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On the fuzzy difference equation $x_{n+1} = A + \Sigma_{i=0}^{k} \frac{B_i}{x_{n-i}}$

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Abstract—In this paper, we study the existence, the boundedness and the asymptotic behavior of the positive solutions of a fuzzy nonlinear difference equations

$$x_{n+1} = A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}}, \quad n = 0, 1, \cdots.$$

where (x_n) is a sequence of positive fuzzy numbers, A, B_i and the initial values $x_{-k}, x_{-k+1}, \cdots, x_0$ are positive fuzzy numbers. $k \in \{0, 1, 2, \cdots\}$.

Keywords—Fuzzy difference equation, Boundedness, Persistence, Equilibrium point, Asymptotic behavior

I. INTRODUCTION

IT is known that difference equation appears naturally as discrete analogous and as numerical solutions of differential equations and delay differential equation having many applications in economics, biology, computer science, control engineering, etc.(see, for example, [1-5] and the references therein). Recently there has been a lot of work concerning the oscillatory behavior, the periodicity, and the boundedness of nonlinear difference equations. Moreover similar results in [6] have been derived for systems of two nonlinear difference equations. A fuzzy difference equation is a difference equation where constants and the initial values are fuzzy numbers, and its' solutions are sequences of fuzzy numbers. Recently there is an increasing interest concerning with investigation of fuzzy difference equation(see, for example, [7-13]).

In [10] G. Papaschinopoulos and B. K. Papadopoulos studied the following fuzzy difference equation

$$x_{n+1} = A + \frac{B}{x_n}, \quad n = 0, 1, \cdots, \tag{1}$$

where (x_n) is a sequence of fuzzy numbers and A, B, x_0 are positive fuzzy numbers.

In this paper we study the following fuzzy nonlinear difference equation

$$x_{n+1} = A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}}, \quad n = 0, 1, \cdots, \tag{2}$$

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where (x_n) is a sequence of positive fuzzy numbers, A, B_i and the initial values $x_{-k}, x_{-k+1}, \cdots, x_0$ are positive fuzzy numbers. $k \in \{0, 1, 2, \cdots\}$.

We need the following definitions:
A is said to be a fuzzy number if $A : R \to [0, 1]$ satisfies the below (i)-(iv)
(i) A is normal, i.e. there exists an $x \in R$ such that $A(x) = 1$;
(ii) A is fuzzy convex, i.e. for all $t \in [0, 1]$ and $x_1, x_2 \in R$ such that

$$A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\};$$

(iii) A is upper semi-continuous;
(iv) The support of A, supp$A$ = $\overline{\bigcup_{\alpha\in(0,1]}[A]_\alpha}$ = $\overline{\{x : A(x) > 0\}}$ is compact.

The $\alpha-$cuts of A are denoted by $[A]_\alpha = \{x \in R : A(x) \geq \alpha\}, \quad \alpha \in [0, 1]$, it is clear that the $[A]_\alpha$ are closed interval. We say that a fuzzy number is positive if supp$A \subset (0, \infty)$.

It is obvious that if A is a positive real number then A is a fuzzy number and $[A]_\alpha = [A, A], \alpha \in (0, 1]$. Then we say that A is a trivial fuzzy number.

Let A, B be fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}], \alpha \in (0, 1]$. We define a norm on fuzzy numbers space as follows:

$$\|A\| = \sup_{\alpha\in(0,1]} \max\{|A_{l,\alpha}|, |A_{r,\alpha}|\}.$$

We take the following metric:

$$D(A, B) = \sup_{\alpha\in(0,1]} \max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}.$$

The fuzzy analog of the boundedness and persistence (see [9,11]) as follows: we say that a sequence of positive fuzzy numbers (x_n) persists (resp. is bounded) if there exists a positive real number M (resp. N) such that

$$\text{supp}x_n \subset [M, \infty)(\text{resp. supp}x_n \subset (0, N]), \quad n = 1, 2, \cdots.$$

We say that x_n is bounded and persists if there exist positive real numbers $M, N > 0$ such that

$$\text{supp}x_n \subset [M, N], \quad n = 1, 2, \cdots.$$

We say $(x_n), n = 1, 2, \cdots$, is an unbounded sequence if the norm $\|x_n\|, n = 1, 2, \cdots$, is an unbounded sequence.

