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One Some Effective Solutions of Stokes Axisymmetric Equation for a Viscous Fluid

N. Khatiashvili, K. Pirumova, and D. Janjgava

***Abstract*—**The Stokes equation connected with the fluid flow over the axisymmetric bodies in a cylindrical area is considered. The equation is studied in a moving coordinate system with the appropriate boundary conditions. Effective formulas for the velocity components are obtained. The graphs of the velocity components and velocity profile are plotted.

***Keywords*—**Stokes system, viscous fluid.

I. INTRODUCTION

THE stationary and non-stationary Newtonian fluids are investigated by numerous of authors by means of Navier-Stokes equation with the specific boundary conditions (see for example [1]-[11]).

We consider the fluid flow over some axisymmetric bodies which moves in the infinite cylindrical channel filled with a viscous fluid. These bodies have the same axis of symmetry. We admit that the pressure fall is a constant. In this case the velocity of the fluid satisfies the linearized Navier-Stokes equation with the appropriate initial-boundary conditions. The solutions of this equation have been obtained. Hence, the velocity components of the Stokes flow are found.

II. STATEMENT OF THE PROBLEM

Let fluid occupied some cylindrical channel of the diameter $d(\mathrm{d} > 0)$ and consider in this channel the motion of some system of axisymmetric bodies at a speed $\vec{V}_0(V_x^0, V_y^0, V_z^0)$. For low Reynolds number the Stokes equation with the equation of continuity are valid [1]-[6]:

$$\frac{\partial \vec{V}}{\partial t} = \vec{F} - \frac{1}{\rho} grad\, P + \nu \Delta \vec{V} \tag{1}$$

$$\frac{\partial V_x}{\partial t} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \tag{2}$$

where $\vec{V}\,(V_x, V_y, V_z)$ is the velocity vector , P is the pressure, $\vec{F}(F_x, F_y, F_z)$ is the external force, ρ is a density of the fluid, ν -is a viscosity .

Equation (1) can be rewritten in terms of velocity components in the form:

$$\frac{\partial V_x}{\partial t} = F_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \Delta V_x \tag{3}$$

$$\frac{\partial V_y}{\partial t} = F_y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \Delta V_y \tag{4}$$

$$\frac{\partial V_z}{\partial t} = F_z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \nu \Delta V_z \tag{5}$$

Also the following boundary conditions are satisfied:

$$V_x\Big|_{S_0} = 0,\ V_y\Big|_{S_0} = 0,\ V_z\Big|_{S_0} = 0 \tag{6}$$

$$V_x\Big|_S = V_x^0(t),\ V_y\Big|_S = V_y^0(t),\ V_z\Big|_S = V_z^0(t) \tag{7}$$

where $V_x^0(t), V_y^0(t), V_z^0(t),$ are the given functions, S is a surface of the moving bodies, S_0 is a surface of the cylindrical channel. The surface S and the width of a channel d will be defined according to the solutions.

Let the axis of symmetry is ox and consider the moving coordinate system. Suppose, that the bodies move parallel to the axis of symmetry at a constant speed $\vec{V}_0(V_x^0, V_y^0, V_z^0)$ and

$$\frac{1}{\rho}\frac{\partial P}{\partial x} - F_x = C_0, \qquad \frac{1}{\rho}\frac{\partial P}{\partial r} - F_r = 0,$$

where C_0 is a definite constant, F_x, F_r are the components of the force in the cylindrical coordinates.

In a cylindrical coordinates (2), (3), (4), (5), becomes

$$\Delta V_x + \frac{1}{r}\frac{\partial V_x}{\partial r} = 2C_1 \tag{8}$$

N. Khatiashvili is with the I. Vekua Institute of. Iv. Javakhishvili Tbilisi State University, University St. 2, 0186 Tbilisi, Georgia (phone: 995 32 230-30-40; e-mail: ninakhatia@gmail.com).

K. Pirumova is with the Iv.Javakhishvili Tbilisi State University, 2,University St., 0186 Tbilisi, Georgia (e-mail: chr4mk@gmail.com).

