

On Quasi Conformally Flat LP-Sasakian Manifolds with a Coefficient α

Jay Prakash Singh

Abstract—The aim of the present paper is to study properties of Quasi conformally flat LP-Sasakian manifolds with a coefficient α . In this paper, we prove that a Quasi conformally flat LP-Sasakian manifold M ($n > 3$) with a constant coefficient α is an η -Einstein and in a quasi conformally flat LP-Sasakian manifold M ($n > 3$) with a constant coefficient α if the scalar curvature tensor is constant then M is of constant curvature.

Keywords—LP-Sasakian manifolds, coefficient α , quasi conformal curvature tensor, concircular vector field, torse forming vector field, η -Einstein manifold.

I. INTRODUCTION

THE notion of LP-Sasakian manifolds has been introduced by Matsumoto [4]. Then in this line, Mihai and Rosca [5] introduced the same notion independently and obtained several results in this manifold. In 2002, De et al. [2] introduced the notion of LP-Sasakian manifolds with a coefficient α which generalizes the notion of LP-Sasakian manifolds. In [3], De et al. studied these manifolds with conformally flat curvature tensor and then Bagewadi et al. [1] investigated it with pseudo projectively flat curvature tensor.

In 1968, Yano and Sawaki [8] defined and studied a tensor field W of type (1,3) which includes both the conformal curvature tensor and concircular curvature tensor as special cases and called Quasi conformal curvature tensor which is given as

$$\begin{aligned} W(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X \\ &- S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &- \frac{r}{n} \left(\frac{a}{n-1} + 2b \right) \{g(Y, Z)X \\ &- g(X, Z)Y\}, \end{aligned} \quad (1)$$

where R, S, Q, r denote curvature tensor, Ricci tensor, Ricci operator, scalar curvature tensor respectively and a, b are arbitrary constant not simultaneously zero. Motivated by these studies in this paper, we have studied some properties of quasi conformally flat LP-Sasakian manifolds with a coefficient α . Here, we prove that in a Quasi conformally flat LP-Sasakian manifolds with a coefficient α , the characteristic vector field ξ is a concircular vector field if and only if the manifold is η -Einstein. Finally, we prove that Quasi conformally flat LP-Sasakian manifolds with a coefficient α is a manifold of constant curvature if the scalar curvature r is constant.

II. PRELIMINARIES

Let M be an n -dimensional differentiable manifold endowed with a (1,1) tensor field ϕ , contravariant vector field ξ , a

covariant vector field η , and a Lorentzian metric g of type (1,2) such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow R$ is a non degenerate inner product of signature $(-, +, +, \dots, +)$ where $T_p M$ denotes the tangent vector space of M at p and R is real number space, which satisfies

$$\eta(\xi) = -1, \quad \phi^2 X = X + \eta(X)\xi \quad (2)$$

$$g(X, \xi) = \eta(X)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (3)$$

for all vector fields X, Y . The structures (ϕ, ξ, η, g) are said to be Lorentzian almost paracontact structure and the manifold M with the structures (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [4]. In the Lorentzian almost paracontact manifold M , the following relations hold [4]:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad (4)$$

$$\Omega(X, Y) = g(Y, X), \quad (5)$$

where $\Omega(X, Y) = g(X, \phi Y)$.

In the Lorentzian almost paracontact manifold M , if the relations

$$\begin{aligned} (D_Z \Omega)(X, Y) &= \alpha[\{g(X, Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y, Z) + \eta(Y)\eta(Z)\}\eta(X)], \end{aligned} \quad (6)$$

$$\Omega(X, Y) = \frac{1}{\alpha}(D_X \eta)(Y), \quad (7)$$

hold where D denotes the operator of covariant differentiation with respect to the Lorentzian metric g , then M is called LP-Sasakian manifolds with a coefficient α [2]. An LP-Sasakian manifolds with a coefficient $\alpha = 1$ is an LP-Sasakian manifolds [4].