We say that x_n is a positive solution of (2) if (x_n) is a sequence of positive fuzzy numbers which satisfies (2). We say a positive fuzzy number x is a positive equilibrium for (2) if

$$x = A + \sum_{i=0}^{k} \frac{B_i}{x}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x is a positive fuzzy number, Suppose that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0,1], \quad n = 0,1,2,\cdots, \tag{3}$$

and

$$[x]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0,1] \tag{4}$$

We say that the sequence (x_n) converges to x with respect to D as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

Suppose that (2) has a unique positive equilibrium x. We say that the positive equilibrium x of (2) is stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every positive solution x_n of (2), which satisfies $D(x_{-i}, x) \le \delta, i = 0,1,\cdots,k$ we have $D(x_n, x) \le \varepsilon$ for all $n > 0$.

Moreover, we say that the positive equilibrium x of (2) is asymptotically stable, if it is stable and every positive solution of (2) tends to the positive equilibrium of (2) with respect to D as $n \to \infty$.

The purpose of this paper is to study the existence of positive solutions of (2). Furthermore, we give some conditions so that every positive solution of (2) is boundedness and persistence. Finally, under some conditions we prove that (2) has a unique positive equilibrium x which is asymptotic stable.

II. Main results

Firstly we study the existence of the positive solutions of (1). We need the following lemma which is a slight generalization of Lemma 2.1 of [11].

Lemma 2.1. *Let $f : {R_+}^{2k+3} \to R_+$ be continuous , $A_i, i = 0,1,\cdots,2k+3$, are fuzzy numbers, Then*

$$[f(A_0,\cdots,A_{2k+3})]_\alpha = f([A_0]_\alpha,\cdots,[A_{2k+3}]_\alpha), \quad \alpha \in (0,1].$$

Theorem 2.1. *Consider equation (2) where A, B_i are positive fuzzy numbers. Then for any positive fuzzy numbers $x_{-i}, i = 0,1,\cdots,k$, there exists a unique positive solution x_n of (2).*

Proof. Suppose that there exists a sequence of fuzzy numbers (x_n) satisfying (2) with the initial values $x_{-k}, x_{-k+1},\cdots,x_0$. Consider the $\alpha-$cuts, $\alpha \in (0,1], n = -k,-k+1,\cdots,$

$$\begin{cases} [x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], [A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], \\ [B_i]_\alpha = [B_{i,l,\alpha}, B_{i,r,\alpha}], i = 0,1,\cdots,k \end{cases} \tag{5}$$

Then from (2), (5) and Lemma 2.1 it follows that

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}}\right]_\alpha \\ &= [A]_\alpha + \sum_{i=0}^{k} \frac{[B_i]_\alpha}{[x_{n-i}]_\alpha} \\ &= [A_{l,\alpha}, A_{r,\alpha}] + \sum_{i=0}^{k} \frac{[B_{i,l,\alpha}, B_{i,r,\alpha}]}{[L_{n-i,\alpha}, R_{n-i,\alpha}]} \\ &= \left[A_{l,\alpha} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha}}{R_{n-i,\alpha}}, A_{r,\alpha} + \sum_{i=0}^{k} \frac{B_{i,r,\alpha}}{L_{n-i,\alpha}}\right] \end{aligned}$$

from which we have that for $n = 0,1,\cdots, \alpha \in (0,1]$

$$\begin{cases} L_{n+1,\alpha} = A_{l,\alpha} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha}}{R_{n-i,\alpha}}, \\ R_{n+1,\alpha} = A_{r,\alpha} + \sum_{i=0}^{k} \frac{B_{i,r,\alpha}}{L_{n-i,\alpha}} \end{cases} \tag{6}$$

Then it is obvious that for any $(L_{j,\alpha}, R_{j,\alpha}), j = -k,-k+1,\cdots,0$, there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$ with the initial conditions $(L_{j,\alpha}, R_{j,\alpha}), j = -k,-k+1,\cdots,0, \alpha \in (0,1]$.