D. Janjgava was with the Iv.Javakhishvili Tbilisi State University, 2, University St., 0186Tbilisi, Georgia. He is now with the Logistic Center Lilo1, Iumashev St. 14, 0198 Tbilisi,Georgia (e-mail: admin@lilo1.com).

$$\Delta V_r + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} = 0 \tag{9}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_r}{\partial r} + \frac{V_r}{r} = 0 \tag{10}$$

where V_x, V_r ,are the components of the velocity,

$$r = \sqrt{y^2 + z^2}\,, \quad C_1 = \frac{C_0}{2\nu}\,.$$

The boundary conditions will be given by:

$$V_x \Big|_{r=\pm h} = -V_x^0, \quad V_x\Big|_{\Gamma} = 0 \tag{11}$$

$$V_r \Big|_{r=\pm h} = 0, \quad V_r\Big|_{\Gamma} = 0 \tag{12}$$

where Γ is the contour of the bodies, $h = d/2$.

In the next chapter we will find the bounded solutions of the system (8), (9), (10), (11), (12), and Γ .

III. Solution of the Problem

The function:

$$U = C^* - q\left\{\frac{1}{\sqrt{(x+c)^2 + r^2}} - \frac{1}{\sqrt{(x-c)^2 + r^2}}\right\};$$
$$C^* = const,$$

where q and c are the certain parameters, is the solution of (8) for $C_1 = 0$, [11].

By direct verification we obtain, that the pair of functions:

$$V_x = \frac{\partial U}{\partial x} + C_1 r^2 - A, \quad V_r = \frac{\partial U}{\partial r}$$

where A $(A > 0)$ is the definite constant, is the solution of the system (8), (9), (10), Also, the pair of functions:

$$V_{xm} = \frac{\partial^m}{\partial x^m}\left(\frac{\partial U}{\partial x}\right) + C_1 r^2 - A, \quad V_{rm} = \frac{\partial^m}{\partial x^m}\left(\frac{\partial U}{\partial r}\right),$$

will be the solution of this system.

1. For odd $m, m = 2n-1, n = 1,2,...$ the solutions of the system (8), (9), (10) are given by the formulas

$$V_{xm} = q\sum_{k=0}^{n}\left(\frac{\alpha_k r^{2k}}{\sqrt{(x+c)^2 + r^2}^{\,2k+1+2n}} - \frac{\alpha_k r^{2k}}{\sqrt{(x-c)^2 + r^2}^{\,2k+1+2n}}\right)$$
$$+ C_1 r^2 - A, \tag{13}$$

$$V_{rm} =$$
$$q\sum_{k=1}^{n}\left(\frac{\alpha_k^* r^{2k-1}(x+c)}{\sqrt{(x+c)^2 + r^2}^{\,2k+1+2n}} - \frac{\alpha_k^* r^{2k-1}(x-c)}{\sqrt{(x-c)^2 + r^2}^{\,2k+1+2n}}\right) \tag{14}$$

where α_k, α_k^*, are the definite constants.

2. For even $m, m = 2n, n = 1,2,...$ the solutions of system (8), (9), (10) are given by

$$V_{xm} = q\sum_{k=0}^{n}\left(\frac{\beta_k r^{2k}(x+c)}{\sqrt{(x+c)^2 + r^2}^{\,2k+3+2n}} - \frac{\beta_k r^{2k}(x-c)}{\sqrt{(x-c)^2 + r^2}^{\,2k+3+2n}}\right)$$
$$+ C_1 r^2 - A, \tag{15}$$

$$V_{rm} =$$
$$q\sum_{k=1}^{n+1}\left(\frac{\beta_k^* r^{2k-1}}{\sqrt{(x+c)^2 + r^2}^{\,2k+1+2n}} - \frac{\beta_k^* r^{2k-1}}{\sqrt{(x-c)^2 + r^2}^{\,2k+1+2n}}\right) \tag{16}$$

where β_k, β_k^*, are the definite constants.

