On the solution of fully fuzzy linear systems

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Abstract—A linear system is called a fully fuzzy linear system (FFLS) if quantities in this system are all fuzzy numbers. For the FFLS, we investigate its solution and develop a new approximate method for solving the FFLS. Observing the numerical results, we find that our method is accurate than the iterative Jacobi and Gauss-Seidel methods on approximating the solution of FFLS.

Keywords—fully fuzzy linear equations, iterative method, homotopy perturbation method, approximate solutions

I. INTRODUCTION

THe topic of fully fuzzy linear systems which attracted growing interest for some time, in particular in relation to option valuation or portfolio selection, have been rapidly develop in recent years. The first step which gives a solution of FFLS is followed by introducing a condition for the existence of unique solution to FFLS. Finally, a numerical method, called block homotopy perturbation method (HPM), are developed for computing approximations of FFLS. Prior to discussing the solution and their associate numerical methods, it is necessary to present an appropriate introduction to pre-liminary topics such as fuzzy numbers and fuzzy arithmetics.

Besides finding the solution of FFLS, Friendman et al. [10] study the solution of the $n \times n$ fuzzy system of linear equations (FSLE) whose coefficient matrix is crisp and the right hand side is an arbitrary fuzzy vector. To find the solution, the original $n \times n$ fuzzy linear system is replaced with a $(2n) \times (2n)$ crisp linear system. They also derive conditions for the existence of a unique fuzzy solution to FSLE and design a numerical procedure for calculating the solution. So far, there are many numerical methods such as [1-3, 5, 12] are proposed to find the solution of the $(2n) \times (2n)$ crisp linear systems.

Recently, many papers are proposed to investigate the solution of FFLS. Buckley and Qu [4] took the α-levels of the equations and obtain linear systems of interval equations. The exact solution of FFLS can be found by solving these interval systems. However, the classical solution to an FFLS often fails to exist [15] since the solution to systems of linear interval equations is not necessarily intervals [11].

For finding the solution of a nonnegative FFLS, Dehahan and Hashemi [6] made modifications to existing methods, such as the direct method, LU decomposition method, linear programming method, Jacobi iterative method and Adomian decomposition method. On the other hand, Dehghan et al. [8] found that their methods are not available for the nonnegative solution.

When quantities are either positive nor negative in FFLS, there is a lack of discussion on their solutions. The purpose of this paper is to propose a solution of an arbitrary FFLS. the FFLS are now transformed into two crisp linear systems. Recently many numerical methods, such as [1-3, 7], are proposed for approximation the solution of large scale linear systems. In this paper, we develop a new method, called the block HPM, for the approximations of the sparse liner systems. The numerical results indicate that the block HPM converges to the exact solution rapidly than the iterative Jacobi and Gauss-Seidel methods.

However, we find that our solution is no longer a triangle fuzzy number. Hence, one of the future studies is to derive conditions for that the solution is a triangle fuzzy number. On the other hand, many financial problems such as portfolio se-lection problems or the option valuation problems are modeled as linear systems. For modeling the uncertainty, these financial models are said to operate as FFLS. Hence, one of the other future directions in our studies is to apply the results toward analyzing uncertain financial models.

This paper is organized as follows. In Section 2, the basic results of the fuzzy numbers and fuzzy calculus are discussed and a definition for the solution of FFLS is derived. In Section 3, the block homotopy perturbation method is adapted find an approximation of the solution. In Section 4, two numerical examples which indicates accuracy of the block HPM are displayed. In Section 5, we close this paper with a concise conclusion.

II. FUZZY NUMBERS AND FUZZY OPERATIONS

In this section, an appropriate brief introduction to prelimi-nary topics such as fuzzy numbers and fuzzy calculus will be introduced and the definition for FFLS will be provided.