If a vector field V satisfies the equation

$$D_X V = \beta X + T(X)V,$$

where β is a non zero scalar function and T is a covariant vector field, then V is called a torse forming vector field [7]. In a Lorentzian manifold M , if we assume that ξ is a unit torse forming vector field, then we have:

$$(D_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad (8)$$

where α is a non zero scalar function. Hence, the manifold admitting a unit torse forming vector field satisfying (8) is an

J. P. Singh is with the Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram, 796004 India (e-mail: jpsmaths@gmail.com).

LP-Sasakian manifolds with a coefficient α . Especially, if η satisfies

$$(D_X\eta)(Y) = \epsilon[g(X, Y) + \eta(X)\eta(Y)], \quad \epsilon^2 = 1 \quad (9)$$

then M is called an LSP-Sasakian Manifold [4]. In particular, if α satisfies (8) and the following equation

$$\alpha(X) = p \eta(X), \quad \alpha(X) + D_X\alpha, \quad (10)$$

where p is a scalar function, then ξ is called a concircular vector field. Let us consider an LP-Sasakian manifolds $M(\phi, \xi, \eta, g)$ with a coefficient α . Then we have the following relations [4]

$$\begin{aligned} \eta(R(X, Y)Z) &= -\alpha(X)\Omega(Y, Z) + \alpha(Y)\Omega(X, Z) \\ &+ \alpha^2\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \end{aligned} \quad (11)$$

$$S(X, \xi) = -\Psi\alpha(X) + (n-1)\alpha^2\eta(X) + \alpha(\phi X), \quad (12)$$

where $\Psi = \text{Trace}(\phi)$.

We now state the following results which will be needed in the later section.

Lemma 1. [2] In an LP-Sasakian manifolds with a coefficient α , one of the following cases occur;

- i) $\Psi^2 = (n-1)^2$
- ii) $\alpha(Y) = -p \eta(Y)$, where $p = \alpha(\xi)$.

Lemma 2. [2] In a Lorentzian almost paracontact manifold M with its structure (ϕ, ξ, η, g) satisfying $\Omega(X, Y) = \frac{1}{\alpha}(D_X\eta)(Y)$, where α is a non-zero scalar function, the vector field ξ is a torse forming if and only if the relation $\Psi^2 = (n-1)^2$ holds good.

III. QUASI CONFORMALLY FLAT LP-SASAKIAN MANIFOLDS WITH A COEFFICIENT α

Let us consider a Quasi conformally flat LP-Sasakian manifolds M ($n > 3$) with a coefficient α . Then, since the quasi conformal curvature tensor W vanishes, (1) reduces to

$$\begin{aligned} R(X, Y, Z, U) &= -\frac{b}{a}[S(Y, Z)g(X, U) - S(X, Z)g(Y, U) \\ &+ S(X, U)g(Y, Z) - g(X, Z)S(Y, U)] \\ &+ \frac{r}{n}(\frac{1}{n-1} + \frac{2b}{a})\{g(Y, Z)g(X, U) \\ &- g(X, Z)g(Y, U)\}. \end{aligned} \quad (13)$$

Putting $U = \xi$ in (13) and using (11) and (12), we get

$$\begin{aligned} -\alpha(X)\Omega(Y, Z) &+ \alpha(Y)\Omega(X, Z) \\ &+ \alpha^2\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \\ &= -\frac{b}{a}\left[\{S(Y, Z)\eta(X) - S(X, Z)\eta(Y)\} \right. \\ &+ g(Y, Z)\{-\Psi\alpha(X) + (n-1)\alpha^2\eta(X) \\ &+ \alpha(\phi X)\} - g(X, Z)\{-\Psi\alpha(Y) \\ &+ (n-1)\alpha^2\eta(Y) + \alpha(\phi Y)\}] \\ &+ \frac{r}{n}(\frac{1}{n-1} + \frac{2b}{a})\{g(Y, Z)\eta(X) \\ &- g(X, Z)\eta(Y)\}. \end{aligned} \quad (14)$$

