On Symmetries and Exact Solutions of Einstein Vacuum Equations for Axially Symmetric Gravitational Fields

Nisha Goyal and R.K. Gupta

Abstract—Einstein vacuum equations, that is a system of nonlinear partial differential equations (PDEs) are derived from Weyl metric by using relation between Einstein tensor and metric tensor. The symmetries of Einstein vacuum equations for static axisymmetric gravitational fields are obtained using the Lie classical method. We have examined the optimal system of vector fields which is further used to reduce nonlinear PDE to nonlinear ordinary differential equation (ODE). Some exact solutions of Einstein vacuum equations in general relativity are also obtained.

Keywords—Gravitational fields, Lie Classical method, Exact solutions.

I. INTRODUCTION

THE adequate description of the gravitational field of an astrophysical object has been an important subject in both relativistic and Newtonian gravity. The problem of how to describe the gravitational field of astrophysical objects has been of central importance in general relativity, both as an issue of principal and as a foundation for observational predictions. The formalism of the general theory of relativity in 1915 by Albert Einstein became a starting point for research in the field of exact solutions of the new theory. The vacuum static Einstein equations for the case of spherical symmetry were considered in 1916 by Schwarzschild [15].

Nonlinear PDEs are ubiquitous in many branches of mathematics, continuum mechanics, mathematical physics and general relativity, where they provide a mathematical description of many phenomena. Apart from their theoretical importance, they have remarkable applications to many physical area such as hydrodynamics, plasma physics etc. Exact solutions of these nonlinear PDEs are useful for the proper understanding of various features of many phenomena.

In search of exact solutions of nonlinear PDEs, the introduction of symmetries served as basic tool. The classification schemes of the solutions of nonlinear PDEs are symmetry methods based on the Lie groups. The most effective methods for finding symmetry reductions and constructing exact solutions have been discussed in Bihlo and Popovych [6], Bluman and Anco [7], Bluman and Cole [8], Bruzon et al. [9], Cicogna et al. [10], Harris and Clarkson [11], Ibragimov [12], Naicker et al. [13], Singh and Gupta [16], Steinberg [17] and Goyal and Gupta [21]. Einstein field equations are at the core of General Relativity and hence their exact solutions play very

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important role in the discussion of this theory. In literature, exact solutions of Einstein field equations have been found by many researchers by using group-theoretic techniques, some important contributions are Ali [1], Ashghar et al. [2], Attalaha et al. [3], Bhutani and Singh [4], Bhutani et al. [5] Stephani et al. [18] and Wilshire [20]. In this paper, the main focus is on the exact solutions of system of nonlinear PDEs in general relativity.

The paper is organized as follows. Section II, is devoted to the derivation of system of PDEs from Weyl metric by using relation between Einstein tensor and metric tensor. In section III, the symmetries in the generalized form are derived, which are then used to obtain associated Lie algebra of vector fields. Exact solutions of system (11)-(12) has also been obtained in this section. In section IV, we record our discussion and concluding remarks.

II. BASIC FIELD EQUATIONS

For axially symmetric vacuum static gravitational fields, the line element reduces to the Weyl [19] metric :

$$ds^2 = \frac{1}{f}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2] - fdt^2. \tag{1}$$

Here ρ, ϕ, z and t are the canonical Weyl coordinates and time respectively; $f(\rho, z)$ and $\gamma(\rho, z)$ are two unknown functions to be determined from the field equations. For this metric the field equations take the form

$$G_{ik} = 0, \tag{2}$$

where G_{ik} is the Einstein tensor related to the Ricci tensor R_{ik} and the curvature scalar R by the relation

$$G_{ik} = R_{ik} - \frac{1}{2}g_{ik}R. \tag{3}$$

To calculate the Ricci tensor and the curvature scalar, one should use the formulae

$$R_{ik} = -\frac{\partial \Gamma_{ik}^l}{\partial x_l} + \frac{\partial \Gamma_{il}^l}{\partial x^k} - \Gamma_{lm}^l \Gamma_{ik}^m + \Gamma_{il}^m \Gamma_{km}^l,$$
$$R = R_{ik}g^{ik}, \tag{4}$$

where the Christoffel symbols Γ_{kl}^i can be obtained from the relations:

$$\Gamma_{kl}^i = \frac{1}{2}g^{im}(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x_m}), \tag{5}$$

where Γ^i_{kl} are Christoffel symbols of second kind.

