On the existence and global attractivity of solutions of a functional integral equation

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Abstract—Using the concept of measure of noncompactness, we present some results concerning the existence, uniform local attractivity and global attractivity of solutions for a functional integral equation. Our results improve and extend some previous known results and based on weaker conditions. Some examples which show that our results are applicable when the previous results are inapplicable are also included.

Keywords—Functional integral equation, Fixed-point, Measure of noncompactness, Attractive solution, Asymptotic stability.

I. INTRODUCTION

Integral equations are one of useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics, where they are not only useful but often indispensable even for numerical computations. Many problems of mathematical physics can be stated in the form of integral equations. In fact, there is almost no area of applied mathematics and mathematical physics where integral equations do not play a role [5], [7].

In this paper we consider the existence and uniform local attractivity of solutions of the following functional integral equation

$$x(t) = f(t, x(\alpha(t)))$$
$$+ h(\int_0^{\beta(t)} g(t, s, x(\gamma(s)))ds), \quad t \in [0, \infty). \quad (1)$$

This equation includes several classes of integral equations. Banaś and Dhage [1] studied the existence and behavior of solutions for the equation (1) where $h(x) = x$ under the following assumptions:

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(1) The functions $\alpha, \beta, \gamma : \mathbb{R}_+ = [0, \infty) \to \mathbb{R}_+$ are continuous and $\alpha(t) \to \infty$ as $t \to \infty$.

(2) The function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist positive constants L, M such that

$$|f(t, x) - f(t, y)| \leq \frac{M|x - y|}{L + |x - y|},$$

for $t \in \mathbb{R}_+$ and for $x, y \in \mathbb{R}$. Moreover, assume that $M < L$.

(3) The function $t \to f(t, 0)$ is bounded on $\mathbb{R}_+$ with $\bar{F} = \sup\{f(t, 0) : t \in \mathbb{R}_+\}$.

(4) The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, x)| \leq a(t)b(s),$$

for $t, s \in \mathbb{R}_+$. Moreover, assume that

$$\lim_{t \to \infty} a(t) \int_0^{\beta(t)} b(s)ds = 0,$$

and gave their main result as:

Theorem *A. Under the above assumptions the functional integral equation*

$$x(t) = f(t, x(\alpha(t)))$$
$$+ \int_0^{\beta(t)} g(t, s, x(\gamma(s)))ds, \quad t \in \mathbb{R}_+, \quad (2)$$

has at least one solution in the space $BC(\mathbb{R}_+)$. Moreover, solutions of equation (2) are globally asymptotically stable.

Here $BC(\mathbb{R}_+)$ is the Banach space of all bounded and continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ equipped with the standard norm

$$||x|| = \sup\{|x(t)| : t \in \mathbb{R}_+\}, \qquad x \in BC(\mathbb{R}).$$

In this paper we study the uniform local attractivity and existence of solution for equation (1) and present some new conditions which our results substantially extend and improve previous results. We

will use the concept of measure of noncompactness and Darbo type fixed-point theorem which proved by Banaś and Goebel [3]. Also among definitions of measure of noncompactness, we take the axiomatic definition, given by Banaś and Goebel [3] which is more useful and convenient in applications.

This paper is organized as follows: in Section 2 we present some definitions and preliminary results and in Section 3 we give our main results and provide some examples to show that these results are applicable where the previous results are inapplicable.

II. PRELIMINARIES

In this section we present some definitions and results which will be needed in this paper. Let $(E, ||.||)$ be an infinite Banach space with zero element θ. We write $B(x, r)$ to denote the closed ball centered at x with radius r and $\bar{X}, ConvX$ to denote the closure and closed convex hull of X, respectively. Moreover let m_E indicates the family of all nonempty bounded subsets of E and n_E indicates the family of all relatively compact sets. We use the following definition of the measure of noncompactness was given in [2].

