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On the Wave Propagation in Layered Plates of General Anisotropic Media

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***Abstract*—**Analysis for the propagation of elastic waves in arbitrary anisotropic plates is investigated, commencing with a formal analysis of waves in a layered plate of an arbitrary anisotropic media, the dispersion relations of elastic waves are obtained by invoking continuity at the interface and boundary of conditions on the surfaces of layered plate. The obtained solutions can be used for material systems of higher symmetry such as monoclinic, orthotropic, transversely isotropic, cubic, and isotropic as it is contained implicitly in the analysis. The cases of free layered plate and layered half space are considered separately. Some special cases have also been deduced and discussed. Finally numerical solution of the frequency equations for an aluminum epoxy is carried out, and the dispersion curves for the few lower modes are presented. The results obtained theoretically have been verified numerically and illustrated graphically.

***Keywords*—**Anisotropic, layered, dispersion, elastic waves, frequency equations.

I. Introduction

ENGINEERING materials such as fiber reinforced composite, graphite and laminate, where high strength-to-weight and stiffness-to-weight ratios are required. These materials are crucial for structural applications, and have resulted in considerable research activities on their behavior. Consequently studies of the propagation of elastic waves in the layered media [1]-[4], which are anisotropic in nature, become very important and have long been of interest to researchers in the fields of geophysics, acoustics, and nondestructive evaluation

Compared to the extensive literature on the elastic waves in infinite anisotropic media; relatively little attention has been given to elastic waves in anisotropic plates. Although a complete review of the extensive literature on this subject cannot be undertaken, several salient contributions should be mentioned in [5]-[9]. Propagation of waves in free isotropic plates were first reported by Lamb in 1917 in his famous work [10], and followed by several authors [3] and [11]-[15]. Propagation of free guided waves in anisotropic homogeneous plate has been studied in detail by authors [16]-[18]. These studies provide an interesting picture of the rich dispersion characteristic of these guided waves. Several others authors [8], [12], [15] and [19] have studied free Lamb waves.

In this paper analysis for the propagation of elastic waves in plates of general anisotropic media is investigated on the basis of an exact theory. Dispersion relations of elastic waves are obtained by invoking continuity at the interface and boundary of conditions on the surfaces of layered plate. The obtained solutions can be used for material systems of higher symmetry such as monoclinic, orthotropic, transversely isotropic, cubic, and isotropic as it is contained implicitly in the analysis. The cases of free layered plate and layered half space are considered special cases have also been deduced and discussed separately. It is also demonstrated that the particle motions for SH modes decouple from rest of the motion, if the propagation occurs along an in-plane axis of symmetry. Some special cases have also been deduced and discussed. Finally numerical solution of the frequency equations for an aluminum epoxy is carried out, and the dispersion curves for the few lower modes are presented and the results obtained theoretically have also been verified numerically and illustrated graphically.

II. Formulation

Consider an infinite generally- anisotropic plate, having thickness d, whose normal is aligned with the x_3 axis of a reference Cartesian coordinate system $x_i = (x_1, x_2, x_3)$. The mid-plane of the plate is chosen to coincide with the $x_1 - x_2$ plane. The equations of motion in the absence of body forces

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{1}$$

where

$$\sigma_{ij} = C_{ijkl} e_{kl} \tag{2}$$

ρ is the density, t is the time, u_i is the displacement in the x_i direction, σ_{ij} and e_{ij} are the stress and strain tensor respectively; and the fourth order tensor of the elasticity C_{ijkl} satisfies the (Green) symmetry conditions:

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jikl} . \tag{3}$$

Strain-displacement relation

$$e_{ij} = (u_{i,j} + u_{j,i})/2 \tag{4}$$

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The displacement, stress components at the surface of the plate are:

$$S(x_3) = \{\bar{\sigma}_{13}, \bar{\sigma}_{23}, \bar{\sigma}_{33}\} \quad (5)$$

$$D(x_3) = \{\bar{u}_3, \bar{u}_2, \bar{u}_3\} \quad (6)$$

and the bar means the amplitudes of the displacement, stress are the function of x_3 only.

The boundary conditions on the plate surfaces are:

$$S(-d/2) = 0 \quad (7)$$

$$S(d/2) = 0, \quad (8)$$

where $\mathbf{0}$ is a zero vector.