Conversely we prove that $[L_{n,\alpha}, R_{n,\alpha}]$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of the system (6) with the initial values $(L_{-i,\alpha}, R_{-i,\alpha}), i = 0,1,\cdots,k$, determines the solution x_n of (2) with the initial values $x_{-k}, x_{-k+1},\cdots,x_0$ such that (3) holds.

From Theorem 2.1 of [14] and since $A, B_i, x_{-i}, i = 0,1,\cdots,k$, are positive fuzzy numbers for any $\alpha \in (0,1]$, $\alpha_1 \le \alpha_2$, we have

$$\begin{cases} 0 < A_{l,\alpha_1} \le A_{l,\alpha_2} \le A_{r,\alpha_2} \le A_{r,\alpha_1} \\ 0 < B_{l,\alpha_1} \le B_{l,\alpha_2} \le B_{r,\alpha_2} \le B_{r,\alpha_1} \\ 0 < L_{-i,\alpha_1} \le L_{-i,\alpha_2} \le R_{-i,\alpha_2} \le R_{-i,\alpha_1} \end{cases} \tag{7}$$

We claim that

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}, \quad n = 0,1,2,\cdots. \tag{8}$$

We prove it by induction. It is obvious from (7) that (8) holds true for $n = 0$. Suppose that (8) are true for $n = m$. Then from (6) and (7) it follows that

$$\begin{aligned} L_{m+1,\alpha_1} &= A_{l,\alpha_1} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha_1}}{R_{m-i,\alpha_1}} \\ &\le A_{l,\alpha_2} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha_2}}{R_{m-i,\alpha_2}} = L_{m+1,\alpha_2} \\ &= A_{l,\alpha_2} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha_2}}{R_{m-i,\alpha_2}} \\ &\le A_{r,\alpha_2} + \sum_{i=0}^{k} \frac{B_{i,r,\alpha_2}}{L_{m-i,\alpha_2}} = R_{m+1,\alpha_2} \\ &= A_{r,\alpha_2} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha_2}}{R_{m-i,\alpha_2}} \\ &\le A_{r,\alpha_1} + \sum_{i=0}^{k} \frac{B_{i,r,\alpha_1}}{L_{m-i,\alpha_1}} = R_{m+1,\alpha_1} \end{aligned}$$

Therefore (8) are satisfied. Moreover from (6) we have, for $\alpha \in (0,1]$,

$$L_{1,\alpha} = A_{l,\alpha} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha}}{R_{-i,\alpha}}, \quad R_{1,\alpha} = A_{r,\alpha} + \sum_{i=0}^{k} \frac{B_{i,r,\alpha}}{L_{-i,\alpha}}, \tag{9}$$

Since $A, B, x_{-i}, i = 0,1,\cdots,k$ are positive fuzzy numbers, then we have that $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}, L_{-i,\alpha}, R_{-i,\alpha}$ are left continuous. So from (9) we have that $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By induction we can get that $L_{n,\alpha}, R_{n,\alpha}, n = 1,2,\cdots$, are left continuous.

We prove now that the support of x_n, $\mathrm{supp}x_n = \overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}$ is compact. It is sufficient to prove $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]$ is bounded. Let $n=1$. Since A, B_i, $x_{-i}, i=0,1,\cdots,k$, are positive fuzzy numbers there exist positive real numbers $K, L, M_i, N_i, P_{-i}, Q_{-i}, i=0,1,\cdots,k$ such that for all $\alpha\in(0,1]$

$$\begin{cases} [A_{l,\alpha},A_{r,\alpha}] & \subset & [K,L], \\ [B_{i,l,\alpha},B_{i,r,\alpha}] & \subset & [M_i,N_i], \\ [L_{-i,\alpha},R_{-i,\alpha}] & \subset & [P_{-i},Q_{-i}]. \end{cases} \quad (10)$$