From equation (3), we obtain all the Einstein equations for an axially symmetric gravitational field outside the sources:

$$f\Delta f = (\overrightarrow{\nabla} f)^2, \tag{6}$$

$$4\frac{\partial\gamma}{\partial\rho} = \rho\frac{1}{f^2}[\left(\frac{\partial f}{\partial\rho}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2], \tag{7}$$
$$2\frac{\partial\gamma}{\partial z} = \rho\frac{1}{f^2}\frac{\partial f}{\partial\rho}\frac{\partial f}{\partial z},$$

$$\frac{\partial^2\gamma}{\partial\rho^2} + \frac{\partial^2\gamma}{\partial z^2} = -\frac{1}{4}\frac{1}{f^2}(\overrightarrow{\nabla} f)^2, \tag{8}$$

The operator Δ and $\overrightarrow{\nabla}$ are defined by the formulae

$$\begin{aligned}\Delta &\equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}, \\ \overrightarrow{\nabla} &\equiv \overrightarrow{\rho}_0\frac{\partial}{\partial\rho} + \overrightarrow{z}_0\frac{\partial}{\partial z},\end{aligned} \tag{9}$$

where $\overrightarrow{\rho}_0$ and $\overrightarrow{z}_0$ being the unit vectors, they are similar to the ordinary Laplacian and gradient operators for flat space expressed in cylindrical coordinates provided that there is no angular coordinate dependence. Among the field equations (6)-(9), equation (6) is independent, whereas (8) is a consequence of both equations in (7).

Solutions of (6)-(8) define the Weyl class. With the substitution

$$f = e^{2\psi}, \tag{10}$$

equation (6) becomes

$$\Delta\psi \equiv \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial z^2} = 0. \tag{11}$$

Equations (7) can be rewritten in the form

$$\gamma(\rho, z) = \int \rho\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right)d\rho + \int 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}dz, \tag{12}$$

We refer the system of equations (11)-(12) as the reduced form of Einstein axially symmetric metric for gravitational fields. To determine the symmetry group and exact solutions for this system, we used Lie classical method in next section. By utilizing this method, we obtained the basic fields of the optimal system which lead to the reductions that are inequivalent with respect to symmetry transformations. For each vector field, some exact solutions are examined for the system.

III. LIE CLASSICAL METHOD AND GROUP INVARIANT SOLUTIONS

In this section we present Lie symmetries of Equation (11). To obtain the Lie symmetries of Equation (11) we consider the one parameter Lie group of infinitesimal transformations in (ρ, z, ψ, γ) given by

$$\begin{aligned}\rho^* &= \rho + \epsilon\xi(\rho, z, \psi) + O(\epsilon^2), \\ z^* &= z + \epsilon\tau(\rho, z, \psi) + O(\epsilon^2), \\ \psi^* &= \psi + \epsilon\phi(\rho, z, \psi) + O(\epsilon^2),\end{aligned} \tag{13}$$

where ϵ is the group parameter, hence the corresponding generator of the Lie algebra is of the form

$$X = \xi(\rho, z, \psi)\frac{\partial}{\partial\rho} + \tau(\rho, z, \psi)\frac{\partial}{\partial z} + \phi(\rho, z, \psi)\frac{\partial}{\partial\psi}, \tag{14}$$

where $X^{[2]}$ denotes the second prolongation of X then using the invariance condition

$$X^{[2]}\left(\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz}\right) = 0, \tag{15}$$

yields the following system of ten determining equations:

$$\begin{aligned}\xi(\rho, z, \psi)_\psi &= 0, \\ \xi(\rho, z, \psi)_\rho &= \frac{\xi(\rho, z, \psi)}{\rho}, \\ \xi(\rho, z, \psi)_{zz} &= 0, \\ \tau(\rho, z, \psi)_z &= 0, \\ \tau(\rho, z, \psi)_\psi &= 0, \\ \tau(\rho, z, \psi)_\rho &= \xi(\rho, z, \psi)_z, \\ \phi(\rho, z, \psi)_{zz} &= \frac{\phi(\rho, z, \psi)_\rho + \rho\phi(\rho, z, \psi)}{\rho}, \\ \phi(\rho, z, \psi)_{z\psi} &= -\frac{\xi(\rho, z, \psi)_z}{2\rho}, \\ \phi(\rho, z, \psi)_{\psi\psi} &= 0, \\ \phi(\rho, z, \psi)_{\psi\rho} &= 0.\end{aligned} \tag{16}$$

On solving this system of determining equations, we have

$$\begin{aligned}\xi(\rho, z, \psi) &= az\rho + b\rho, \\ \tau(\rho, z, \psi) &= a\frac{(\rho^2+z^2)}{2} + bz + c, \\ \phi(\rho, z, \psi) &= -\frac{1}{2}az\psi + l\psi,\end{aligned} \tag{17}$$

where a, b, c and l are arbitrary constants.