Definition 1.2. *A mapping* $\mu : m_E \to \mathbb{R}_+$ *is said to be a measure of noncompactness in E if it satisfies the following conditions:*

1^0 *The family* $ker\mu = \{X \in m_E : \mu(X) = 0\}$ *is nonempty and* $ker\mu \subset n_E$,

2^0 $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

3^0 $\mu(\bar{X}) = \mu(X)$,

4^0 $\mu(ConvX) = \mu(X)$,

5^0 $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0, 1]$,

6^0 *If* (X_n) *is a sequence of close sets from m_E such that* $X_{n+1} \subset X_n(n = 1, 2, ...)$ *and if* $\lim_{n\to\infty}\mu(X_n) = 0$, *then the intersection set* $X_\infty = \bigcap_{n=1}^\infty X_n$ *is nonempty.*

In the following we state a fixed-point theorem of Darbo type proved by Banaś and Goebel [2].

Theorem 1.2. *Let C be a nonempty, closed, bounded, and convex subset of the Banach space E and let $F : C \to C$ be a continuous mapping. Assume that there exist a constant $k \in [0, 1)$ such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset of C. Then F has a fixed-point in the set C.*

For any nonempty bounded subset X of $BC(\mathbb{R}_+)$, $x \in X$, $T > 0$ and $\varepsilon \geq 0$ let

$$w^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T],$$
$$|t - s| \leq \varepsilon\},$$
$$w^T(X, \varepsilon) = \sup\{w^T(x, \varepsilon) : x \in X\},$$
$$w_0^T(X) = \lim_{\varepsilon\to 0} w^T(X, \varepsilon),$$
$$w_0(X) = \lim_{T\to\infty} w_0^T(X),$$
$$X(t) = \{x(t) : x \in X\},$$
$$diamX(t) = \sup\{|x(t) - y(t)| : x, y \in X\},$$

and

$$\mu(X) = w_0(X) + \limsup_{t\to\infty} diamX(t). \quad (3)$$

Banaś has shown in [4] that the function μ is a measure of noncompatness in the space $BC(\mathbb{R}_+)$. Let F be an operator from $\Omega \subset BC(\mathbb{R}_+)$ into itself and consider the solutions of equation

$$(Fx)(t) = x(t). \quad (4)$$

Now we review the concept of attractivity of solutions for equation (4).

Definition 2.2. (See [1].) *Solutions of equation (4) are locally attractive if there exist a ball $B(x_0, r)$ in the space $BC(\mathbb{R}_+)$ such that for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equations (4) belonging to $B(x_0, r) \bigcap \Omega$ we have that*

$$\lim_{t\to\infty}(x(t) - y(t)) = 0. \quad (5)$$

When the limit (5) is uniform with respect to $B(x_0, r) \bigcap \Omega$, solutions of equation (4) are said to be locally attractive (or equivalently that solutions of (4) are asymptotically stable).

Definition 3.2. (See [1].) *The solution $x = x(t)$ of equation (4) is said to be globally attractive if (5) hold for each solution $y = y(t)$ of (4).*

If condition (5) is satisfied uniformly with respect to the set Ω, solutions of equation (4) are said to be globally asymptotically stable (or uniformly globally attractive).

III. Main results and examples

In this section we study the functional integral equation (1). Here we consider the following condition:

$(4')$ The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $y_0 \in \mathbb{R}$ and positive constant D such that

$$\int_0^{\beta(t)} |g(t, s, y_0)| ds \leq D, \quad t \in \mathbb{R}_+. \qquad (6)$$

Moreover,

$$\lim_{t \to \infty} \int_0^{\beta(t)} |g(t, s, x(s)) - g(t, s, y(s))| ds = 0,$$
$$\int_0^{\beta(t)} |g(t, s, x(s)) - g(t, s, y(s))| ds < \infty, \quad (7)$$

for any $t \in \mathbb{R}_+$ and uniformly respect to $x, y \in BC(\mathbb{R}_+)$.

Remark 1.3. If a function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies the condition (4) in Theorem A then we infer that

$$\int_0^{\beta(t)} |g(t, s, x)| ds \leq \sup_{t \in \mathbb{R}_+} \{ a(t) \int_0^{\beta(t)} b(s) ds : \} < \infty,$$

uniformly respect to x in $\mathbb{R}$, also

$$\lim_{t \to \infty} \int_0^{\beta(t)} |g(t, s, x(s)) - g(t, s, y(s))| ds$$
$$\leq 2 \lim_{t \to \infty} a(t) \int_0^{\beta(t)} b(s) ds = 0,$$

for $x, y \in BC(\mathbb{R}_+)$. Consequently the condition $(4')$ holds.