Therefore from (9) and (10) it can follows easily that

$$[L_{1,\alpha},R_{1,\alpha}]\subset\left[K+\sum_{i=0}^{k}\frac{M_i}{Q_{-i}},L+\sum_{i=0}^{k}\frac{N_i}{P_{-i}}\right],\quad \alpha\in(0,1]$$

from which it is obvious that, $\alpha\in(0,1]$,

$$\bigcup_{\alpha\in(0,1]}[L_{1,\alpha},R_{1,\alpha}]\subset\left[K+\sum_{i=0}^{k}\frac{M_i}{Q_{-i}},L+\sum_{i=0}^{k}\frac{N_i}{P_{-i}}\right]. \quad (11)$$

Relation (11) implies that $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha},R_{1,\alpha}]}$ is compact and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha},R_{1,\alpha}]}\subset(0,\infty)$. Working inductively we can easily prove that $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}$ is compact and

$$\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}\subset(0,\infty),\quad n=1,2,\cdots. \quad (12)$$

Therefore from Theorem 2.1 of [14], relations (8), (12) and since $L_{n,\alpha}, R_{n,\alpha}$ are left continuous we have that $[L_{n,\alpha},R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers (x_n) such that (3) holds.

We prove now that x_n is the solution of (2) with the initial conditions $x_{-i}, i=0,1,\cdots,k$. Since for all $\alpha\in(0,1]$

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha},R_{n+1,\alpha}] \\ &= \left[A_{l,\alpha}+\sum_{i=0}^{k}\frac{B_{i,l,\alpha}}{R_{n-i,\alpha}},A_{r,\alpha}+\sum_{i=0}^{k}\frac{B_{i,r,\alpha}}{L_{n-i,\alpha}}\right] \end{aligned}$$

It follows that x_n is the solution of (2) with the initial conditions $x_{-i}, i=0,1,\cdots,k$.

Suppose that there exists another solution $\bar{x}_n$ of (2) with the initial conditions $x_{-i}, i=0,1,\cdots,k$. Then arguing as above we can easily prove that

$$[\bar{x}_n]_\alpha=[L_{n,\alpha},R_{n,\alpha}],\quad \alpha\in(0,1],\quad n=0,1,\cdots. \quad (13)$$

Then from (3) and (13) we have $[x_n]_\alpha=[\bar{x}_n]_\alpha$, $\alpha\in(0,1]$, $n=0,1,\cdots$ from which it follows that $x_n=\bar{x}_n, n=0,1,\cdots$. Therefore the proof of theorem 2.1 is completed.

In the following theorem we study the boundedness and persistence of the positive solution of (2). We first give a lemma of [10].

Lemma 2.2 *Let X, Y be fuzzy numbers and $[X]_\alpha=[X_{l,\alpha},X_{r,\alpha}], [Y]_\alpha=[Y_{l,\alpha},Y_{r,\alpha}], \alpha\in(0,1]$ be the $\alpha-$cuts of X, Y respectively. Let Z be a fuzzy number such that $[Z]_\alpha=[Z_{l,\alpha},Z_{r,\alpha}], \alpha\in(0,1]$.Then*

$$MIN\{X,Y\}=Z,(resp.MAX\{X,Y\}=Z)$$

if and only if

$$min\{X_{l,\alpha},Y_{l,\alpha}\}=Z_{l,\alpha},\quad min\{X_{r,\alpha},Y_{r,\alpha}\}=Z_{r,\alpha}$$

$$(resp.\ max\{X_{l,\alpha},Y_{l,\alpha}\}=Z_{l,\alpha},\quad max\{X_{r,\alpha},Y_{r,\alpha}\}=Z_{r,\alpha}).$$

Theorem 2.2. *Every positive solution of (2) is bounded and persists, where $A, B_i, i=0,1,\cdots,k$ are positive fuzzy numbers.*