The Lie algebra associated with the system (17) consists of following four vector fields:

$$\begin{aligned}V_1 &= \frac{\partial}{\partial z}, \\ V_2 &= \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{z}, \\ V_3 &= \psi\frac{\partial}{\partial\psi}, \\ V_4 &= z\rho\frac{\partial}{\partial\rho} + \frac{(\rho^2+z^2)}{2}\frac{\partial}{\partial z} - \frac{z\psi}{2}.\end{aligned} \tag{18}$$

The commutation relations and the Adjoint Table for the Lie algebra G, determined by V_1, V_2, V_3, V_4 are shown in Table I and Table II respectively.

COMMUTATOR TABLE I

	V_1	V_2	V_3	V_4
V_1	0	V_1	0	V_2
V_2	$-V_1$	0	0	V_4
V_3	0	0	0	0
V_4	$-V_2$	$-V_4$	0	0

ADJOINT TABLE II

Ad	V_1	V_2	V_3	V_4
V_1	V_1	$V_2 - \epsilon V_1$	V_3	$V_4 - \epsilon V_2 + \frac{\epsilon^2}{2}V_1$
V_2	$V_1 e^\epsilon$	V_2	V_3	$V_4 e^{-\epsilon}$
V_3	V_1	V_2	V_3	V_4
V_4	$V_1 + \epsilon V_2 - \frac{\epsilon^2}{2}V_4$	$V_2 + \epsilon V_4$	V_3	V_4

The optimal system described by Olver [14] consists of the following four basic vector fields:

$$\begin{aligned}&(i) V_1, \\ &(ii) V_2, \\ &(iii) V_3 + \alpha V_2, \\ &(iv) V_4 + \beta V_2 + \delta V_1,\end{aligned} \tag{19}$$

where α, β and δ are arbitrary constants.

Now, the similarity variables and similarity solutions can be obtained by solving characteristic equations (14). The

general solution of these equations involves two constants; one becomes the new independent variable ζ and the other is F as dependent variable.

In this section, we have found similarity variables and similarity solutions for all the four essential vector fields with optimal system (19). The reduction of equation (11) into ODEs are obtained corresponds to each vector field in the optimal system. Some exact solutions of these ODEs and system (11)-(12) are also obtained.

Vector field (i) V_1
Corresponding to this vector field, the forms of the similarity variable and similarity solution are as follows:
$\zeta = \rho$, $\psi(\rho, z) = F(\zeta)$
By using these similarity variable and similarity solution in equation (11), the reduced ODE is as follows:

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} = 0, \qquad (20)$$

On solving this ODE, we have

$$F(\zeta) = c_1 + c_2 \log(\zeta). \qquad (21)$$

Thus the solution of PDE (11) is

$$\psi(\rho, z) = c_1 + c_2 \log(\rho) \qquad (22)$$

and after substituting value of ψ in (12), we have the metric function γ as

$$\gamma(\rho, z) = c_2^2 \log(\rho), \qquad (23)$$

also the metric function $f(\rho, z)$, from (1.10) is

$$f(\rho, z) = e^{(2c_1 + c_2 \log(\rho))} \qquad (24)$$

where c_1 and c_2 are arbitrary constants.

Vector field (ii) V_2
In this case, the forms of the similarity variable and similarity solution are as follows:
$\zeta = \frac{\rho}{z}$, $\psi(\rho, z) = F(\zeta)$.
Using these substitutions in equation (11), we get following reduced ODE:

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \zeta^2 F''(\zeta) - 2\zeta F'(\zeta) = 0, \qquad (25)$$

solution of this ODE is

$$F(\zeta) = c_1 + c_2 \arctan\left(\frac{1}{\sqrt{-1 + \zeta^2}}\right) \qquad (26)$$

and solution of system (11)and (12) and is

$$\psi(\rho, z) = c_1 + c_2 \arctan\left(\frac{1}{\sqrt{-1 + \left(\frac{\rho}{z}\right)^2}}\right) \qquad (27)$$

$$\gamma(\rho, z) = c_2^2((\log(\rho + z) - \log(\rho)) \\ + c_2^2 (\log(\rho - z) + \log((z - \rho)(z + \rho))). \qquad (28)$$