Substituting eqs. (2) and (4) into eq. (1), equation of motion are expressed by displacements as follows:

$$C_{jkl} u_{k,jl} = \rho \ddot{u}_i \quad (9)$$

III. Analysis

If Assume that solutions to eqs. (9) are expressed by

$$u_j = U_j \exp[\iota\xi(n_1x_1 + n_2x_2 + n_3x_3 - ct)], \; j = 1, 2, 3 \quad (10)$$

where ξ is the wave number, $i = \sqrt{-1}$ c is the phase velocity (= ω/ξ), ω is the circular frequency, U_j are the constants related to the amplitudes of displacement, n_k (k = 1,2,3) are the components of the unit vector giving the direction of propagation. Substituting eq. (10) into eq. (9), this leads to the three coupled equations

$$\prod_{ik} U_k = 0 \quad (11)$$

where $\prod_{ik} = \Gamma_k - \rho c^2 \delta_{ik}$, δ_{ik} is the Kronecker delta, and Γ_{ik} are the Christofied stiffness as follows:

$$e_{ik} = \sum_{l=1}^{6} p_{k_{(l)}} B_l \exp(\iota\xi\alpha_l x_3) \exp[\iota\xi(n_1x_1 + n_2x_2 - ct)] \quad (12)$$

where

$$p_{ik(l)} = \iota\xi(n_k q_{i(l)} + n_i q_{k(l)})/2, (i, k = 1, 2, 3) \quad (13)$$

The stress tensor is

$$\sigma_k = \sum_{l=1}^{6} D_{k_{(l)}} B_l \exp(\iota\xi\alpha_l x_3) \exp[\iota\xi(n_1x_1 + n_2x_2 - ct)] \quad (14)$$

where

$$D_{ik(l)} = C_{jkl} p_{pq(l)}, (i, k, p, q = 1, 2, 3) \quad (15)$$

In eqs. (12) and (19), $n_3 = \alpha_l$ (l = 1, 2 . . . . 6) . With eqs. (14) and (18), we have

$$u_j = \bar{u}_{j(l)} \exp[\iota\xi(n_1x_1 + n_2x_2 - ct)], \quad (16)$$

$$\sigma_{i3} = \bar{\sigma}_{i3(l)} \exp[\iota\xi(n_1x_1 + n_2x_2 - ct)], \quad (17)$$

where

$$\bar{u}_{j(l)} = \sum_{l=1}^{6} q_{j_{(l)}} \exp(\iota\xi\alpha_l x_3) B_l, \quad (18)$$

$$\bar{\sigma}_{ik} = \sum_{l=1}^{6} D_{ik_{(l)}} \exp(\iota\xi\alpha_l x_3) B_l, \quad (19)$$

Eqs. (18)-(19) can be expressed in the matrix form :

$$\mathbf{F}^{(n)} = \mathbf{R}^{(n)} \mathbf{B}^{(n)}, \quad (20)$$

in which $\mathbf{B}^{(n)} = (A_1, A_2, A_3, A_4, A_5, A_6)$, $\mathbf{F}^{(n)} = \{\bar{\sigma}_{33}, \bar{\sigma}_{13}, \bar{\sigma}_{23}, \bar{u}_1, \bar{u}_2, \bar{u}_3\}$ and $\mathbf{R}^{(n)}$ is the matrix, the elements of which are specified in the (18)-(19).

Considering the origin of co-ordinates at the $(n-1)$th interface, the relation between the stresses, displacements and vector $\mathbf{F}^{(n),(n\text{-}1)}$ within the n th layer at the $(n-1)$th interface ,and the vector $\mathbf{B}^{(n)}$

$$\mathbf{F}^{(n),(n)} = \mathbf{S}^{(n)} \mathbf{B}^{(n)} \quad (21)$$

where $\mathbf{S}^{(n)}$ is derived from $\mathbf{R}^{(n)}$ by putting $x_3 = 0$.

Similarly stresses, displacements vector $\mathbf{F}^{(n),(n)}$ within the n th layer, at the interface is

$$\mathbf{F}^{(n),(n)} = \mathbf{U}^{(n)} \mathbf{B}^{(n)} \quad (22)$$

where $\mathbf{U}^{(n)}$ is derived from $\mathbf{R}^{(n)}$ by putting $x_3 = h^{(n)}$

Using (20) to (22), $\mathbf{F}^{(n),(n-1)}$ and $\mathbf{F}^{(n),(n)}$ at the upper and bottom surface of n th layer we can related,

$$\mathbf{F}^{(n),(n)} = \mathbf{G}^{(n)} \mathbf{F}^{(n),(n-1)},$$

where $\mathbf{G}^{(n)} = \mathbf{U}^{(n)} \mathbf{S}^{(n)-1}$ (23)

Using the boundary and continuity conditions $(\mathbf{F}^{(n),(n-1)} = \mathbf{F}^{(n-1),(n-1)})$ at each interface and considering that no slips occur at the interface.