Proof. Let x_n be a positive solution of (2). Suppose (5) is satisfied. From (6) it is clear that $n=k+2,k+3,\cdots$,

$$A_{l,\alpha}\le L_{n,\alpha},\quad A_{r,\alpha}\le R_{n,\alpha},\quad \alpha\in(0,1], \quad (14)$$

Then from (14) we get

$$[\min\{L_{n,\alpha},A_{l,\alpha}\},\min\{R_{n,\alpha},A_{r,\alpha}\}]=[A_{l,\alpha},A_{r,\alpha}] \quad (15)$$

So from (15) and Lemma 2.2 it follows that

$$\mathrm{MIN}\{x_n,A\}=A,\quad n\ge k+2. \quad (16)$$

Moreover relations (6) and (14) imply that $n=k+2,k+3,\cdots$,

$$L_{n,\alpha}\le D_{l,\alpha},\quad R_{n,\alpha}\le D_{r,\alpha},\quad \alpha\in(0,1], \quad (17)$$

where

$$\begin{cases} D_{l,\alpha}=A_{l,\alpha}+\frac{1}{A_{r,\alpha}}\sum_{i=0}^{k}B_{i,l,\alpha}, \\ \\ D_{r,\alpha}=A_{r,\alpha}+\frac{1}{A_{l,\alpha}}\sum_{i=0}^{k}B_{i,r,\alpha} \end{cases} \quad (18)$$

Using relation (7) for $0<\alpha_1\le\alpha_2$ we get

$$0<D_{l,\alpha_1}\le D_{l,\alpha_2}\le D_{r,\alpha_2}\le D_{r,\alpha_1} \quad (19)$$

From Theorem 2.1 of [14] and (18) we have $D_{l,\alpha}, D_{r,\alpha}$ are left continuous. Moreover from (18) and (7) we have

$$[D_{l,\alpha},D_{r,\alpha}]\subset\left[K+\frac{1}{L}\sum_{i=0}^{k}M_i,L+\frac{1}{K}\sum_{i=0}^{k}N_i\right]$$

from which it is obvious that $\overline{\bigcup_{\alpha\in(0,1]}[D_{l,\alpha},D_{r,\alpha}]}$ is compact. Hence from Theorem 2.1 of [14], (19) and since $D_{l,\alpha}, D_{r,\alpha}$ are left continuous there exists a fuzzy number D such that $[D]_\alpha=[D_{l,\alpha},D_{r,\alpha}]$. Using (17) and Lemma 2.2 it follows that

$$\mathrm{MAX}\{x_n,D\}=D,\quad n\ge k+2. \quad (20)$$

Hence from (16) and (20) it follows that every positive solution x_n of (2) is bounded and persists. The proof is complete.

In the following we study the existence of a unique positive equilibrium x of (2) which is asymptotically stable. We need the following lemma.

Lemma 2.3. *Consider the following system of difference equations $n=0,1,2,\cdots$,*

$$y_{n+1}=a+\sum_{i=0}^{k}\frac{p_i}{z_{n-i}},\quad z_{n+1}=b+\sum_{i=0}^{k}\frac{q_i}{y_{n-i}}, \quad (21)$$

where $k\in\{0,1,2,\cdots\}, y_{-k},y_{-k+1},\cdots,y_0,z_{-k},z_{-k+1},\cdots,z_0$ are positive constants and $a,b,p_i,q_i, i=0,1,\cdots,k$ are

positive real numbers. Then the following statements are true:
(i) Every positive solution (y_n, z_n) *of (21) satisfies* $n \geq k+2$,

$$a \leq y_n \leq a + \frac{1}{b}\sum_{i=0}^{k} p_i, \quad b \leq z_n \leq b + \frac{1}{a}\sum_{i=0}^{k} q_i. \tag{22}$$

(ii) System (21) has a unique positive equilibrium (y, z) *given by*

$$\begin{cases} y &= \frac{ab-\sum_{i=0}^{k}(q_i-p_i)}{2b} \\ & + \frac{\sqrt{\left[\sum_{i=0}^{k}(q_i-p_i)-ab\right]^2+4ab\sum_{i=0}^{k}q_i}}{2b} \\ z &= \frac{ab-\sum_{i=0}^{k}(p_i-q_i)}{2a} \\ & + \frac{\sqrt{\left[\sum_{i=0}^{k}(p_i-q_i)-ab\right]^2+4ab\sum_{i=0}^{k}p_i}}{2a} \end{cases} \tag{23}$$

(iii) Every positive solution of system (21) converges the positive equilibrium (y, z) *of (21) as* $n \to \infty$.