Definition 2.1: A fuzzy number A is of LR-type if there exist reference function L (for left), R (for right), and scalars $\alpha > 0$, $\beta > 0$ with $A = (a, \alpha, \beta)$, if its membership function has the following forms:

$$\mu_A(x) = \begin{cases} L(\frac{a-x}{\alpha}) & \text{if } x \leq a, \\ 1 & \text{if } x = a, \\ R(\frac{x-a}{\beta}) & \text{if } x \geq a. \end{cases}$$

where a is the center of F and $\alpha \geq 0$ and $\beta \geq 0$ are the left and right spreads, respectively.
Here, we say that $\tilde{A} = (a, \alpha, \beta)$ is positive if $a - \alpha > 0$ and that two fuzzy numbers $A = (a, \alpha, \beta)$ and $B = (b, \gamma, \delta)$ is equal if $a = b$, $\alpha = \gamma$ and $\beta = \delta$.

For any two fuzzy numbers, we define the following oper-ations (Dubois, 1980).
Definition 2.2: Let $\mathcal{F}$ be the set of LR-fuzzy numbers. If $A = (a, \alpha, \beta)$ and $B = (b, \gamma, \delta)$ are two fuzzy numbers in $\mathcal{F}$, then two operations, say addition and multiplication, are defined as:

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- Addition

$$(a, \alpha, \beta) \oplus (b, \gamma, \delta) = (a + b, \alpha + \gamma, \beta + \delta).$$

- Approximate multiplication
  If $A > 0$ and $B > 0$, then

$$(a, \alpha, \beta) \otimes (b, \gamma, \delta) = (ab, b\alpha + a\gamma, b\beta + a\delta).$$

  If $A < 0$ and $B > 0$, then

$$(a, \alpha, \beta) \otimes (b, \gamma, \delta) = (ab, b\alpha - a\delta, b\beta - a\gamma).$$

  If $A < 0$ and $B < 0$, then

$$(a, \alpha, \beta) \otimes (b, \gamma, \delta) = (ab, -b\beta - a\delta, -b\alpha - a\gamma).$$

On the other hand, the fuzzy matrix is defined as follows.

Definition 2.3: A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix if each element in A is a fuzzy number.

In matrix representation, the matrix $\tilde{A}$ can be represented as $\tilde{A} = (A, M, N)$ since $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})$. Three crisp matrices $A = (a_{ij})$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$ are called the center matrix and the right and left spread matrices, respectively,

III. FULLY FUZZY LINEAR SYSTEMS

In this section, we give a difinition for FFLS whse solution is proposed following the definition.

Definition 3.1: We consider FFLS as the form:

$$(\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1,$$
$$(\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2,$$
$$\vdots \qquad\qquad (1)$$
$$(\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \cdots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n.$$

where $\tilde{a}_{ij}$, $\tilde{x}_i$ and $\tilde{b}_i$ are all fuzzy numbers.

The matrix form of this linear system is represented as

$$\tilde{A} \otimes \tilde{x} = \tilde{b}$$

where $\tilde{A} = (\tilde{a}_{ij}) = (A, M, N)$, $1 \le i, j \le n$ is an $n \times n$ fuzzy matrix, $\tilde{x} = (\tilde{x}_i) = (x, y, z)$ and $\tilde{b} = (\tilde{b}_i) = (b, g, h)$, $1 \le i \le n$ are $n \times 1$ fuzzy matrices.

In the following theorem, we derive a solution of arbitrary FFLS.

Theorem 3.2: A fuzzy number vector $(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ driven by

$$\tilde{x}_i = (\tilde{x}_i, \tilde{y}_i, \tilde{z}_i), \quad 1 \le i \le n, \quad 0 \le r \le 1,$$

is called a solution of the fuzzy system of

$$Ax = b,$$
$$A^+ y - A^- z = g - (M^+ x^+ + N^+ x^-)$$
$$\qquad\qquad - (M^- x^+ - N^- x^-),$$
$$-A^- y + A^+ z = h - (N^+ x^+ - M^+ x^-)$$
$$\qquad\qquad - (N^- x^+ - M^- x^-),$$

where $A = A^+ + A^-$, $x = x^+ + x^-$ and $M^\pm$, $N^\pm$ are defined in the proof.