Now we formulate our main theorem as:

Theorem 1.3. *Suppose that the conditions* $(1), (3)$ *and* $(4')$ *hold and f is Lipschitz continuous with constant* $k \in [0, 1)$. *Also* $h : \mathbb{R} \to \mathbb{R}$ *is a continuous function such that*

$$|h(x) - h(y)| \leq \rho |x - y|^\sigma, \qquad x, y \in \mathbb{R}. \qquad (8)$$

for some positive constants ρ, σ. Then the equation (1) *has at least one solution in $BC(\mathbb{R}_+)$. Moreover, the solutions of* (1) *are uniformly locally attractive.*

Proof. First of all we define operator F, such that for any $x \in BC(\mathbb{R}_+)$

$$\begin{aligned}(Fx)(t) &= f(t, x(\alpha(t))) \\ &+ h(\int_0^{\beta(t)} g(t, s, x(\gamma(s))) ds),\end{aligned}$$

By considering conditions of theorem we infer that Fx is continuous on $\mathbb{R}_+$. Now we prove that $Fx \in BC(\mathbb{R}_+)$ for any $x \in BC(\mathbb{R}_+)$. For arbitrarily fixed $t \in \mathbb{R}_+$ we have

$$\begin{aligned}|(Fx)(t)| &\leq |f(t, x(\alpha(t))) - f(t, 0)| + |f(t, 0)| \\ &+ |h(\int_0^{\beta(t)} g(t, s, x(\gamma(s))) ds)|,\end{aligned}$$

By using condition $(4')$ and inequality (8) we obtain

$$|h(\int_0^{\beta(t)} g(t, s, x(\gamma(s))) ds)|$$
$$\leq |h(\int_0^{\beta(t)} g(t, s, x(\gamma(s))) ds)$$
$$- h(\int_0^{\beta(t)} g(t, s, y_0) ds)| + |h(\int_0^{\beta(t)} g(t, s, y_0) ds)|$$
$$\leq \rho |\int_0^{\beta(t)} \{g(t, s, x(\gamma(s))) - g(t, s, y_0)\} ds|^\sigma$$
$$+ |h(\int_0^{\beta(t)} g(t, s, y_0) ds)| \leq M_1,$$

where

$$\begin{aligned}M_1 &= \sup\{\rho| \int_0^{\beta(t)} \{g(t, s, y(s)) - g(t, s, y_0)\} ds|^\sigma \\ &: t \in \mathbb{R}_+, \ y \in BC(\mathbb{R}_+)\} \\ &+ \max\{|h(r)| : r \in [-D, D]\}, \qquad (9)\end{aligned}$$

and D is given by (6). Thus

$$|(Fx)(t)| \leq k|x(\alpha(t))| + M_0. \qquad (10)$$

Here $M_0 = \sup\{|f(t, 0)| : t \in \mathbb{R}_+\} + M_1$. Hence $Fx \in BC(\mathbb{R}_+)$. Equation (10) yields that F transforms the ball $B_r = B(\theta, r)$ into itself where $r = \frac{M_0}{1-k}$. Now we show that F is continuous on the ball B_r. Let us fix arbitrary $\varepsilon > 0$ and take $x, y \in B_r$ such that $||x - y|| \leq \varepsilon$. Then

$$\begin{aligned}&|(Fx)(t) - (Fy)(t)| \\ &\leq |f(t, x(\alpha(t))) - f(t, y(\alpha(t)))| \\ &+ |h(\int_0^{\beta(t)} g(t, s, x(\gamma(s))) ds) \\ &- h(\int_0^{\beta(t)} g(t, s, y(\gamma(s))) ds)| \\ &\leq k|x(\alpha(t)) - y(\alpha(t))| \\ &+ \rho\{\int_0^{\beta(t)} |g(t, s, x(\gamma(s))) \\ &- g(t, s, y(\gamma(s)))| ds\}^\sigma, \qquad (11)\end{aligned}$$