Proof. (i) Let (y_n, z_n) be a positive solution of (21), from (21) it is obvious that

$$a \leq y_n, \quad b \leq z_n, \quad n \geq k+2. \tag{24}$$

Moreover from (25) and using (21) we have $n \geq k+1$,

$$\begin{cases} y_{n+1} = a + \frac{\sum_{i=0}^{k} p_i}{z_{n-i}} \leq a + \frac{1}{b}\sum_{i=0}^{k} p_i, \\ z_{n+1} = b + \frac{\sum_{i=0}^{k} q_i}{y_{n-i}} \leq b + \frac{1}{a}\sum_{i=0}^{k} q_i. \end{cases} \tag{25}$$

From (24) and (25) it follows that (22) holds.

(ii) Let y, z be positive real numbers such that

$$y = a + \frac{\sum_{i=0}^{k} p_i}{z}, \quad z = b + \frac{\sum_{i=0}^{k} q_i}{y} \tag{26}$$

Then from (26) we have that the positive real numbers y, z are given by (23).

(iii) From (22) we have

$$\begin{cases} \lim_{n\to\infty}\sup y_n = L_1 < \infty, \ \lim_{n\to\infty}\inf y_n = l_1 > 0, \\ \lim_{n\to\infty}\sup z_n = L_2 < \infty, \ \lim_{n\to\infty}\inf z_n = L_1 > 0. \end{cases} \tag{27}$$

from (21) and (27) we have

$$L_1 \leq a + \frac{1}{l_2}\sum_{i=0}^{k} p_i, \ l_1 \geq a + \frac{1}{L_2}\sum_{i=0}^{k} p_i,$$

$$L_2 \leq b + \frac{1}{l_1}\sum_{i=0}^{k} q_i, \quad l_2 \geq b + \frac{1}{L_1}\sum_{i=0}^{k} q_i.$$

From which it follows that

$$bL_1+\sum_{i=0}^{k} q_i \leq al_2+\sum_{i=0}^{k} p_i, \quad aL_2+\sum_{i=0}^{k} p_i \leq bl_1+\sum_{i=0}^{k} q_i. \tag{28}$$

Then relation (28) implies that $bL_1 + aL_2 \leq al_2 + bl_1$, from which it follows that

$$bL_1 + aL_2 = al_2 + bl_1. \tag{29}$$

We claim that

$$L_1 = l_1, \quad L_2 = l_2. \tag{30}$$

Suppose on contrary that $l_1 < L_1$. Then from (29) it follows that $bL_1 + aL_2 = al_2 + bl_1 < al_2 + bL_1$ and so $L_2 < l_2$ which is a contradiction. Hence $L_1 = l_1$. Similarly we can prove that $L_2 = l_2$. Therefore (30) are true. Hence from (21) and (30) there exist the $\lim y_n$ and $\lim z_n$ as $n \to \infty$ such that

$$\lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} z_n = z$$

where (y, z) is the unique positive equilibrium of (21). The proof is completed.