Proof: Let $\tilde{a}_i = \{\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in}\}$ be a fuzzy vector with $\tilde{a}_{ij} = (a_{ij}, m_{ij}, n_{ij})$, $1 \le j \le n$, then two fuzzy vectors $\tilde{a}_i^+ = \{\tilde{a}_{i1}^+, \tilde{a}_{i2}^+, \ldots, \tilde{a}_{in}^+\}$ and $\tilde{a}_i^- = \{\tilde{a}_{i1}^-, \tilde{a}_{i2}^-, \ldots, \tilde{a}_{in}^-\}$ with

$$\tilde{a}_{ij}^+ = (a_{ij}^+, m_{ij}^+, n_{ij}^+) \quad \text{and} \quad \tilde{a}_{ij}^- = (a_{ij}^-, m_{ij}^-, n_{ij}^-)$$

are defined as follows.

(1) If $a_{ij} \ge 0$ then

$$a_{ij}^+ = a_{ij}, \quad m_{ij}^+ = m_{ij}, \quad n_{ij}^+ = n_{ij}, \text{ and}$$
$$a_{ij}^- = 0, \qquad m_{ij}^- = 0, \qquad n_{ij}^- = 0.$$

(2) If $a_{ij} < 0$ then

$$a_{ij}^+ = 0, \qquad m_{ij}^+ = 0, \qquad n_{ij}^+ = 0, \text{ and}$$
$$a_{ij}^- = a_{ij}, \quad m_{ij}^- = m_{ij}, \quad n_{ij}^- = n_{ij}.$$

This implies

$$\tilde{a}_i = \tilde{a}_i^+ \oplus \tilde{a}_i^-.$$

That is $\tilde{a}_i^+$ and $\tilde{a}_i^-$ are consisted of the components in $\tilde{a}_i$ with the nonnegative center and the negative center, respectively.

Using the same method to $\tilde{x}_i = (x_i, y_i, z_i)$, we define two fuzzy vectors $\tilde{x}_i^+ = (x_i^+, y_i^+, z_i^+)$ and $\tilde{x}_i^- = (x_i^-, y_i^-, z_i^-)$ such that

$$\tilde{x}_i = \tilde{x}_i^+ \oplus \tilde{x}_i^-.$$

Applying the approximate multiplication rule, we get

$$\tilde{a}_i \otimes \tilde{x}$$
$$= (\tilde{a}_i^+ \oplus \tilde{a}_i^-) \otimes (\tilde{x}^+ \oplus \tilde{x}^-)$$
$$= [(a_i^+, m_i^+, n_i^+) \oplus (a_i^-, m_i^-, n_i^-)]$$
$$\quad \otimes [(x^+, y^+, z^+) \oplus (x^-, y^-, z^-)]$$
$$= (C, L, R),$$

where

$$C = \sum_{j=1}^n a_{ij} x_j^+ + \sum_{j=1}^n a_{ij} x_j^- = \sum_{j=1}^n a_{ij} x_j,$$
$$L = \sum_{j=1}^n a_{ij}^+ y_j$$
$$\quad + \sum_{j=1}^n (-a_{ij}^-) z_j + m_{ij}^+ x_j^+ + m_{ij}^- x_j^+ + n_{ij}^+ (-x_j^-) + n_{ij}^- (-x_j^-),$$
$$R = \sum_{j=1}^n a_{ij}^+ z_j$$
$$\quad + \sum_{j=1}^n (-a_{ij}^-) y_j + n_{ij}^+ x_j^+ + n_{ij}^- x_j^+ + m_{ij}^+ (-x_j^-) + m_{ij}^- (-x_j^-).$$

Here, $y_i = y_i^+ + y_i^-$ and $z_i = z_i^+ + z_i^-$.