On applying (23) to each interface in turns gives

$$\mathbf{F}^{(n),(n)} = \mathbf{G}^{(n)} \mathbf{G}^{(n-1)} \ldots \mathbf{G}^{(1)} \mathbf{F}^{(1),(0)} = \mathbf{G} \mathbf{F}^{(1),(0)} \quad (24)$$

A. Free layered plate:

If the semi-infinite medium is absent, consider a free layered plate. The characteristic equation for such a situation is obtained by invoking stress-free $(\bar{\sigma}_{33} = \bar{\sigma}_{13} = \bar{\sigma}_{23} = 0)$ upper and bottom surfaces. Partitioning the 6×6 matrix $\mathbf{G}$ into four 3×3 sub-matrices $\mathbf{G}_1, \mathbf{G}_2$ etc., and partitioning the vector conformably

We obtain the characteristic equation as (24)

$$\{\{0,0,0\}, \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}\}^{(n),(n)} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \\ \mathbf{G}_3 & \mathbf{G}_4 \end{bmatrix} \{\{0,0,0\}, \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}\}^{(1),(0)} \quad (25)$$

that is

$$\mathbf{G}_2 \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}^{(1),(0)} = 0 \quad (26)$$

and

$$\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}^{(n),(n)} = \mathbf{G}_4 \{\bar{u}_1, \bar{u}_2, \bar{u}_3\}^{(1),(0)} \quad (27)$$

If $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}^{(1),(0)}$ is not to be null, therefore, $\mathbf{G}_2$ must be singular, that is

$$\det(\mathbf{G}_2) = 0 \quad (28)$$

The elements of $\mathbf{G}_2$ are functions of the elastic constants and thickness of the layers. Equation (28) is the desired dispersion equation. The numerical values of det($\mathbf{G}_2$) can be computed by the successive multiplication of the matrices $\mathbf{G}^{(n)}$ for each layer, and equation (28) can be solved by applying a suitable iterative method.

B. Semi-infinite medium

If the Semi-infinite medium is present, then the problem reduces to that of a layered half space, pre-multiplying (3.29) by $\left(\mathbf{S}^{(\mathbf{n+1})}\right)^{-1}$ gives

$$\mathbf{B}^{(\mathbf{n+1})} = \left(\mathbf{S}^{(\mathbf{n+1})}\right)^{-1} \mathbf{G}\mathbf{F}^{(\mathbf{1}),(\mathbf{0})} = \mathbf{J}\mathbf{F}^{(\mathbf{1}),(\mathbf{0})} \tag{29}$$

where $\mathbf{J} = \left(\mathbf{S}^{(\mathbf{n+1})}\right)^{-1} \mathbf{G}$.

The conditions on the semi-infinite medium are that there be no sources at infinity and this implies that $\overline{u}_1, \overline{u}_2,$ and $\overline{u}_3$ are expressible in terms of $\exp(-\alpha_i x_3)$, respectively , i.e. $A_1 = A_2, A_3 = A_4,$ and $A_5 = A_6$. Again partitioning J into four 3×3 sub-matrices

$$\left\{\{A_1, A_2, A_3\}, \{A_1, A_2, A_3\}\right\} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{J}_2 \\ \mathbf{J}_3 & \mathbf{J}_4 \end{bmatrix} \left\{\{0,0,0\}, \{\overline{u}_1, \overline{u}_2, \overline{u}_3\}\right\}^{(1),(0)} \tag{30}$$

so that if is not to be null, $\mathbf{J}_2 - \mathbf{J}_4$ must be singular. The dispersion equation for layered half space is therefore

$$\det(\mathbf{J}_2 - \mathbf{J}_4) = 0 \tag{31}$$

Eq. (31) is to be solved with some iteration process, when this has been done; the displacement ratio on the top surface is given by

$$\left[\mathbf{J}_2 - \mathbf{J}_4\right]\left\{\overline{u}_1, \overline{u}_2, \overline{u}_3\right\} = 0 \tag{32}$$

and the displacements, ,stresses at other depths can be obtained as for the free plate.

In both the half-space and free plate problems, the dispersion equation involves only the last two columns of $\mathbf{J}$ or $\mathbf{G}$, and there is no need to compute more than this. Further, the dispersion equation for a half space involves the matrices $\mathbf{J}_2 - \mathbf{J}_4$. This can be obtained directly by pre-multiplying $\mathbf{J}$ by the matrix

$$\mathbf{I}_2 = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \tag{33}$$

in which $\mathbf{I}$ represents a 3×3 unit matrix. Thus

$$\mathbf{I}_2\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 - \mathbf{J}_3 & \mathbf{J}_2 - \mathbf{J}_4 \\ \mathbf{J}_1 - \mathbf{J}_3 & \mathbf{J}_2 - \mathbf{J}_4 \end{bmatrix} = \mathbf{I}_2\left(\mathbf{S}^{(\mathbf{n+1})}\right)^{-1} \mathbf{G}^{(n)}\mathbf{G}^{(n-1)}\ldots\mathbf{G}^{(1)} \tag{34}$$