Furthermore, with due attention to the condition $(4^{'})$ there exist $T > 0$ such that for $t \geq T$ we have

$$\int_0^{\beta(t)} |g(t,s,x(\gamma(s))) - g(t,s,y(\gamma(s)))|ds \leq (\frac{\varepsilon}{\rho})^{\frac{1}{\sigma}},$$
$$(12)$$

and then from (11) and (12) for $t \geq T$ we have

$$|(Fx)(t) - (Fy)(t)| \leq (k+1)\varepsilon.$$

Now we assume that $t \in [0, T]$, then by using continuity of g on $[0, T] \times [0, \beta_T] \times [-r, r]$, where $\beta_T = \sup\{\beta(t) : t \in [0, T]\}$, we can obtain

$$\int_0^{\beta(t)} |g(t,s,x(\gamma(s))) - g(t,s,y(\gamma(s)))|ds \to 0.$$

as $\varepsilon \to 0$. Thus F is continuous on B_r. In the sequel we show that for any nonempty set $X \subset B_r$, $\mu(FX) \leq k\mu(X)$. To do this fix arbitrarily $T > 0$ and $\varepsilon > 0$ now let us choose $x \in X$ and $t_1, t_2 \in [0, T]$ with $|t_2 - t_1| \leq \varepsilon$, also without lose of generality suppose that $\beta(t_1) \leq \beta(t_2)$ thus we have

$$|(Fx)(t_2) - (Fx)(t_1)|$$
$$\leq |f(t_2, x(\alpha(t_2))) - f(t_2, x(\alpha(t_1)))|$$
$$+ |f(t_2, x(\alpha(t_1))) - f(t_1, x(\alpha(t_1)))|$$
$$+ |h(\int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))ds$$
$$- h(\int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))ds)|$$
$$\leq k|x(\alpha(t_2)) - x(\alpha(t_1))| + w_r^T(f, \varepsilon)$$
$$+ \rho|\int_0^{\beta(t_2)} g(t_2, s, x(\gamma(s)))ds$$
$$- \int_0^{\beta(t_1)} g(t_1, s, x(\gamma(s)))ds|^{\sigma}$$
$$\leq kw^T(x, w^T(\alpha, \varepsilon)) + w_r^T(f, \varepsilon)$$
$$+ \rho 2^{\sigma}(\int_0^{\beta(t_2)} |g(t_2, s, x(\gamma(s)))$$
$$- g(t_1, s, x(\gamma(s)))|ds)^{\sigma}$$
$$+ \rho 2^{\sigma}(\int_{\beta(t_1)}^{\beta(t_2)} |g(t_1, s, x(\gamma(s)))|ds)^{\sigma}$$
$$\leq kw^T(x, w^T(\alpha, \varepsilon)) + w_r^T(f, \varepsilon)$$
$$+ \rho 2^{\sigma}\{(\int_0^{\beta_T} w_r^T(g, \varepsilon)ds)^{\sigma} + (w^T(\beta, \varepsilon)G_T^r)^{\sigma}\},$$

where

$$w_r^T(f, \varepsilon) = \sup\{|f(t_2, x) - f(t_1, x)| : t_1, t_2 \in [0, T],$$
$$|t_2 - t_1| \leq \varepsilon, x \in [-r, r]\},$$

$$\beta_T = \sup\{\beta(t) : t \in [0, T]\},$$

$$w_r^T(g, \varepsilon) = \sup\{|g(t_2, s, x) - g(t_1, s, x)| :$$
$$t_1, t_2 \in [0, T], |t_2 - t_1| \leq \varepsilon, s \in [0, \beta_T], x \in [-r, r]\}$$

and

$$G_r^T = \sup\{|g(t, s, x)| : t \in [0, T], s \in [0, \beta_T],$$
$$x \in [-r, r]\}.$$

Now by using the above estimate we have

$$w^T(FX, \varepsilon) \leq kw^T(X, w^T(\alpha, \varepsilon)) + w_r^T(f, \varepsilon)$$
$$+ \rho 2^{\sigma}\{(\int_0^{\beta_T} w_r^T(g, \varepsilon)ds)^{\sigma} + (w^T(\beta, \varepsilon)G_T^r)^{\sigma}\}.$$