Again putting $X = \xi$ in (14) and using (4) and (12), we obtain by straightforward calculations

$$\begin{aligned} S(Y, Z) &= \left\{\frac{ar}{n(n-1)b} + \frac{2r}{n} - p\Psi - (n-1)\alpha^2 \right. \\ &- \frac{a}{b}\alpha^2\}g(Y, Z) + \left\{\frac{ar}{n(n-1)b} \right. \\ &+ \frac{2r}{n} - 2(n-1)\alpha^2 - \frac{a}{b}\alpha^2\}\eta(Y)\eta(Z) \\ &+ \{\Psi\alpha(Z) - \alpha(\phi Z)\}\eta(Y) \\ &+ \{\Psi\alpha(Y) - \alpha(\phi Y)\}\eta(Z) \\ &- \frac{a}{b}p \Omega(Y, Z), \end{aligned} \quad (15)$$

where $p = \alpha(\xi)$. If an LP-Sasakian manifolds M with a coefficient α satisfies the relation

$$S(X, Y) = cg(X, Y) + d\eta(X)\eta(Y),$$

where c, d are associated functions on the manifold, then the manifold M is said to be an η -Einstein manifold. Now we have [2]

$$\begin{aligned} S(Y, Z) &= \left[\frac{r}{(n-1)} - \alpha^2 + \frac{p\Psi}{n-1}\right]g(X, Y) \\ &+ \left[\frac{r}{(n-1)} - \alpha^2 + \frac{np\Psi}{n-1}\right]\eta(X)\eta(Y). \end{aligned} \quad (16)$$

Contracting (16), we obtain

$$r = n(n-1)\alpha^2 + n p\Psi. \quad (17)$$

By virtue of (15) and (16), we get

$$\begin{aligned} &\left[\frac{\{a + (n-2)b\}r}{n(n-1)b} - \{a + (n-2)b\}\frac{\alpha^2}{b} \right. \\ &+ \frac{(2-n)p\Psi}{n-1}\}g(Y, Z) + \left[\frac{\{a + (n-2)b\}r}{n(n-1)b} \right. \\ &- \{a + (n-2)b\}\frac{\alpha^2}{b} + \frac{np\Psi}{n-1}\}\eta(Y)\eta(Z) \\ &+ \{\Psi\alpha(Z) - \alpha(\phi Z)\}\eta(Y) \\ &+ \{\Psi\alpha(Y) - \alpha(\phi Y)\}\eta(Z) \\ &- p\frac{a}{b}\Omega(Y, Z) = 0. \end{aligned} \quad (18)$$

Putting $Z = \xi$ in (18), we obtain

$$\Psi\alpha(Y) - \alpha(\phi Y) = -\Psi p \eta(Y), \quad (19)$$

for all vector fields Y . In consequence of (17) and (19), (18) becomes

$$\begin{aligned} \frac{a}{b}\left[\frac{\Psi}{n-1}\{g(Y, Z) + \eta(Y)\eta(Z)\} \right. \\ \left. - \Omega(Y, Z)\right] = 0. \end{aligned} \quad (20)$$

If $p=0$, then from (19) we have $\alpha(\phi Y) = \Psi\alpha(Y)$. Thus, since Ψ is an eigenvalue of the matrix ϕ , Ψ is equal to ± 1 . Hence by Lemma 1, we get $\alpha(Y) = 0$ for all Y and hence α is constant which contradict to our assumption. Consequently, we have $p \neq 0$ and hence from (20) we get

$$\begin{aligned} \frac{a}{b}\left[\frac{\Psi}{n-1}\{g(Y, Z) + \eta(Y)\eta(Z)\} \right. \\ \left. - \Omega(Y, Z)\right] = 0. \end{aligned} \quad (21)$$

Replacing Y by ϕY in (21) and using (4), we get

$$\frac{a}{b} [\Omega(Y, Z) - \frac{\Psi}{n-1} \{g(Y, Z) + \eta(Y)\eta(Z)\}] = 0, \quad (22)$$