Finally,

$$\left[\mathbf{J}_2 - \mathbf{J}_4\right] = \mathbf{M}^{(n+1)}\mathbf{G}^{(n)}\ldots\mathbf{E}^{(1)}$$

in which $\mathbf{M}^{(n+1)}$ is 3×6 matrix formed by the top three rows of $\mathbf{I}_2\left(\mathbf{S}^{(\mathbf{n+1})}\right)^{-1}$ and $\mathbf{M}^{(1)}$ is 6×3 matrix formed by the last four columns of $\mathbf{E}^{(1)}$. With this modification, the dispersion equations both for free plate and a half space involve only a 3×3 matrix and only this part of this matrix need to be computed. To make sure on these equations, it may be noted that they yield the dispersion equation for a uniform plate in the form

$$\left[\mathbf{G}^{(\mathbf{2})}\right] = 0 \tag{35}$$

which is in agreement with the exiting solutions by Abubakar [11] when coupling constant is zero. Similarly, the dispersion equation for a medium half-space reduces to the usual Rayleigh wave equation.

IV. Free Single Layer Plate

A. The Monoclinic Case

For monoclinic materials having x_1-x_2 as a plane of mirror symmetry , and x_1-x_3 as the plane of incidence, the equations of motion for monoclinic plate can be written as :

$$\begin{aligned} &c_{11}u_{1,11} + c_{55}u_{1,33} + (c_{13} + c_{55})u_{3,13} \\ &\qquad = \rho\ddot{u}_1 - c_{16}u_{2,11} - c_{45}u_{2,33} \\ &c_{16}u_{1,11} + c_{45}u_{1,33} + (c_{36} + c_{45})u_{3,13} \\ &\qquad = \rho\ddot{u}_2 - c_{66}u_{2,11} - c_{44}u_{2,33} \\ &(c_{13} + c_{55})u_{1,13} + c_{55}u_{3,11} + c_{33}u_{33} \\ &\qquad = \rho\ddot{u}_3 - (c_{36} + c_{45})u_{2,13} \end{aligned} \tag{36}$$

The use of solutions (10) in the form

$$(u_1, u_2, u_3) = (U_1, U_2, U_3)\exp[i\xi(n_1x_1 + n_2x_2 + n_3x_3 - ct)], \tag{37}$$

j = 1, 2, 3

where (n_1, n_2 , n_3) = ($\sin\theta$, 0 , α) , θ is the angle of incidence, α is still an unknown parameters, U_1, U_2 , and U_3 are respectively the amplitudes of the displacements u_1, u_2 and u_3 . Although solutions (37) are explicitly independent of x_2 , an implicit dependence is contained in the transformation and the transverse displacement component u_2 is non-vanishing in eq. (37). The choice of solutions leads to three coupled eqs.

$$M_{mn}(\alpha)U_n = \mathbf{0}, \quad m, n=1,2,3 \tag{38}$$

where

$$M_{11} = F_{11} + c_2\alpha^2,\ M_{12} = F_{12} + c_5\alpha^2,\ M_{13} = F_{13}\alpha,$$
$$M_{22} = F_{22} + c_6\alpha^2,\ M_{23} = F_{23}\alpha,\ M_{33} = F_{33} + c_1\alpha^2$$
$$F_{11} = \sin^2\theta - \zeta^2,$$
$$F_{12} = c_4\sin^2\theta, F_{22} = c_3\sin^2\theta - \zeta^2, \quad F_{23} = c_8\sin\theta\alpha,$$
$$F_{33} = c_2\sin^2\theta - \zeta^2,\ F_{13} = c_7\sin\theta,$$

and

$$c_1 = c_{33}/c_{11}, c_2 = c_{55}/c_{11}, c_3 = c_{66}/c_{11},$$
$$c_4 = c_{16}/c_{11}, c_5 = c_{45}/c_{11}, c_6 = c_{44}/c_{11},$$
$$c_7 = (c_{13} + c_{55})/c_{11}, c_8 = (c_{36} + c_{45})/c_{11},$$
$$\zeta^2 = c^2\rho/c_{11}. \tag{39}$$