Now, writing $\tilde{A} = \tilde{A}^+ \oplus \tilde{A}^-$ and $\tilde{x} = \tilde{x}^+ \oplus \tilde{x}^-$ yields

$$(b, g, h) = \tilde{A} \otimes \tilde{x} = (C_1, L_1, R_1),$$

where

$$C_1 = Ax,$$
$$L_1 = A^+ y - A^- z + M^+ x^+ + N^+ x^- + M^- x^+ - N^- x^-,$$
$$R_1 = A^+ z - A^- y + N^+ x^+ - M^+ x^- + N^- x^+ - M^- x^-.$$

Here $\tilde{A}^+ = (A^+, M^+, N^+)$, $\tilde{A}^- = (A^-, M^-, N^-)$, $\tilde{x}^+ = (x^+, y^+, z^+)$ and $\tilde{x}^- = (x^-, y^-, z^-)$.

Therefore, we get the solution of FFLS. ∎

Remark 3.3: Using matrix notation, Theorem 3.2 can be written as

$$Ax = b,$$
$$SY = B - S_1 X - S_2 X, \qquad (2)$$

where

$$S = \begin{bmatrix} A^+ & -A^- \\ -A^- & A^+ \end{bmatrix},$$

$$S_1 = \begin{bmatrix} M^+ & -N^+ \\ N^+ & -M^+ \end{bmatrix}, \quad (3)$$

$$S_2 = \begin{bmatrix} M^- & -N^- \\ N^- & -M^- \end{bmatrix},$$

and

$$X = [x^+, x^-], \quad Y = [y, z]^T \text{ and } B = [g, h]^T.$$

Theorem 3.4: The matrix S is nonsingular if and only if the matrices A and $A^+ - A^-$ are both nonsingular.

Proof: Since

$$\det \begin{bmatrix} A^+ & -A^- \\ -A^- & A^+ \end{bmatrix}$$

$$= \det \begin{bmatrix} A^+ + (-A^-) & A^+ + (-A^-) \\ -A^- & A^+ \end{bmatrix}$$

$$= \det \begin{bmatrix} A^+ + (-A^-) & 0 \\ -A^- & A^+ - (-A^-) \end{bmatrix}$$

$$= \det(A^+ + (-A^-))\det(A^+ - (-A^-)),$$

we have

$$\det \begin{bmatrix} A^+ & -A^- \\ -A^- & A^+ \end{bmatrix} \neq 0$$

if and only if

$$\det(A^+ + (-A^-))\det(A^+ - (-A^-)) \neq 0$$

which completes the proof. ∎

Now the solution of FFLS has been proposed. However to find the spread of the solution we should solve a $(2n) \times (2n)$ crisp sparse linear system with density less than 0.5. Recently, many iterative methods (Saad, 2000) such as iterative Jacobi and Gauss-Seidel methods have been proposed for solving a linear sparse system.

In the next section, we will develop a new approximate method, called the homotopy perturbation method, for approximating the solution of the sparse linear systems.

IV. THE HOMOTOPY PERTURBATION METHODS

First of all, we introduce the homotopy perturbation method (HPM) in a general mathematical setting. The convex homotopy $H(u, p)$ is defined by

$$H(u, p) = (1 - p)(Su - Z) + p(Qu - Z) = 0, \quad (4)$$

where

$$Z = B - S_1 X - S_2 X. \quad (5)$$

Hence, we have

$$H(u, 0) = Su - Z = 0, \text{ and}$$
$$H(u, 1) = Qu - Z = 0.$$

Using the homotopy parameter p as an expending parameter yields

$$u = u_0 + u_1 p + u_2 p^2 + \cdots \quad (6)$$

which gives an approximation to the solution of (2) as

$$v = \lim_{p \to 1}(u_0 + u_1 p + u_2 p^2 + \cdots).$$

By substituting (6) in (4) and equaling the terms with the identical power of p, we obtain

$$p^0: \quad Qu_0 - Z = 0$$
$$p^i: \quad Qu_i + (S - Q)u_{i-1} = 0, \ i = 1, 2, 3, \ldots$$