From continuity of f and g on compact sets $[0, T] \times [-r, r]$ and $[0, T] \times [0, \beta_T] \times [-r, r]$ respectively, we find $w_r^T(f, \varepsilon) \to 0$, $w_r^T(g, \varepsilon) \to 0$ as $\varepsilon \to 0$ and G_r^T is a finite constant. Similarly we get $w^T(\beta, \varepsilon) \to 0$, $w^T(\alpha, \varepsilon) \to 0$ as $\varepsilon \to 0$ thus we obtain

$$w_0^T(FX) \leq kw_0^T(X),$$

by taking $T \to \infty$ we have

$$w_0(FX) \leq kw_0(X), \qquad (13)$$

and for an arbitrarily fixed $t \in \mathbb{R}$ and $x, y \in X$ we obtain the following estimate:

$$|(Fx)(t) - (Fy)(t)| \leq k|x(\alpha(t)) - y(\alpha(t))|$$
$$+ \rho\{\int_0^{\beta(t)} |g(t,s,x(\gamma(s))) - g(t,s,y(\gamma(s)))|ds\}^{\sigma}$$
$$\leq k\text{diam}X(\alpha(t)) + M(t),$$

where

$$M(t) = \sup\{\rho(\int_0^{\beta(t)} |g(t,s,x(s))$$
$$- g(t,s,y(s))|ds)^{\sigma} : x, y \in BC(\mathbb{R}_+)\}.$$

By using conditions $(4^{'})$ and $\alpha(t) \to \infty$ as $t \to \infty$, we deduce that

$$\lim_{t \to \infty} \sup \dim(FX)(t) \leq k \lim_{t \to \infty} \sup \dim X(t). \quad (14)$$

Consequently by considering μ defined by (3) we have

$$\mu(FX) \leq k\mu(X), \qquad (15)$$

where $k \in [0, 1)$. Thus from Theorem 1.2 we obtain that the operator F has a fixed point x in B_r and thus the functional integral equation (1) has at least one solution in $BC(\mathbb{R}_+)$.

Now we should investigate uniform local attractivity

for solutions of equation (1). Let us assume that x_0 is a solution of equation (1) with conditions of Theorem 1.3. Consider ball $B(x_0, r_0)$ with $r_0 = \frac{M_2}{1-k}$ where

$$M_2 = \sup\{M(t) : t \in \mathbb{R}_+\}.$$

Take y in $B(x_0, r_0)$ we have

$$
\begin{aligned}
|(Fy)(t) - x_0(t)| &= |(Fy)(t) - (Fx_0)(t)| \\
&\leq k\|y - x_0\| + M_2 \leq r_0,
\end{aligned}
$$

thus we observe that F is continuous function such that $F(B(x_0, r_0)) \subset B(x_0, r_0)$. Similar to the calculations in (13) and (14) we can show that

$$\mu(FX) \leq k\mu(X), \tag{16}$$

for any nonempty subset X of $B(x_0, r_0)$. Let us taking $C_0 = B(x_0, r_0)$, $C_n = Conv(FC_{n-1})$ for $n = 1, 2, ...$, thus we obtain that $C_{n+1} \subset C_n$ (n=1,2,...) and $C_n s$ are nonempty, closed and convex sets. By using (16) we have

$$\mu(C_n) \leq k^n \mu(C_0), \tag{17}$$

for any $n = 1, 2, ...$ But from the definition of $w^T(X, \varepsilon)$ and $\text{diam}X(t)$ we can easily understand that $w^T(X, \varepsilon) \leq 2r_0 + w^T(x_0, \varepsilon)$, $\text{diam}X(t) \leq 2r_0$ and then $\mu(X) \leq 4r_0$ for any nonempty subset X of $B(x_0, r_0)$. Consequently we obtain

$$\lim_{n\to\infty} \mu(C_n) = 0,$$

thus from condition 6^0 of Definition 1.2 we get that the set

$$C_\infty = \bigcap_{n=0}^{\infty} C_n,$$

is nonempty, bounded, closed, convex and $\mu(C_\infty) = 0$. Then

$$\lim_{t\to\infty} \sup \text{diam} C_\infty(t) = 0,$$

and this yields that

$$\lim_{t\to\infty} |z(t) - y(t)| = 0, \qquad \forall z, y \in C_\infty.$$

We deduce that all solutions of the functional integral equation (1) with conditions of Theorem 1.3 are uniformly locally attractive in the sense of Definition 2.2. $\square$