The system of eqs. (38) has a non-trivial solution if the determinant of the coefficients of U_1, U_2, and U_3 vanishes, which yields a sixth-degree polynomial equation relating α to c, which can be written as

$$\alpha^6 + A_1\alpha^4 + A_2\alpha^2 + A_3 = 0. \tag{40}$$

where

$$A_1 = \begin{bmatrix}(c_1c_6F_{11} - c_6F_{13}^2 + c_5F_{13}F_{23} - c_5^2F_{33}\\ -c_2F_{23}^2 - 2c_1c_5F_{12} + c_5F_{21}F_{23} + c_2c_6F_{33})\end{bmatrix}\Big/\Delta$$
$$A_2 = \begin{bmatrix}(F_{11}F_{23}^2 - F_{13}^2F_{22} + c_6F_{11}F_{33} + c_1F_{11}F_{22}\\ -c_1F_{12}^2 + 2F_{12}F_{13}F_{23} + 2c_5F_{12}F_{33} + c_2F_{22}F_{33})\end{bmatrix}\Big/\Delta$$
$$A_3 = \left\{(F_{11}F_{22} - F_{12}^2)F_{33}\right\}\Big/\Delta$$
$$\Delta = (c_2c_6 - c_5^2)c_1$$

This equation admits eight solutions for α (having the properties)

$$\alpha_2 = -\alpha_1,\ \alpha_4 = -\alpha_3,\ \alpha_6 = -\alpha_5. \tag{41}$$

Incorporating eqs. (36) in (15) to (28), and inspecting the resulting relations, we conclude that monoclinic symmetry implies further restrictions

$$q_{2(l+1)} = q_{2(l)},\ q_{3(l+1)} = -q_{3(l)},\ l = 1, 3, 5$$
$$\mathrm{r}_{33(l+1)} = \mathrm{r}_{33(l)},\ r_{13(l+1)} = -r_{13(l)},\ r_{23(l+1)} = -r_{23(l)}, \tag{42}$$

where now

$$r_{33(l)} = i\xi[(c_7 - c_2)\sin\theta + (c_8 - c_5)q_{2(l)} + c_1\alpha_l q_{3(l)_l}],$$
$$r_{13(l)} = i\xi[c_2(\alpha_l + q_{3(l)}\sin\theta) + c_5\alpha_l q_{2(l)}],$$
$$r_{23(l)} = i\xi[c_5(\alpha_l + q_{3(l)}\sin\theta) + c_6\alpha_l q_{2(l)}]. \tag{43}$$

$$q_{1(l)} = 1, q_{2(l)} = \frac{\Gamma_{12}\Gamma_{13} - \Gamma_{23}\prod_{11}}{\Gamma_{12}\Gamma_{23} - \Gamma_{13}\prod_{22}},$$

and $q_{3(l)} = (\frac{-\prod_{11}}{\Gamma_{13}}) - (\frac{\Gamma_{13}}{\Gamma_{12}})q_{2(l)}$ are defined in (11)

Using the analysis of above section and Eqs. (42) and (43) into equation (28), then after algebraic manipulations and reductions to the determinant equation (28), it reduce and partitioned to a 2×2 diagonal matrix whose entries comprise of 3×3 square matrices. The determinant can therefore be separated, leading to the two uncoupled characteristic equations

$$\sum_{k=1}^{3}(-1)^{k+1}r_{33(k)}G_k\tan^{\mp}(\gamma\alpha_k) = 0. \tag{44}$$

Consequent to symmetric and antisymmetric modes of vibrations, respectively, with

$$G_1 = r_{13(3)}r_{23(5)} - r_{13(5)}r_{23(3)}, G_3 = r_{13(1)}r_{23(5)} - r_{13(5)}r_{23(1)},$$
$$G_5 = r_{13(1)}r_{23(3)} - r_{13(3)}r_{23(1)}, \quad \gamma = \xi d/2 = \omega d/2. \tag{45}$$

B. Orthotropic Materials

1) Propagation off Principal Axes

If x_1 and x_2 are chosen to coincide with the in-plane principal axes for orthotropic symmetry, then we have

$$c_{j6} = 0,\ c_{45} = 0,\ j = 1,2,3. \tag{46}$$

Results for possessing transverse isotropy, can be easily obtained by noting the additional conditions imposed by symmetry, namely

$$c_{33} = c_{22},\ c_{13} = c_{12},\ c_{55} = c_{66},\ 2c_{44} = c_{22} - c_{23}$$

and $c_{11} = c_{22} = c_{33},\ c_{12} = c_{13} = c_{23},\ c_{44} = c_{55} = c_{66}$

(for cubic symmetry)

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu,\ c_{12} = c_{13} = c_{23} = \lambda,$$
$$c_{44} = c_{55} = c_{66} = \mu \tag{47}$$

(for the isotropic case)

2) Axis of Rotational Symmetry

Returning to the case of orthotropic symmetry, we substitute from eq. (46), which particularize the constitutive relations to orthotropic media, into the coefficients of the Appendix A. Inspection of the resulting entries leads to the conclusion that, for propagation along rotational symmetry axes, the matrix elements c16 , c26 , c36 ,and c45 vanish implies that $c_4 = c_5 = c_8 = 0$, $F_{12} = F_{23} = 0$, also vanish, consequently, $M_{12} = M_{23} = 0$ in eq. (38). As a consequence, eq. (40), reduces to

$$c_1c_2\alpha^4 + (c_2F_{33} - c_1F_{11} - F_{13}^2)\alpha^2 + F_{11}F_{33} = 0, \tag{48}$$

and

$$\alpha_5 = -\alpha_6 = \sqrt{(\zeta^2 - c_3\sin^2\theta)/c_6} \tag{49}$$

Here α_5, α_6 corresponds to SH motion, it means that SH wave motion uncouple from the rest of the motion and gives a purely transverse wave, which propagates without dispersion or damping.