For the different values of q, c, and A we obtain the different fluid flow over some axisymmetric bodies, the shape of which will be defined by the formulas

$$\frac{\partial^m}{\partial x^m}\left(\frac{\partial U}{\partial x}\right) + C_1 r^2 - A = 0 \tag{17}$$

$$\frac{\partial^m}{\partial x^m}\left(\frac{\partial U}{\partial r}\right) \approx 0 \tag{18}$$

In the following chapter some examples are given and the graphics of velocity components and velocity profile are plotted by using Maple.

Note. The Stokes equation has a real physical sense for low velocities only. So not for each parameters q, c, and A, the solutions of (8), (9), (10), (11) ,(12), are suitable according to the physical viewpoint.

IV. The Case of $m = 1$ and $m = 2$, Examples

1. In case of $m = 1$, by (13), (14), we obtain

$$V_{x1} = -\frac{2q}{(r^2 + (x+c)^2)^{\frac{3}{2}}} + \frac{2q}{(r^2 + (x-c)^2)^{\frac{3}{2}}}$$
$$+ \frac{3qr^2}{(r^2 + (x+c)^2)^{\frac{5}{2}}} - \frac{3qr^2}{(r^2 + (x-c)^2)^{\frac{5}{2}}} + C_1 r^2 - A,$$

$$V_{r1} = -\frac{3qr(x+c)}{(r^2 + (x+c)^2)^{\frac{5}{2}}} + \frac{3qr(x-c)}{(r^2 + (x-c)^2)^{\frac{5}{2}}}.$$

In Fig. 1, the lateral cross-section of the cylindrical area with the axisymmetric body is represented.

In Fig. 2 graphics of the corresponding velocity components are given (the black surface is V_{x1}, the gray surface is V_{r1}) in case of $c = 1/5; C_1 = 1; A = 9; q = 1/10$. In Fig. 3 the corresponding velocity profile $|V|, |V| = \sqrt{V_{x1}{}^2 + V_{r1}{}^2}$, is plotted.

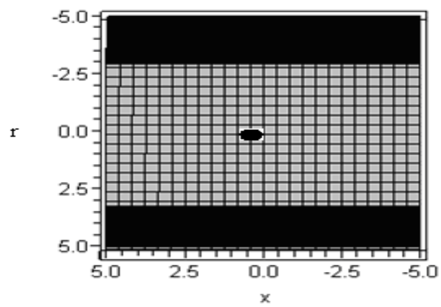

Fig. 1 The lateral cross-section of the cylinder of width $1 < d < 2$ in case of $c = 1/5; C_1 = 1; A = 9; q = 1/10$

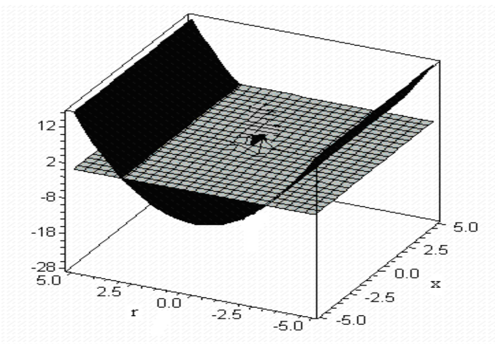

Fig. 2 The graphics of V_{x1} (black surface), and V_{r1} (gray surface) in case of $c = 1/5; C_1 = 1; A = 9; q = 1/10$

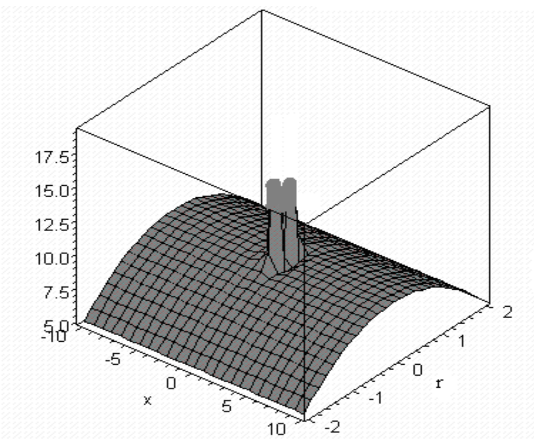

Fig. 3 The graphic of the velocity profile in case of $c = 1/5; C_1 = 1; A = 9; q = 1/10$

