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On the Differential Geometry of the Curves in Minkowski Space-Time II

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Abstract**—**In the first part of this paper [6], a method to determine Frenet apparatus of the space-like curves in Minkowski space-time is presented. In this work, the mentioned method is developed for the time-like curves in Minkowski space-time. Additionally, an example of presented method is illustrated.

Keywords**—**Frenet Apparatus, Time-like Curves, Minkowski Space-time.

I. Introduction

IN the case of a differentiable curve, at each point, a tetrad of mutually orthogonal unit vectors (called tangent, normal, first binormal and second binormal) was defined and constructed, and the rates of change of these vectors along the curve define curvatures of the curve in four dimensional space [2]. In [3], regular curves in Euclidean 4-space are studied and to determine Frenet frame and Frenet elements, a way, which is similar to E^3, is also given.

At the beginning of the twentieth century, Einstein's theory opened a door to use of degenerate submanifolds. Then, the researchers treated some of classical differential geometry topics and extended to Lorentz manifolds. For instance, in [5], the author introduces a method to calculate Frenet apparatus of space-like curves in Minkowksi 4-space according to signature $(+,+,+,-)$. Thereafter, in [6], in an analogous way, mentioned method is adapted to space-like curves in Minkowski space-time (with respect to signature $(-,+,+,+)$, e.g.).

In this paper, using vector product expressed as in [6], the method is developed for the time-like curves in Minkowski space-time E_1^4. Moreover, an example of this method will be provided.

II. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented (A more complete elementary treatment can be found in [2].)

Minkowski space-time E_1^4 is an Euclidean space E_1^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^3 + dx_3^2 + dx_4^2, \qquad (1)$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4. Since g is an indefinite metric, recall that a vector $\vec{v} \in E_1^4$ can have one of the three causal characters; it can be space-like if $g(\vec{v},\vec{v})\rangle 0$ or $\vec{v}=0$, time-like if $g(\vec{v},\vec{v})\langle 0$ and null (light-like) if $g(\vec{v},\vec{v})=0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\vec{\alpha}'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v},\vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v},\vec{v}) = \pm 1$. Next, vectors $\vec{v}, \vec{w}$ in E_1^4 are said to be orthogonal if $g(v,w)=0$. The velocity of the curve $\vec{\alpha}$ is given by $\|\vec{\alpha}'\|$. Thus, a space-like or a time-like curve $\vec{\alpha}$ is said to be parametrized by arclength function s, if $g(\vec{\alpha}',\vec{\alpha}') = \pm 1$. The Lorentzian hypersphere of center $\vec{m} = (m_1, m_2, m_3, m_4)$ and radius $r \in R^+$ in the space E_1^4 defined by

$$S_1^3 = \left\{ \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E_1^4 : g(\vec{\alpha}-\vec{m}, \vec{\alpha}-\vec{m}) = r^2 \right\} \qquad (2)$$

Denote by $\left\{ \vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s) \right\}$ the moving Frenet frame along the curve $\vec{\alpha}$ in the space E_1^4. Then $\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2$ are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. For a time-like unit speed curve $\vec{\alpha}$ in E_1^4 the following Frenet equations are given in [1], [2]

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{N}} \\ \dot{\vec{B}}_1 \\ \dot{\vec{B}}_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}, \qquad (3)$$

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where $\vec{T}, \vec{N}, \vec{B}_1$ and $\vec{B}_2$ are mutually orthogonal vectors satisfying equations

$$g(\vec{T},\vec{T}) = -1$$
$$g(\vec{N},\vec{N}) = g(\vec{B}_1,\vec{B}_1) = g(\vec{B}_2,\vec{B}_2) = 1.$$

And here, $\kappa(s), \tau(s)$ and $\sigma(s)$ are first, second and third curvature of the curve $\vec{\alpha}$, respectively.

Suffice it to say that, for an arbitrary parametrized time-like curve the following Frenet equations are hold:

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}_1' \\ \vec{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & v\kappa & 0 & 0 \\ v\kappa & 0 & v\tau & 0 \\ 0 & -v\tau & 0 & v\sigma \\ 0 & 0 & -v\sigma & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B}_1 \\ \vec{B}_2 \end{bmatrix}, \qquad (4)$$

where v denotes speed of the curve.