Equation (48) correspond to the sagittal plane waves, and for this, it is noticed that for each α_l (l =1, 2, 3, 4) the displacements, stress amplitudes reduce to

$$q^{/}_{3(l)} = -(F_{11} + c_2\alpha_l^2)/F_{13}\alpha_l \,, \tag{50}$$

$$r^{/}_{33(l)} = [i\xi\{(c_7 - c_2)\sin\theta + c_1\alpha_l q^{/}_{3(l)}\}],$$

$$r^{/}_{13(l)} = i\xi[c_2(\alpha_l + q^{/}_{3(l)}\sin\theta)] \,. \tag{51}$$

For the SH type wave, one now has

$$r_{23(6)} = -r_{23(5)} = c_6\alpha_5 \,. \tag{52}$$

By employing the new relations (47), and following the matrix reduction steps used in obtaining the results of eqs. (44), one gets the reduced coupled characteristic equations

$$r^{/}_{33(1)}r^{/}_{13(3)}\tan^{\mp}(\gamma\alpha_1) - r^{/}_{33(3)}r^{/}_{13(1)}\tan^{\mp}(\gamma\alpha_3) = 0, \tag{53}$$

$$\sin(2\gamma\alpha_5) = 0 \tag{54}$$

where γ is defined in eq. (45), and eqs. (53) Constitute the characteristic equations for symmetric and antisymmetric modes of vibrations, propagating along an in-plane axis of symmetry of a plate. Equation (54) corresponds to SH motion and studied in detail by [20]. Furthermore, the relations (53) implicitly contain corresponding results for transversely isotropic, cubic, and isotropic material. Here one needs only to exploit the appropriate restrictions on the elastic properties as described in eqs. (47).

V. Semi-Infinite Medium Single Layer Plate

A. Wave Propagates in an Arbitrary Direction of a Monoclinic Material

In order to have surface wave, the roots α_i^2 , *l =1,2,3* of (40) must be either negative (so that square roots are purely imaginary) or complex numbers: this ensure that the superposition of partial waves has the properties of "exponential decay." There are two cases :

(a) α_i^2 i = 1,2,3 all are negative; and

(bi) α_1^2, is negative $\alpha_2^2 = \alpha_3^{2*}$, are complex conjugates.

For the case (a), as thickness tends to infinity, $\{\tan(\gamma\alpha_i)\}^{\pm} \rightarrow \pm i$ so that we have eq. (44a)

For the case (b), as thickness tends to infinity ,$\{\tan(\gamma\alpha_i)\}^{\pm} \rightarrow \pm i$ and if $\alpha_2^2 = a + ib, \alpha_3^2 = a - ib$, $b > 0$, then $\{\tan(\gamma\alpha_l)\}^{\pm} \rightarrow \pm i$ and $\{\tan(\gamma\alpha_{l+1})\}^{\pm} \rightarrow \mp i$, thus become eq. (44b) as

$$r_{33(1)}G_1 + r_{33(3)}G_3 \pm r_{33(5)}G_5 = 0 \,. \tag{55}$$

Surface wave velocity can be obtained by solving these equations.

B. *Wave Propagation in Principal Direction* (say x_1 direction)

Similar to the situation described in subsection 1, we have two cases :

(a) α_i^2, i = 1, 2 all are negative; and

(b) $\alpha_1^2 = \alpha_2^{2*}$, are complex conjugates.

Equations (53) become

$$r^{/}_{33(1)}r^{/}_{13(3)} \pm r^{/}_{33(3)}r^{/}_{13(1)} = 0, \tag{56}$$

This equation reduces to the well known Rayleigh wave equation for isotropic media.

VI. Special Case

When $\theta = 90\,\text{deg.}$ results obtained agree with the corresponding result obtained and discussed in detail by Nayfeh and Chementi [13] and Yan Li & R. B Thomson[19].