This implies

$$u_0 = Q^{-1}Z$$
$$u_i = (I - Q^{-1}S)^i u_{i-1}, \ i = 2, 3, \ldots$$

Therefore,

$$u = \sum_{i=0}^{\infty} u_i p^i = \sum_{i=0}^{\infty}[(I - Q^{-1}S)^i Q^{-1}Z]p^i.$$

Setting p=1, the solution can be of the form

$$u = \sum_{i=0}^{\infty}(I - Q^{-1}S)^i (Q^{-1}Z).$$

To verify that the series u converges, we use following theorems. Let $\rho(A)$ denote the spectral radius, that is

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|,$$

where $\sigma(A)$ denotes the set of all eigenvalues of A and is called the spectrum of A.

Saad (2000) shows that the series $\sum_{k=0}^{\infty} A^k$ converges if and only if $\rho(A) < 1$. Consequently, we obtain the following results.

Theorem 4.1: The series

$$u = \sum_{i=0}^{\infty}(I - Q^{-1}S)^i Q^{-1}Z$$

converges if and only if

$$\rho(I - Q^{-1}S) < 1.$$

A. Block homotopy perturbation methods

To find the solution of

$$SY = Z,$$

where

$$S = \begin{bmatrix} A^+ & -A^- \\ -A^- & A^+ \end{bmatrix},$$

the splitting matrix in the homotopy method is selected as follows

$$Q = \begin{bmatrix} A^+ & 0 \\ 0 & A^+ \end{bmatrix}$$

of which the inverse can be obtained as

$$Q^{-1} = \begin{bmatrix} (A^+)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{bmatrix}$$

for the special structure of S. This implies that

$$I - Q^{-1}S$$

$$= I_{2n} - \begin{bmatrix} (A^+)^{-1} & 0 \\ 0 & (A^+)^{-1} \end{bmatrix} \begin{bmatrix} A^+ & -A^- \\ -A^- & A^+ \end{bmatrix}$$

$$= I_{2n} - \begin{bmatrix} I_n & (A^+)^{-1}(-A^-) \\ (A^+)^{-1}(-A^-) & I_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(A^+)^{-1}(-A^-) \\ -(A^+)^{-1}(-A^-) & 0 \end{bmatrix}$$

and

$$(I - Q^{-1}S)^{-i} = \begin{cases} \begin{bmatrix} 0 & M^i \\ M^i & 0 \end{bmatrix}, & \text{for odd } i, \\ \begin{bmatrix} M^i & 0 \\ 0 & M^i \end{bmatrix}, & \text{for even } i, \end{cases}$$

where $M = -(A^+)^{-1}A^-$. Hence the homotopy series can be represented as

$$u = \sum_{i=0}^{\infty} \begin{bmatrix} M^{2i}(A^+)^{-1} & M^{2i+1}(A^+)^{-1} \\ M^{2i+1}(A^+)^{-1} & M^{2i}(A^+)^{-1} \end{bmatrix} Z.$$

Using the structure of S we now propose a simple method for calculating the homotopy series. The numerical results will be displayed in the next section.

V. NUMERICAL RESULTS

In this section, we first demonstrate the solution for the following a FFLS.