Combining (21) and (22), we get

$$\{\Psi^2 - (n-1)^2\} [g(Y, Z) + \eta(Y)\eta(Z)] = 0,$$

which gives by virtue $n > 3$

$$\Psi^2 = (n-1)^2. \quad (23)$$

Hence, Lemma 2 proves that ξ is torse forming. Again, we have

$$(D_X \eta)(Y) = \beta \{g(X, Y) + \eta(X)\eta(Y)\}.$$

Now from (7) we get

$$\begin{aligned} \Omega(X, Y) &= \frac{\beta}{\alpha} \{g(X, Y) + \eta(X)\eta(Y)\} \\ &= g\left(\frac{\beta}{\alpha}(X + \eta(X)\xi), Y\right), \end{aligned}$$

and $\Omega(X, Y) = g(X, \phi Y)$.

Since g is non singular, we have

$$\phi(X) = \frac{\beta}{\alpha}(X + \eta(X)\xi)$$

and

$$\phi^2(X) = \left(\frac{\beta}{\alpha}\right)^2 (X + \eta(X)\xi).$$

It follows from (2) that $\left(\frac{\beta}{\alpha}\right)^2 = 1$ and hence $\alpha = \pm\beta$. Thus, we have

$$\phi(X) = \pm(X + \eta(X)\xi).$$

By virtue of (19) we see that $\alpha(Y) = -p\eta(Y)$. Thus, we conclude that ξ is a concircular vector field. Conversely suppose that ξ is a concircular vector field. Then we have:

$$(D_X \eta)(Y) = \beta \{g(X, Y) + \eta(X)\eta(Y)\},$$

where β is a certain function and $(D_X \beta)(Y) = q\eta(X)$ for a certain scalar function q . Hence by virtue of (7), we have $\alpha = \pm\beta$. Thus

$$\begin{aligned} \Omega(X, Y) &= \epsilon \{g(X, Y) + \eta(X)\eta(Y)\}, \epsilon^2 = 1, \\ \Psi &= \epsilon(n-1), D_X \alpha = \alpha(X) + p\eta(X), p = \epsilon q. \end{aligned}$$

Using these relations and (19) in (15), it can be easily seen that M is η -Einstein manifold. This leads to the following theorem:

Theorem 1. In a Quasi conformally flat LP-Sasakian manifold $M(n > 3)$ with a non constant coefficient α , the characteristic vector field η is a concircular vector field if and only if M is η -Einstein manifold.

Next we consider the case when α is constant. In this case, the following relations hold:

$$\eta(R(X, Y)Z) = \alpha^2 \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}, \quad (24)$$

$$S(X, \xi) = (n-1)\eta(X). \quad (25)$$

Putting $U = \xi$ in (13) and then using (31) and (25), we get

$$\begin{aligned} &\alpha^2 \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \\ &= -\frac{b}{a} [S(Y, Z)\eta(X) - S(Y, Z)\eta(Y) \\ &+ (n-1)\alpha^2 g(Y, Z)\eta(X) \\ &- (n-1)\alpha^2 g(X, Z)\eta(Y)] \\ &+ \frac{r}{n} \left(\frac{1}{n-1} + \frac{2b}{a}\right) \{g(Y, Z)\eta(X) \\ &- g(X, Z)\eta(Y)\}. \end{aligned}$$

Again putting $U = \xi$ in above and making use of (25) we get

$$\begin{aligned} S(Y, Z) &= \left[\frac{\{a + 2b(n-1)\}r}{bn(n-1)} \right. \\ &- \frac{\alpha^2}{b} \{a + b(n-1)\} \left. \right] g(Y, Z) \\ &+ \left[\frac{\{a + 2b(n-1)\}r}{bn(n-1)} \right. \\ &- \frac{\alpha^2}{b} \{a + 2b(n-1)\} \left. \right] \eta(Y)\eta(Z). \quad (26) \end{aligned}$$

Thus, we can state the following theorem:

Theorem 2. A Quasi conformally flat LP-Sasakian manifold $M(n > 3)$ with a constant coefficient α is an η -Einstein.