2. In case of m=2 by (15), (16), we obtain

$$V_{x2} = \frac{6q(x+c)}{(r^2+(x+c)^2)^{\frac{5}{2}}} - \frac{6q(x-c)}{(r^2+(x-c)^2)^{\frac{5}{2}}} - \frac{15qr^2(x+c)}{(r^2+(x+c)^2)^{\frac{7}{2}}} + \frac{15qr^2(x-c)}{(r^2+(x-c)^2)^{\frac{7}{2}}} + C_1 r^2 - A,$$

$$V_{r2} = \frac{12qr}{(r^2+(x+c)^2)^{\frac{5}{2}}} - \frac{12qr}{(r^2+(x-c)^2)^{\frac{5}{2}}} - \frac{15qr^3}{(r^2+(x+c)^2)^{\frac{7}{2}}} + \frac{15qr^3}{(r^2+(x-c)^2)^{\frac{7}{2}}}.$$

In Fig. 4, the lateral cross-section of the cylindrical area with the axisymmetric body is represented.

In Fig. 5 graphics of the corresponding velocity components are given (the black surface is V_{x2}, the gray surface is V_{r2}) in case of $c = 1; C_1 = 1; A = 9; q = 1/10$. In Fig. 6 the corresponding velocity profile $|V|, |V| = \sqrt{V_{x2}{}^2 + V_{r2}{}^2}$, is plotted.

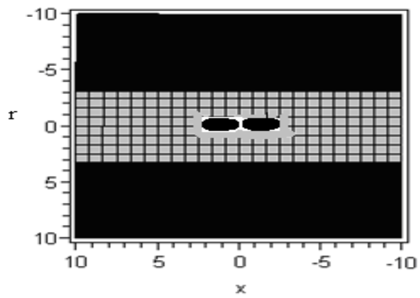

Fig. 4 The lateral cross-section of the cylinder of width $1 < d < 3$, in case of $c = 1; C_1 = 1; A = 9; q = 1/10$

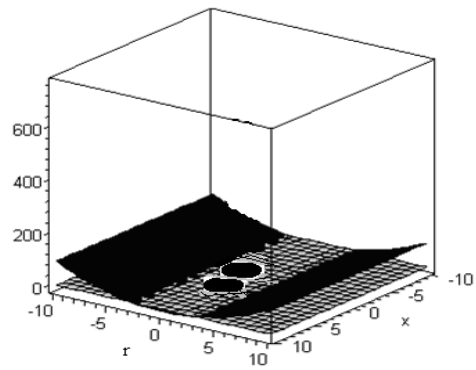

Fig. 5 The graphics of V_{x2} (black surface), and V_{r2} (gray surface) in case of $c = 1/5; C_1 = 1; A = 9; q = 1/10$

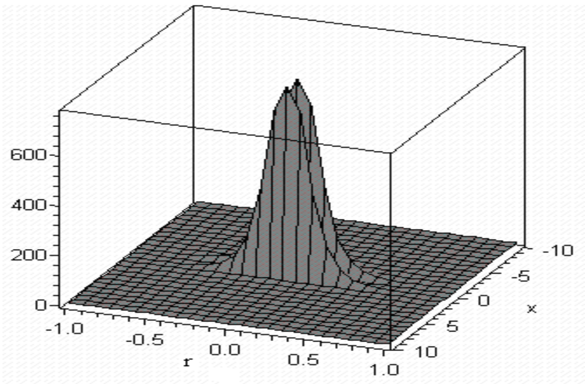

Fig. 6 The graphic of the velocity profile in case of $c = 1; C_1 = 1; A = 9; q = 1/10$

V. Conclusion

The effective solutions of the system (1), (2), with the initial-boundary conditions (6), (7), in the axisymmetric case are given by the formulas 1. (13), (14); or 2. (15), (16); and these solutions represent fluid flow over the system of axisymmetric bodies, contours of which are given by the formulas (17), (18), respectively.

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