Example 5.1: Let $\tilde{A} = (A, M, N)$ and $\tilde{b} = (b, h, g)$ be a fully fuzzy matrix and a fully fuzzy vector with

$$A = \begin{bmatrix} 3 & 2 & 5 & 8 \\ -2 & 4 & 1 & -3 \\ 1 & 3 & 4 & 5 \\ 1 & -1 & 6 & 2 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.1 & 0.3 & 0.2 & 0.5 \\ 0.1 & 0.2 & 0.4 & 0.1 \\ 0.2 & 0.3 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{bmatrix},$$

$$N = \begin{bmatrix} 0.2 & 0.3 & 0.1 & 0.2 \\ 0.3 & 0.1 & 0.2 & 0.4 \\ 0.3 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.1 & 0.4 \end{bmatrix},$$

$$b = \begin{bmatrix} 3 & 6 & 2 & 8 \end{bmatrix}^T,$$

$$h = \begin{bmatrix} 0.8 & 0.9 & 0.4 & 0.7 \end{bmatrix}^T,$$

$$g = \begin{bmatrix} 0.9 & 0.7 & 0.8 & 0.4 \end{bmatrix}^T.$$

Then we consider the following FFLS

$$\tilde{A} \otimes \tilde{x} = \tilde{b}. \tag{7}$$

To find the solution of FFLS, we solve the following two crisp linear systems

$$Ax = b$$
$$SY = Z,$$

of which the solutions are

$$x = \begin{bmatrix} 3.9579832 & 1.2016807 & 1.697479 & -2.4705882 \end{bmatrix}$$
$$Y = \begin{bmatrix} y & z \end{bmatrix},$$

where

$$y = \begin{bmatrix} 0.441667 & -0.21529 & -0.289685 & -0.0169875 \end{bmatrix}$$
$$z = \begin{bmatrix} -0.5676463 & -0.4031918 & -0.2137236 & 0.2275997 \end{bmatrix}.$$

Here S, S_1 and S_2 are defined in (3) and Z is defined in (5). The solution of (7) is obtained. However, the negative spreads of its solution are available in this example. This implies that the solution of (7) is not a strong fuzzy number.

Example 5.2: To compare the block homotopy perturbation method with the direct methods and iteration methods (Yang, 2005) we applied these methods to solve the system

$$SX = Z,$$

where $S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_1 \end{bmatrix}$ is a 1000×1000 matrix with $\rho(I - S) = 0.5167$ and Z is a 1000×1 vector. The errors, computation times and iterations for the direct methods, iterative methods and the block HPM are displayed in Table I. As the real solution is not known, errors are computed by the maximum absolute difference (MAD) between SX and Z. As we can see the errors for the methods are all less than 10^{-6}. The computation times for the direct methods, such as Gauss-Jordan (GJ) elimination and LU decomposition, are greater than 10^4 but the computation time for the block HPM is less than 10^{-2}. This implies that the block HPM converge rapidly for large sparse systems with small spectrum radius. To compare the block HPM with iterative methods, we find that under the same degree of error the computation time of the block HPM is far less than that of Gauss-Seidel iterative method. On the other hand, the degree of error is about 10^{-15} for the block HPM after 30 iterations, but the degree of error is only 10^{-6} for the Jacobi iterative method after 30 iterations. Hence we get that the accuracy of the block HPM is higher than that of Jacobi iterative method.

TABLE I
THE DIRECT METHOD, ITERATIVE METHOD AND HOMOTOPY
PERTURBATION METHOD

Direct methods		
Method	GJ	LU
Iterations		
MAD	3.52×10^{-15}	3.44×10^{-15}
Time (Sec.)	1.42×10^4	1.19×10^4

Iterative methods		
Method	Jacobi	Gauss-Seidel
Iteration	30	30
MAD	7.41×10^{-6}	9.97×10^{-8}
Time (Sec.)	0.0129	0.4019

Block HPM		
Iteration	15	30
MAD	1.51×10^{-8}	2.33×10^{-15}
Time (Sec.)	0.0262	0.0492

VI. Conclusion

In this paper, we have proposed a solution of FFLS and developed a block HPM for finding an approximation of the solution. The numerical results display that the block HPM converges to the exact solution rapidly than the direct method and the iterative Jacobi and Gauss-Seidel methods on solving the FFLS. However, we find that the solution for a triangle FFLS is no longer a triangle fuzzy vector. One of the future studies is to derive conditions for that the solution of a triangle FFLS is a triangle fuzzy vector.

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