VII. Numerical Discussion And Conclusions

Numerical calculations of phase velocity verses wave number are carried out based on the expression (53) are computed for orthotropic, transversely isotropic and aluminum-epoxy composite plates, whose physical data are given as orthotropic [13]

C_{11} = 128GPa, C_{12} = 7 GPa, C_{13} = 6GPa, C_{22} = 72GPa,
C_{23} = 25GPa, C_{33} = 32GPa, C_{44} = 18GPa, C_{55} = 12.25 GPa,
C_{66} = 8 GPa, C_{13} = 6 GPa, ρ = 2.0 g/cm^3.

Transversely isotropic (Graphite-Epoxy)
C_{11} = 155.6GPa, $C_{12} = C_{13}$ = 3.7GPa, C_{23} = 4.33 GPa,
$C_{22} = C_{33}$ = 15.95GPa, C_{44} = 5.81GPa,
$C_{55} = C_{66}$ = 7.46 GPa, ρ = 1.6 g/cm^3.

Dispersion curves the symmetric and antisymmetric modes for orthotropic plate are plotted in Fig. 1and Fig 2, for transversely isotropic plate in Fig.3 and Fig 4, and finally for isotropic plates are illustrated in Fig 5 and Fig 6.

Phase velocity vs. wave number curves plotted as a function of wave number ξ. It is observed that the phase velocities of lowers, symmetric and antisymmetric modes, is more effected at zero wave number limits, and little variation is observed as the wave number increases, and all the curves approaches each others at high ξ where the phase velocity tends towards the Rayleigh surface wave speed. of phase velocity verses wave number are carried out based on the expression Using expression (35) phase velocity verses wave number giving the surface wave speed is plotted in Fig.7. Each of figure exhibit coupled wave speeds (quasi-longitudinal, quasi-transverse etc.) due to the anisotropic effect as the distinction between the mode types is somewhat artificial. At zero wave number limits, for the higher value wave numbers higher modes appear in both cases (symmetric and antisymmetric) with ξ increases. One of the modes seems to be associated with quick change in the slope of the mode. Lower modes are found to highly influence at low values of wave number both in symmetric and antisymmetric modes, while in higher modes, change is observed at high values of wave number.

In this article, exact formal solution for the displacements, and stresses, in infinite plates of arbitrary anisotropy of finite thickness are derived. Dispersion relations are derived for elastic waves for more specialized case of a monoclinic plate in closed form and separate the mathematical conditions for symmetric and antisymmetric are obtained. The cases of free layered plate and layered half space are considered special

cases have also been deduced and discussed separately. Results for elastic plates of orthotropic, transversely isotropic, cubic and isotropic materials are implicitly contained in the analysis. The SH wave gets decoupled from the rest of motion and if propagation occurs along an in-plane axis of symmetry and it propagates without dispersion or damping.

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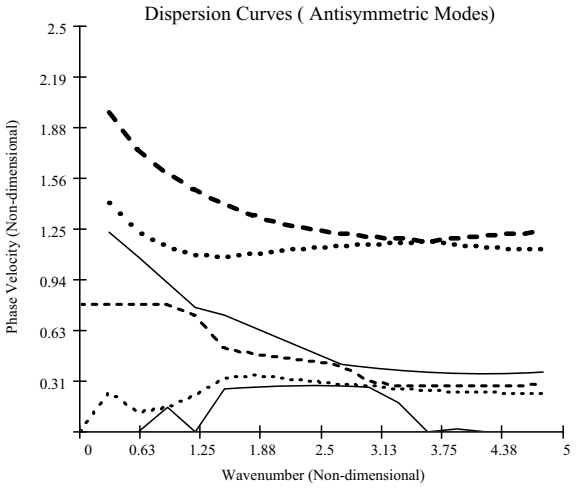

