_{\omega_{z_k,\varepsilon} \subset \Omega} j(\Omega\backslash \cup_{k=1}^m \overline{\omega_{z_k,\varepsilon}}). \end{cases}$$

To solve this optimization problem $(\mathcal{P})$ we shall use the topological gradient method. It consists in studying the variation of the design function j with respect to a small topological perturbation of the domain Ω.

A. *Topological sensitivity analysis*

In this section we derive a topological asymptotic expansion of the design function j with respect to the insertion of a small injector $\omega_{z,\varepsilon} = z + \varepsilon\omega$ inside the domain Ω.

Next we assume that J_ε satisfies the following assumptions.

Hypothesis 3.1: i) J_0 is differentiable with respect to u, its derivative being denoted by $DJ_0(u)$.
ii) There exists a real number δJ such that $\forall\, \varepsilon \ge 0$

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = DJ_0(u_0)(\widehat{u}_\varepsilon - u_0) + \varepsilon\delta J + o(\varepsilon)\tag{7}$$

where $\widehat{u}_\varepsilon$ denotes the extension of u_ε in Ω defined by $u_\varepsilon = u_{inj}$ in $\omega_{z,\varepsilon}$.

We are now ready to derive the topological asymptotic expansion of the design function j. It consists in computing the variation $j(\Omega\backslash\overline{\omega_{z,\varepsilon}}) - j(\Omega)$ when inserting a small injector inside the domain. The asymptotic expansion described in Theorem 3.1 is valid for arbitrary shaped holes and all cost function verifying the Hypothesis 3.1.

Theorem 3.1: If Hypothesis 3.1 holds, the function j has the following asymptotic expansion

$$j(\Omega\backslash\overline{\omega_{z,\varepsilon}}) - j(\Omega) = \varepsilon\left[\Big(-\int_{\partial\omega}\eta(y)\ \mathrm{ds(y)}\Big).v_0(z) + \delta J\right] + o(\varepsilon),$$

where v_0 is the solution to the adjoint problem

$$\begin{cases} \alpha v_0 - \nu\Delta v_0 + \nabla q_0 & = -DJ_0(u_0) & \text{in } \Omega \\ \nabla.v_0 & = 0 & \text{in } \Omega \\ v_0 & = 0 & \text{on } \Gamma. \end{cases}$$

and $\eta \in H^{-1/2}(\partial\omega)^3$ is the solution to the boundary integral equation.

$$\int_{\partial\omega} E(y-x)\,\eta(x)\ \mathrm{ds}(x) = u_{inj} - u_0(z), \quad \forall y \in \partial\omega. \tag{8}$$

with $(E,\,\Pi)$ the fundamental solution of the Stokes equations

$$E(y) = \frac{1}{8\pi\nu r}\Big(I + e_r e_r^T\Big), \quad \Pi(y) = \frac{y}{4\pi r^3},$$

with $r = ||y||$, $e_r = y/r$ and e_r^T is the transposed vector of e_r.

Corollary 3.1: If $\omega = B(0,1)$, the density η is given explicitly : $\eta(y) = -\dfrac{3\nu}{2}u_0(z)\ \ \forall y \in \partial\omega$ and under the hypothesis of theorem 3.1, we have

$$j(\Omega\backslash\overline{\omega_{z,\varepsilon}}) - j(\Omega) = \varepsilon\left[6\pi\nu\, u_0(z).v_0(z) + \delta J\right] + o(\varepsilon).$$

B. Numerical results

Aeration is considered as the best remedial action against eutrophication. This process consists in inserting some injector holes ω_k in the bottom layer of the reservoir in order to create a dynamic and aerate the water. We suppose that a "good" water reservoir aeration can be described by a target velocity $\mathcal{U}_g$ (see figure 4). Our aim is to determine the optimal location in Ω of some injector holes ω_k, $1 \le k \le m$ in order to minimize the function

$$J_\varepsilon(u_\varepsilon) = \int_{\Omega_m} |u_\varepsilon - \mathcal{U}_g|^2\ dx, \tag{9}$$

where $\Omega_m \subset \Omega$ is the measurement domain (the top layer).

The following Proposition describes the variation of the associated design function j with respect to the insertion of a small injector $\omega_{z,\varepsilon} = z + \varepsilon B(0,1)$ inside the domain Ω.

Proposition 3.1: The cost function J_ε defined in (9) satisfies the Hypothesis 3.1 with

$$DJ_0(u_0)(w) = 2\int_{\Omega_m}(u_0 - \mathcal{U}_g)w\ \mathrm{dx}, \quad \forall w \in \mathcal{V}_0,\ \text{and } \delta J = 0. \tag{10}$$

Then, the design function j has the following expansion

$$j(\Omega\backslash\overline{\omega_{z,\varepsilon}}) - j(\Omega) = 6\pi\nu\, u_0(z).v_0(z) + o(\varepsilon).$$

The optimal location of the injectors $\omega_k = z_k + \varepsilon B(0,1)$, $1 \le k \le m$ is obtained using the following topological optimization algorithm.