Theorem 2.3 *Consider Eq.(2) where* $A, B_i, i = 0, 1, \cdots k$ *are positive fuzzy numbers. Suppose that*

$$\frac{L^2+\sum_{i=0}^{k}(N_i-M_i)}{2K} + \frac{\sqrt{\left[\sum_{i=0}^{k}(M_i-N_i)-L^2\right]^2+4L^2\sum_{i=0}^{k}N_i}}{2K} < \left(\sum_{i=0}^{k} M_i\right)^{\frac{1}{2}} \tag{31}$$

Then the following statements are true

(i) Eq.(2) has a unique positive equilibrium x.
(ii) The unique equilibrium x *is asymptotically stable.*

Proof. (i) Consider the following system, for $\alpha \in (0, 1]$,

$$\begin{cases} L_\alpha = A_{l,\alpha} + \frac{1}{R_\alpha}\sum_{i=0}^{k} B_{i,l,\alpha}, \\ R_\alpha = A_{r,\alpha} + \frac{1}{L_\alpha}\sum_{i=0}^{k} B_{i,r,\alpha}, \end{cases} \tag{32}$$

Then the positive solution (L_α, R_α) of (32) is given by

$$\begin{cases} L_\alpha = \frac{A_{l,\alpha}A_{r,\alpha}-\sum_{i=0}^{k}(B_{i,r,\alpha}-B_{i,l,\alpha})}{2A_{r,\alpha}} \\ + \frac{\sqrt{\left[\sum_{i=0}^{k}(B_{i,r,\alpha}-B_{i,l,\alpha})-A_{l,\alpha}A_{r,\alpha}\right]^2+4A_{l,\alpha}A_{r,\alpha}\sum_{i=0}^{k}B_{i,r,\alpha}}}{2A_{r,\alpha}} \\ R_\alpha = \frac{A_{l,\alpha}A_{r,\alpha}-\sum_{i=0}^{k}(B_{i,l,\alpha}-B_{i,r,\alpha})}{2A_{l,\alpha}} \\ + \frac{\sqrt{\left[\sum_{i=0}^{k}(B_{i,l,\alpha}-B_{i,r,\alpha})-A_{l,\alpha}A_{r,\alpha}\right]^2+4A_{l,\alpha}A_{r,\alpha}\sum_{i=0}^{k}B_{i,l,\alpha}}}{2A_{l,\alpha}} \end{cases} \tag{33}$$

Let x_n be a positive solution of (2) such that $[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}]$, $\alpha \in (0, 1], n = 0, 1, \cdots$. Then using Lemma 2.3 to the system (6) we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{34}$$

Hence relations (7) and (34), for $0 < \alpha_1 \leq \alpha_2 \leq 1$, imply that

$$0 < L_{\alpha_1} \leq L_{\alpha_2} \leq R_{\alpha_2} \leq R_{\alpha_1} \tag{35}$$

Since $A_{l,\alpha}, A_{r,\alpha}, B_{i,l,\alpha}, B_{i,r,\alpha}, i = 0, 1, \cdots, k$ are left continuous from (33) it follows that L_α, R_α are also left continuous. Moreover from (33) and (10) we get

$$\begin{aligned} R_\alpha \quad \leq \quad d = & \frac{L^2+\sum_{i=0}^{k}(N_i-M_i)}{2K} \\ & + \frac{\sqrt{\left[\sum_{i=0}^{k}(M_i-N_i)-L^2\right]^2+4L^2\sum_{i=0}^{k}N_i}}{2K}. \end{aligned} \tag{36}$$

Then from (10), (32) and (36) we have

$$L_\alpha \geq c = K + \frac{\sum_{i=0}^{k} M_i}{d}. \quad (37)$$

Hence relations (36) and (37) imply that $[L_\alpha, R_\alpha] \subset [c, d]$ and so $\bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \subset [c, d]$. From which it is clear that

$$\overline{\bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha]} \ \textit{is compact and} \ \overline{\bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha]} \subset (0, \infty) \quad (38)$$

So from Theorem 2.1 of [14], relations (35), (38), (5), (32) and since L_α, R_α are left continuous we have that $[L_\alpha, R_\alpha], \alpha \in (0, 1]$ determines a fuzzy number x such that

$$x = A + \frac{1}{x}\sum_{i=0}^{k} B_i, \quad [x]_\alpha = [L_\alpha, R_\alpha], \alpha \in (0, 1]$$

and so x is a positive equilibrium of (2).