In the same space, the authors, in [6], expressed a vector product with the following definition.

Definition 1. Let $\vec{a} = (a_1, a_2, a_3, a_4)$, $\vec{b} = (b_1, b_2, b_3, b_4)$ and $\vec{c} = (c_1, c_2, c_3, c_4)$ be vectors in the space E_1^4. The vector product in Minkowski space-time is defined with the determinant

$$\vec{a} \wedge \vec{b} \wedge \vec{c} = -\begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \qquad (5)$$

where $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{e}_4$ are coordinate direction vectors satisfying

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_4,\ \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 = \vec{e}_1,\ \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_1 = \vec{e}_2$$
$$\vec{e}_4 \wedge \vec{e}_1 \wedge \vec{e}_2 = -\vec{e}_3.$$

III. Determination of Frenet Apparatus of the Time-like Curves in Minkowski Space-Time

Let $\vec{\varphi} = \vec{\varphi}(s)$ be an arbitrary time-like curve in E_1^4 with curvatures κ, τ and σ for each $s \in I \subset R$. If we calculate 1st, 2nd, 3rd and 4th order derivatives (with respect to s) of this time-like curve, we have, respectively,

$$\vec{\varphi}' = v\vec{T}, \qquad (6)$$

$$\vec{\varphi}'' = v'\vec{T} + v^2\kappa\vec{N}, \qquad (7)$$

$$\vec{\varphi}''' = (v'' + v^3\kappa^2)\vec{T} + (3vv' + v^2\kappa')\vec{N} + (v^3\kappa\tau)\vec{B}_1, \qquad (8)$$

$$\vec{\varphi}^{(IV)} = (...)\vec{T} + (...)\vec{N} + (...)\vec{B}_1 + (v^4\kappa\tau\sigma)\vec{B}_2. \qquad (9)$$

Considering (6), we immediately arrive

$$\vec{T} = \frac{\vec{\varphi}'}{v}. \qquad (10)$$

From definition of velocity, we may express that, due to time-like tangent vector

$$v^2 = -g(\vec{\varphi}', \vec{\varphi}'). \qquad (11)$$

Differentiating both sides of (11), we have

$$v' = -\frac{g(\vec{\varphi}', \vec{\varphi}'')}{v}. \qquad (12)$$

Using (12) in (7) and taking the norm of both sides this expression, we have, respectively,

$$\kappa = \frac{\left\| \|\vec{\varphi}'\|^2 .\vec{\varphi}'' + g(\vec{\varphi}', \vec{\varphi}'').\vec{\varphi}' \right\|}{\|\vec{\varphi}'\|^4} \qquad (13)$$

and

$$\vec{N} = \frac{\|\vec{\varphi}'\|^2 .\vec{\varphi}'' + g(\vec{\varphi}', \vec{\varphi}'').\vec{\varphi}'}{\left\| \|\vec{\varphi}'\|^2 .\vec{\varphi}'' + g(\vec{\varphi}', \vec{\varphi}'').\vec{\varphi}' \right\|}. \qquad (14)$$

The vector product of $\vec{T}, \vec{N}$ and $\vec{\varphi}'''$ gives us

$$\vec{T} \wedge \vec{N} \wedge \vec{\varphi}''' = -\begin{vmatrix} -\vec{T} & \vec{N} & \vec{B}_1 & \vec{B}_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ v'' + v^3\kappa^2 & 3vv' + v^2\kappa' & v^3\kappa\tau & 0 \end{vmatrix} \qquad (15)$$
$$= \mu v^3 \kappa\tau \vec{B}_2$$

where μ is taken ± 1 to make $+1$ determinant of $\left[\vec{T}, \vec{N}, \vec{B}_1, \vec{B}_2\right]$ matrix. By this way, Frenet frame provides positively oriented.