The algorithm :

- Initialization: choose $\Omega_0 = \Omega_b$, and set $k = 0$.
- Repeat until target is reached:
  - compute u_k and v_k, respectively solutions to direct and adjoint problems in Ω_k,
  - compute the topological sensitivity $\delta j_k = (u_k - u_{inj}).v_k$,
  - set $\Omega_{k+1} = \{x \in \Omega_k,\quad \delta j_k(x) \ge c_{k+1}\}$ where c_{k+1} is chosen in such a way that the cost function decreases,
  - $k \longleftarrow k+1$.

This algorithm can be seen as a descent method where the descent direction is determined by the topological sensitivity δj_k and the step length is given by the volume variation $\Omega_k\backslash\Omega_{k+1}$.

We propose an adaptation of the previous algorithm to our context. We consider the set $\{x \in \Omega_k;\ \delta j_k(x) < c_{k+1}\}$ Each connected component of this set is a hole created by the algorithm. Our idea is to replace each hole by an injector located at the local minimum of $\delta j_k(x)$.

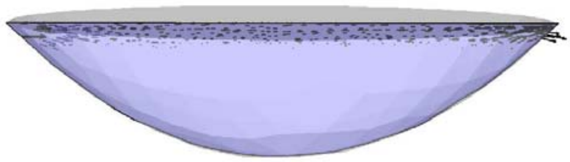


Fig. 3. The initial flow u^0

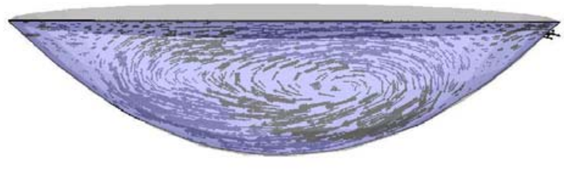
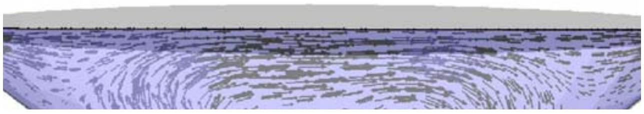


Fig. 4. The wanted flow $\mathcal{U}_g$

a) The wanted velocity $\mathcal{U}_g$ in the measurement domain Ω_m



b) The obtained velocity $u_{|\Omega_m}$

Fig. 5. Velocities field obtained in Ω_m (measurement domain) at the end of the optimization process.

For this numerical test, we consider in figure 4 a constructed solution representing the velocity field $\mathcal{U}_g$. This solution is obtained by the dynamic aeration process using more than 1000 injectors located at the bottom layer Ω_b. We aim to find the optimal location of a fixed number of injectors m in order to approximate the wanted flow $\mathcal{U}_g$.

Using our algorithm with 25 injectors (i.e. $m = 25$), we show in figure 7 the obtained flow during the optimization process at iterations 1, 3 and 5. The optimal injectors location is given

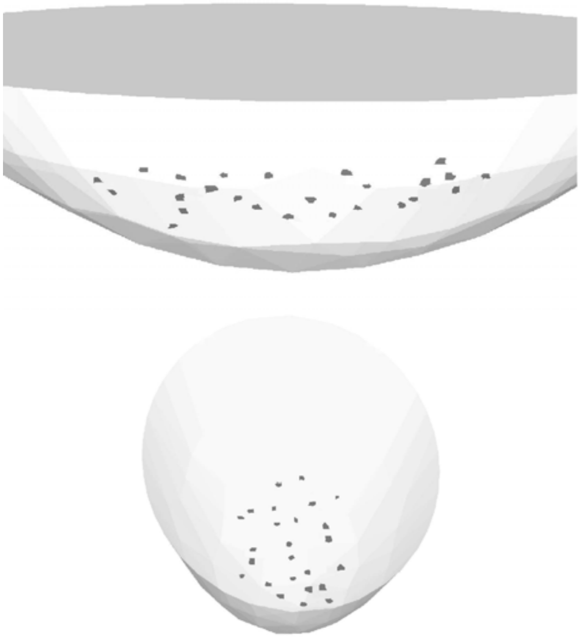
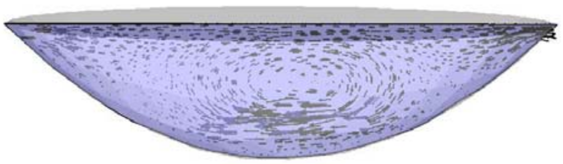
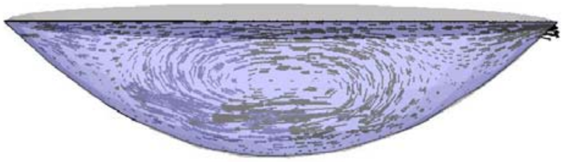


Fig. 6. Injectors location obtained during the optimization process: lateral view (left) and top view (right)

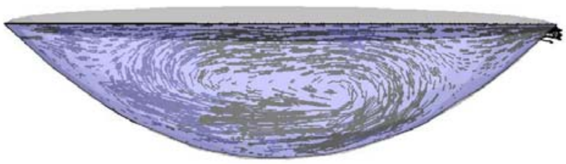
in figure 6. Figure 5 shows a vertical cut of the wanted and obtained flows in the measurement domain Ω_m. We remark that we obtain approximately the same flow.



first iteration



third iteration



fifth iteration

Fig. 7. Velocities field obtained during the optimization process.

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