Suppose there exists another positive equilibrium $\overline{x}$ of (2). Then there exist functions $\overline{L_\alpha} : (0, 1] \to (0, \infty), \overline{R_\alpha} : (0, 1] \to (0, \infty)$ such that

$$\overline{x} = A + \frac{1}{x}\sum_{i=0}^{k} B_i, \quad [\overline{x}]_\alpha = [\overline{L_\alpha}, \overline{R_\alpha}], \quad \alpha \in (0, 1]. \quad (39)$$

From (39) it follows that

$$\overline{L_\alpha} = A_{l,\alpha} + \frac{1}{\overline{R_\alpha}}\sum_{i=0}^{k} B_{i,l,\alpha}, \quad \overline{R_\alpha} = A_{r,\alpha} + \frac{1}{\overline{L_\alpha}}\sum_{i=0}^{k} B_{i,r,\alpha} \quad (40)$$

and so $L_\alpha = \overline{L_\alpha}, R_\alpha = \overline{R_\alpha}, \alpha \in (0, 1]$. Therefore $x = \overline{x}$. This completes part (i).

(ii) From (34) we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} \quad (41)$$

Let ε be a positive real number, we consider the positive real number δ as follows

$$\delta < \min\{\varepsilon, c, c + K - d\} \quad (42)$$

where c, d are defined in (36) and (37).

Let x_n be a positive solution of (2) such that

$$D(x_{-i}, x) \leq \delta < \varepsilon, \quad i = 0, 1, 2, \cdots, k. \quad (43)$$

From (43) it follows that $i = 0, 1, \cdots, k$,

$$|L_{-i,\alpha} - L_\alpha| \leq \delta, \quad |R_{-i,\alpha} - R_\alpha| \leq \delta, \ \alpha \in (0, 1] \quad (44)$$

From (6), (7), (32) and (44) we have

$$\begin{aligned} L_{1,\alpha} - L_\alpha &= A_{l,\alpha} + \sum_{i=0}^{k} \frac{B_{i,l,\alpha}}{R_{-i,\alpha}} - L_\alpha \\ &\leq A_{l,\alpha} + \frac{1}{R_\alpha - \delta}\sum_{i=0}^{k} B_{i,l,\alpha} - L_\alpha \\ &\leq \delta\frac{L_\alpha - A_{l,\alpha}}{R_\alpha - \delta} \leq \delta\frac{R_\alpha - A_{l,\alpha}}{R_\alpha - \delta} \quad (45) \\ &\leq \delta\frac{R_\alpha - K}{R_\alpha - \delta} \end{aligned}$$

from (42) and (45) we get

$$|L_{1,\alpha} - L_\alpha| < \delta < \varepsilon, \quad \alpha \in (0, 1]. \quad (46)$$

Moreover from (6), (7), (32) and (44) we have

$$\begin{aligned} R_{1,\alpha} - R_\alpha &= A_{r,\alpha} + \sum_{i=0}^{k} \frac{B_{i,r,\alpha}}{L_{-i,\alpha}} - R_\alpha \\ &\leq \delta\frac{R_\alpha - A_{r,\alpha}}{L_\alpha - \delta} \leq \delta\frac{d - K}{c - \delta} \quad (47) \end{aligned}$$

From (31), (42) and (47) we get

$$|R_{1,\alpha} - R_\alpha| < \varepsilon, \quad \alpha \in (0, 1] \quad (48)$$

From (47) and (48), working inductively we can easily prove that

$$|L_{n,\alpha} - L_\alpha| < \varepsilon, \quad |R_{n,\alpha} - R_\alpha| < \varepsilon, \quad \alpha \in (0, 1] \quad (49)$$

and so $D(x_n, x) < \varepsilon, n \geq 0$. Therefore the positive equilibrium x is stable, and noting (41). So the positive equilibrium x is asymptotically stable. The proof is complete.

III. Conclusion

In this paper, we study the existence of positive solution to fuzzy difference equation $x_{n+1} = A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}}, \quad n = 0, 1, \cdots$. Under certain conditions, we prove that the positive solutions are bounded and persists. Furthermore, we prove that the equation has a unique positive equilibrium which is asymptotically stable.

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