Fig. 1 Phase velocity vs wave number antisymmetric modes for orthotropic plate

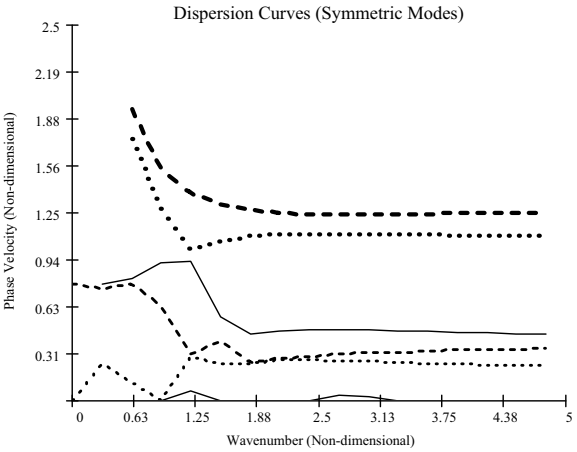

Fig. 2 Phase velocity vs wave number symmetric modes for orthotropic plate

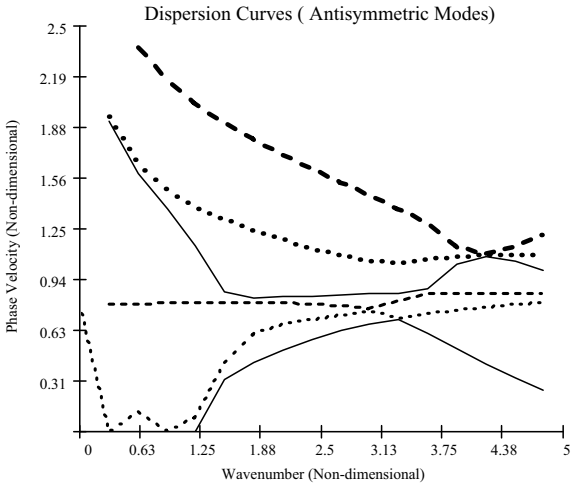

Fig. 3 Phase velocity vs wave number antisymmetric modes for transversely isotropic plate

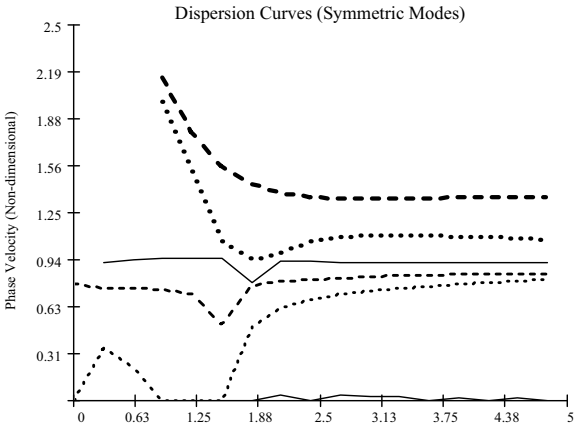

Fig. 4 Phase velocity vs wave number symmetric modes for transversely isotropic plate

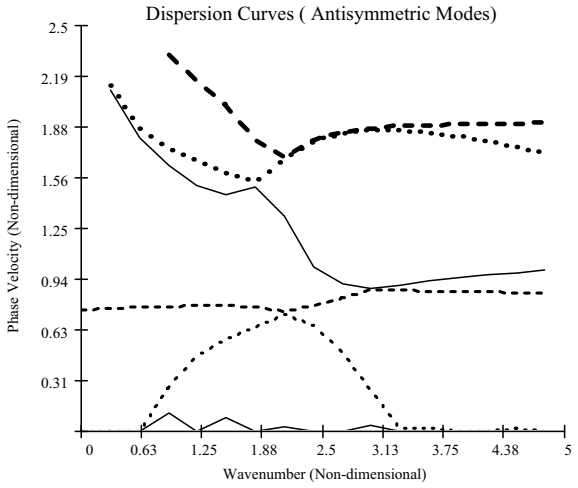

Fig. 5 Phase velocity vs wave number antisymmetric modes for aluminum-epoxy plate

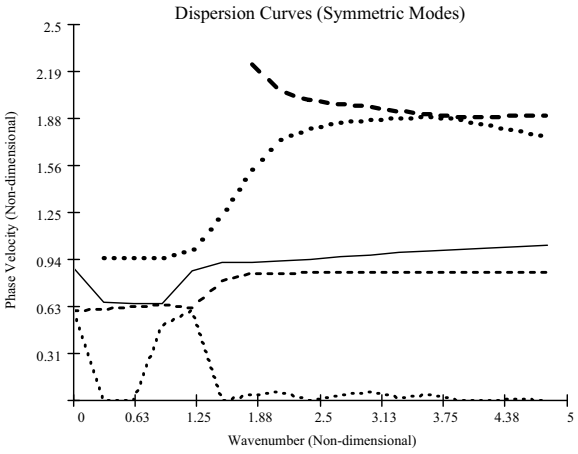

Fig. 6 Phase velocity vs wave number symmetric modes for aluminum-epoxy plate

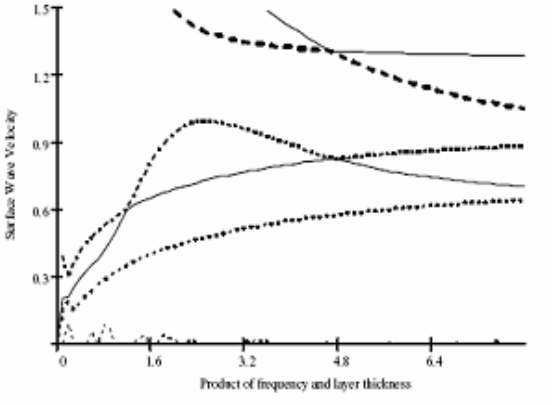

Fig. 7 Surface wave velocity vs product of frequency and layer thickness