Remark 2.3. Condition $(4')$ can be satisfied in various ways. For instance, one can assume that

The function $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous and there exist $y_0 \in \mathbb{R}$ and positive constant D such that

$$\int_0^{\beta(t)} |g(t, s, y_0)|ds \leq D, \quad t \in \mathbb{R}_+. \tag{18}$$

Moreover, there exist function $M(t, s) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, s, x) - g(t, s, y)| \leq M(t, s), \quad \forall x, y \in \mathbb{R},$$

$$\lim_{t\to\infty} \int_0^{\beta(t)} M(t, s)ds = 0.$$

We can easily deduce condition (4) in Theorem A implies this condition but the converse is not true. To investigate this claim we give some examples which show that our theorem can be applied but the previous results [1], [2], [6] are inapplicable.

Example 1.3. Consider the integral equation

$$
\begin{aligned}
x(t) = & \frac{t}{1+t^2} \sin(x(t)) \\
& + \int_0^t \frac{sx^2(s) + s^3(x^4(s) + 1)}{(x^4(s) + 1)(t^4 + 1)} ds,
\end{aligned} \tag{19}
$$

where $\alpha(t) = \beta(t) = \gamma(t) = t$, and

$$f(t, x) = \frac{t}{1+t^2} \sin(x),$$

$$g(t, s, x) = \frac{sx^2 + s^3(x^4 + 1)}{(x^4 + 1)(t^4 + 1)}.$$

By simple calculation we obtain that

$$\lim_{t\to\infty} \int_0^t |g(t, s, x)|ds = \lim_{t\to\infty} \frac{t^2 x^2}{2(x^4 + 1)(t^4 + 1)}$$

$$+ \lim_{t\to\infty} \frac{t^4}{4(t^4 + 1)} = \frac{1}{4}, \quad x \in \mathbb{R}.$$

So the condition (4) in Theorem A is not hold and this theorem is inapplicable for equation (19). Also by considering $M(t, s) = \frac{s}{1+t^4}$, for every $x, y \in \mathbb{R}$ we have

$$|g(t, s, x) - g(t, s, y)| \leq \frac{s}{1+t^4} = M(t, s).$$

$$\lim_{t\to\infty} \int_0^t M(t, s)ds = \lim_{t\to\infty} \frac{t^2}{2(t^4 + 1)} = 0,$$

We can easily satisfy another conditions in Theorem 1.3. Then the integral equation (1) has at least one solution and solutions of this equation are uniformly locally attractive.

Remark 3.3. In Example 1.3 we observe that the condition (4) in Theorem A cannot be satisfied. In addition to if g has the form

$$g(t, s, x) = \psi_1(x)\psi_2(t, s) + \psi_3(t, s), \qquad (20)$$

where ψ_i, $i = 1, 2, 3$ are continuous and positive functions such that

$$\psi_1(x) \leq k_1, \quad \lim_{t \to \infty} \int_0^t \psi_2(t, s)ds = 0,$$

$$\lim_{t \to \infty} \int_0^t \psi_3(t, s)ds < \infty,$$

where k_1 is a positive constant. Then we have

$$\lim_{t \to \infty} \int_0^t |g(t, s, x)|ds = \lim_{t \to \infty} \int_0^t \psi_3(t, s)ds,$$

$$|g(t, s, x) - g(t, s, y)| \leq k_1\psi_2(t, s) = M(t, s),$$

$$\lim_{t \to \infty} \int_0^t M(t, s)ds = k_1 \lim_{t \to \infty} \int_0^t \psi_2(t, s)ds = 0,$$

for $x, y \in \mathbb{R}$. If

$$0 < \lim_{t \to \infty} \int_0^t \psi_3(t, s)ds, \qquad (21)$$

then we cannot use Theorem A for the integral equation (1) with this g. The functions below are the examples of function g which satisfied the above assumptions and inequality (21) :

$$g_n(t, s, x) = \frac{|x|s^{n-2} + s^{n-1}(x^2 + 1)}{(1 + x^2)(t^n + 1)}, \quad n > 1,$$

$$g(t, s, x) = \frac{\ln(1 + s|x|) + \sinh(s)(1 + x^2)}{(1 + x^2)(1 + \cosh(t))}.$$