and the metric function $f(\rho, z)$ is

$$f(\rho, z) = e^{\left(2c_1 + c_2 \arctan\left(\frac{1}{\sqrt{-1 + \left(\frac{\rho}{z}\right)^2}}\right)\right)} \qquad (29)$$
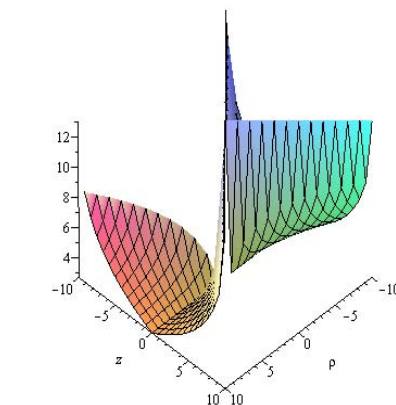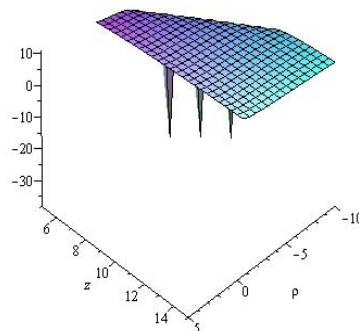

Fig. 1. Behavior of solution (28)

Fig. 2. Behavior of solution (29)

where c_1, c_2 and c_3 are arbitrary constants. The behavior of solutions (28) and (29) are represented by Fig1. and Fig2. respectively.

Vector field (iii) $V_3 + \alpha V_2$
For this vector field, the forms of the similarity variable and similarity solution are as follows:
$\zeta = \frac{\rho}{z}$, $\psi(\rho, z) = z^{\frac{1}{\alpha}} F(\zeta)$.
Using these substitutions, equation (11) reduces to

$$F''(\zeta) + \frac{F'(\zeta)}{\zeta} - \frac{F}{\alpha^2} + \frac{F}{\alpha} + \frac{2F(\zeta)F'(\zeta)}{\alpha} - \zeta^2 F''(\zeta) - 2\zeta F'(\zeta) = 0, \qquad (30)$$

by solving above ODE, we get

$$F(\zeta) = c_1 \left(\left(-\frac{1}{2\alpha}, \frac{1}{2}\frac{(\alpha-1)}{\alpha} \right), \left[\frac{1}{2} \frac{-2+\alpha}{\alpha} \right], -\zeta^2 + 1 \right)$$
$$+ c_2 (\zeta^2 - 1)^{\frac{\alpha+2}{2\alpha}} \left(\left[\frac{1+2\alpha}{2\alpha}, \frac{1+\alpha}{2\alpha} \right], \left[\frac{1}{2} \frac{3\alpha+2}{\alpha} \right], -\zeta^2 + 1 \right),$$
$$(31)$$

where c_1 and c_2 are arbitrary constants and is hypergeometric function.

Thus, the solution of equation (11) is

$$\psi(\rho, z) = c_1 \left(\left[-\frac{1}{2\alpha}, \frac{1}{2}\frac{(\alpha-1)}{\alpha} \right], \left[\frac{1}{2}\frac{-2+\alpha}{\alpha} \right], -(\frac{\rho}{z})^2 + 1 \right)$$
$$+ c_2 ((\frac{\rho}{z})^2 - 1)^{\frac{\alpha+2}{2\alpha}} \left(\left[\frac{1+2\alpha}{2\alpha}, \frac{1+\alpha}{2\alpha} \right], \left[\frac{1}{2}\frac{3\alpha+2}{\alpha} \right], -(\frac{\rho}{z})^2 + 1 \right),$$
$$(32)$$

solution of equation (12) will be in the form of integration and the metric function $f(\rho, z)$ is as follows

$$f(\rho, z) = e^{2\psi}, \qquad (33)$$

where ψ is given by (32) and c_1 and c_2 are constants of integration.

Vector field (iv) $V_4 + \beta V_3 + \delta V_2$

Corresponding to this vector field reduction is not possible.

IV. CONCLUSION

In order to completely describe the gravitational field of a body, one must know the corresponding exact solutions. Keeping in view the efficacy and physical importance of Weyl metric, we have in the foregone sections, first derived a system of PDEs from Weyl metric by using Einstein tensor. After that by Lie classical approach, we have investigated the symmetries of Einstein vacuum equations and utilized these symmetries for obtaining group infinitesimals which are helpful in the reduction of PDE to ODE. It is worth to mention here that the solutions obtained can be further utilized for physical applications. The authenticity of solutions has been checked with aid of software Maple.

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