Differentiating covariantly (26) along X and making use of (7), we obtain

$$\begin{aligned} (D_X S)(Y, Z) &= \frac{dr(X)}{b(n-1)n} \{a + 2b(n-1)\} \times \\ &\{g(Y, Z) + \eta(Y)\eta(Z)\} \\ &+ \frac{\alpha\{a + 2b(n-1)\}}{b} \left(\frac{r}{n(n-1)} - \alpha^2\right) \times \\ &\{\Omega(X, Y)\eta(Z) + \Omega(X, Z)\eta(Y)\}. \quad (27) \end{aligned}$$

where $dr(X) = D_X r$. This implies that

$$\begin{aligned} (D_X S)(Y, Z) &- (D_Y S)(X, Z) \\ &= \frac{dr(X)}{n} \left(2 + \frac{a}{b(n-1)}\right) \{g(Y, Z) \\ &+ \eta(Y)\eta(Z)\} - \frac{dr(Y)}{n} \left(2 + \frac{a}{b(n-1)}\right) \{g(X, Z) \\ &+ \eta(X)\eta(Z)\} \\ &+ \frac{\alpha}{b} \{a + 2b(n-1)\} \left(\frac{r}{n(n-1)} - \alpha^2\right) \times \\ &\{\Omega(X, Z)\eta(Y) + \Omega(Y, Z)\eta(X)\}. \quad (28) \end{aligned}$$

On the other hand, we also have for a Quasi conformally flat curvature tensor [6]

$$\begin{aligned} &(D_X S)(Y, Z) - (D_Y S)(X, Z) \\ &= \frac{\{2a - (n-1)(n-4)b\}}{2(a+b)n(n-1)} [dr(X)g(Y, Z) \\ &- dr(Y)g(X, Z)], \quad (29) \end{aligned}$$

provided $a + 2b(n-1) \neq 0$. From (28) and (29), it follows that

$$\begin{aligned} & \frac{dr(X)}{n} \left(2 + \frac{a}{b(n-1)} \right) \{ g(Y, Z) + \eta(Y)\eta(Z) \} \\ & - \frac{dr(Y)}{n} \left(2 + \frac{a}{b(n-1)} \right) \{ g(X, Z) + \eta(X)\eta(Z) \} \\ & + \frac{\alpha}{b} \{ a + 2(n-1)b \} \left(\frac{r}{n(n-1)} - \alpha^2 \right) \times \\ & \quad \{ \Omega(X, Z)\eta(Y) + \Omega(Y, Z)\eta(X) \} \\ & = \frac{\{ 2a - (n-1)(n-4)b \}}{2(a+b)n(n-1)} [dr(X)g(Y, Z) \\ & \quad - dr(Y)g(X, Z)]. \end{aligned} \quad (30)$$

If r is constant then (30) yields

$$r = n(n-1)\alpha^2.$$

Hence from (13), it follows that

$$R(X, Y, Z, U) = \alpha^2 [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)], \quad (31)$$

which shows that the manifold is of constant curvature. Thus, we can state the following:

Theorem 3. In a Quasi conformally flat LP-Sasakian manifold M ($n > 3$) with a constant coefficient α , if the scalar curvature tensor is constant then M is of constant curvature.

IV. CONCLUSION

The present paper is about the study of some geometrical properties of quasi conformally flat LP-Sasakian manifolds with a coefficient α . It is established that a quasi conformally flat LP-Sasakian manifold M ($n > 3$) with a constant coefficient α is an η -Einstein and in a quasi conformally flat LP-Sasakian manifold M ($n > 3$) with a non coefficient α , the characteristic vector field η is a concircular vector field if and only if M is η -Einstein manifold.

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Jay Prakash Singh completed his B.Sc., M.Sc. (Mathematics) and Ph.D. from Banaras Hindu University, Varanasi, U.P., India. After completing his Ph.D., he joined Mizoram University, Aizawl-796004, Mizoram, India as Assistant Professor in 2007.