Taking the norm of both sides of (15) and using (13), we have

$$\tau = \frac{\left\|\vec{T} \wedge \vec{N} \wedge \vec{\varphi}'''\right\| \|\vec{\varphi}'\|}{\left\| \|\vec{\varphi}'\|^2 .\vec{\varphi}'' + g(\vec{\varphi}', \vec{\varphi}'').\vec{\varphi}' \right\|}. \qquad (16)$$

Substituting (16) and (13) into (15), we have second binormal vector field

$$\vec{B}_2 = \mu \frac{\vec{T} \wedge \vec{N} \wedge \vec{\varphi}'''}{\left\|\vec{T} \wedge \vec{N} \wedge \vec{\varphi}'''\right\|}. \qquad (17)$$

And using inner product $g(\vec{\varphi}^{(IV)}, \vec{B}_2)$ and considering obtained equations, we obtain the third curvature

$$\sigma = \frac{g(\vec{\varphi}^{(IV)}, \vec{B}_2)}{\left\|\vec{T} \wedge \vec{N} \wedge \vec{\varphi}'''\right\| \|\vec{\varphi}'\|}. \qquad (18)$$

Finally, vector product of $\vec{N}, \vec{T}$ and $\vec{B}_2$ gives us the first binormal vector field as

$$\vec{B}_1 = \mu\, \vec{N} \wedge \vec{T} \wedge \vec{B}_2. \qquad (19)$$

Since, we calculate Frenet apparatus $\left\{\vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s), \kappa(s), \tau(s), \sigma(s)\right\}$ of an arbitrary time-like curve $\vec{\varphi} = \vec{\varphi}(s)$ in Minkowski space-time.

IV. An Example

In this section, we illustrate an example of presented method.

Let us consider the following curve in the space E_1^4

$$\vec{\psi} = \vec{\psi}(s) = (\sqrt{2}s,\ 1,\ \sin s,\ \cos s). \tag{20}$$

Differentiating both sides of (20) with respect to s, we have

$$\vec{\psi}' = (\sqrt{2},\ 0,\ \cos s,\ -\sin s). \tag{21}$$

The inner product $g(\vec{\psi}', \vec{\psi}')$ follows that $g(\vec{\psi}', \vec{\psi}') = -1$, which shows $\vec{\psi}(s)$ is an unit speed time-like curve. Thus, $\|\vec{\psi}'\| = v = 1$. Then, we decompose tangent vector of $\vec{\psi}$ as follows:

$$\vec{T} = (\sqrt{2},\ 0,\ \cos s,\ -\sin s). \tag{22}$$

And, considering equation (14), we have principal normal vector

$$\vec{N} = (0,\ 0,\ -\sin s,\ -\cos s). \tag{23}$$

Writing vector product $\vec{T} \wedge \vec{N} \wedge \vec{\psi}'''$, we have

$$\vec{T} \wedge \vec{N} \wedge \vec{\psi}''' = (0,\ -\sqrt{2},\ 0,\ 0). \tag{24}$$

Since, we have the second binormal vector

$$\vec{B}_2 = (0,\ -1,\ 0,\ 0). \tag{25}$$

Using vector product of $\vec{N}, \vec{T}$ and $\vec{B}_2$, we express

$$\vec{B}_1 = (-1,\ 0,\ \sqrt{2}\cos s, -\sqrt{2}\sin s). \tag{26}$$

Finally, using fourth order derivative of $\vec{\psi}(s)$, (13), (16) and (18), we write curvatures of the curve, respectively,

$$\kappa = 1, \tag{27}$$

$$\tau = \sqrt{2}, \tag{28}$$

$$\sigma = 0. \tag{29}$$

Corollary 1. Suffice it to say that, $\{\vec{T}(s), \vec{N}(s), \vec{B}_1(s), \vec{B}_2(s)\}$ is an orthonormal frame of E_1^4.

V. Conclusion and Further Remarks

Throughout the presented paper, we present a method to calculate all Frenet apparatus of a time-like curve which lies fully in E_1^4. Here, using vector product, we give formulas of frame vectors (and therefore curvatures).

Via this method, some of classical differential geometry topics can be treated. Relations among spherical indicators, Bertrand curves and Involute-evolute curve couple may be easily calculated. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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