We can easily see

$$\lim_{t \to \infty} \int_0^t |g_n(t, s, x)|ds = \frac{1}{n},$$

$$|g_n(t, s, x) - g_n(t, s, y)| \leq \frac{s^{n-2}}{t^n + 1} = M(t, s)$$

$$\lim_{t \to \infty} \int_0^t M(t, s)ds = \lim_{t \to \infty} \frac{t^{n-1}}{1 + t^n} = 0,$$

and

$$\lim_{t \to \infty} \int_0^t |g(t, s, x)|ds = \lim_{t \to \infty} \frac{\cosh(t)}{1 + \cosh(t)} = 1,$$

$$|g(t, s, x) - g(t, s, y)| \leq \frac{2 + 2s}{1 + \cosh(t)} = M(t, s),$$

$$\lim_{t \to \infty} \int_0^t M(t, s)ds = \lim_{t \to \infty} \frac{2t + t^2}{(1 + \cosh(t))} = 0.$$

Thus all of results presented by Banaś and Rezepka [2], Banaś and Dhgae [1] and Liu and Kang [6]

which in their theorem have the condition (4), are inapplicable to integral equation (1) with $h(x) = x$ and g satisfied in above examples. But by using our theorem we can find this integral equation has at least one solution which is uniformly locally attractive.

Corollary 1.3. *If in Theorem 1.3 in addition to Lipschitz continuous with constant $k \in [0, 1)$, f be bounded then the solutions of functional integral equation* (1) *are globally attractive.*

Proof. Suppose that

$$|f(t, x)| \leq M_3, \qquad \forall t \in \mathbb{R}_+, \quad \forall x \in \mathbb{R},$$

then for any $x \in BC(\mathbb{R}_+)$ we have

$$|(Fx)(t)| \leq M_3 + M_1 = r_1, \qquad (22)$$

where M_1 is the constant in (9). The inequality (22) yields that $F(BC(\mathbb{R}_+)) \subset B_{r_1} = B(\theta, r_1)$ then all solutions of the equation (1) are in B_{r_1}. Similar to the proof of Theorem 1.3 we can show that

$$\mu(FX) \leq k\mu(X),$$

for any nonempty subset of B_{r_1} and we can find the subset C_∞ such that this set includes all solutions of equation (1) and $\mu(C_\infty) = 0$ i.e.

$$\lim_{t \to \infty} |x(t) - y(t)| = 0, \qquad \forall x, y \in C_\infty,$$

thus solutions of functional integral equation (1) are globally attractive. $\quad\square$

Example 2.3. Consider the following integral equation:

$$x(t) = \frac{\ln(1 + x^2)}{3(1 + x^2)(1 + t^2)}$$

$$+ \arctan(\int_0^{t^2} \frac{\ln(1 + s|x(\sqrt{s})|) + s(1 + x^2(\sqrt{s}))}{(1 + t^4)(1 + x^2(\sqrt{s}))}ds).$$

$$(23)$$

Notice that the equation (23) is a special case of equation (1) where

$$h(x) = \arctan(x), \ \alpha(t) = \gamma(t) = \sqrt{t},$$

$$\beta(t) = t^2, f(t, x) = \frac{\ln(1 + x^2)}{3(1 + x^2)(1 + t^2)},$$

$$g(t, s, x) = \frac{\ln(1 + s|x|) + s(1 + x^2)}{(1 + t^4)(1 + x^2)}.$$

By simple calculation we can find that f, α, β, γ and h satisfy in conditions of Theorem 1.3 and f is a bounded function and

$$\lim_{t\to\infty} \int_0^{t^2} |g(t,s,x)|ds = \lim_{t\to\infty} \int_0^{t^2} \frac{s}{1+t^4} ds = \frac{1}{2},$$

$$|g(t,s,x) - g(t,s,y)| \leq \frac{1+s}{1+t^4} = M(t,s),$$

$$\lim_{t\to\infty} \int_0^{t^2} M(t,s)ds = \lim_{t\to\infty} \frac{2t+t^2}{2+2t^4} = 0.$$

Hence the conditions in corollary 1.3 are provided thus the equation (23) has at least one solution and solutions of this equation are globally attractive.

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