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# On Quasi Conformally Flat LP-Sasakian Manifolds with a Coefficient $\alpha$

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*Abstract*—The aim of the present paper is to study properties of Quasi conformally flat LP-Sasakian manifolds with a coefficient $\alpha$. In this paper, we prove that a Quasi conformally flat LP-Sasakian manifold M $(n > 3)$ with a constant coefficient $\alpha$ is an $\eta-$Einstein and in a quasi conformally flat LP-Sasakian manifold M $(n > 3)$ with a constant coefficient $\alpha$ if the scalar curvature tensor is constant then M is of constant curvature.

*Keywords*—LP-Sasakian manifolds, coefficient $\alpha$, quasi conformal curvature tensor, concircular vector field, torse forming vector field, $\eta$-Einstein manifold.

## I. Introduction

THE notion of LP-Sasakian manifolds has been introduced by Matsumoto [4]. Then in this line, Mihai and Rosca [5] introduced the same notion independently and obtained several results in this manifold. In 2002, De et al. [2] introduced the notion of LP-Sasakian manifolds with a coefficient $\alpha$ which generalizes the notion of LP-Sasakian manifolds. In [3], De et al. studied these manifolds with conformally flat curvature tensor and then Bagewadi et al. [1] investigated it with pseudo projectively flat curvature tensor.

In 1968, Yano and Sawaki [8] defined and studied a tensor field $W$ of type (1,3) which includes both the conformal curvature tensor and concicular curvature tensor as special cases and called Quasi conformal curvature tensor which is given as

$$\begin{aligned} W(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X \\ &- S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &- \frac{r}{n}(\frac{a}{n-1} + 2b)\{g(Y,Z)X \\ &- g(X,Z)Y\}, \end{aligned} \tag{1}$$

where $R, S, Q, r$ denote curvature tensor, Ricci tensor, Ricci operator, scalar curvature tensor respectively and $a, b$ are arbitrary constant not simultaneously zero. Motivated by these studies in this paper, we have studied some properties of quasi conformally flat LP-Sasakian manifolds with a coefficient $\alpha$. Here, we prove that in a Quasi conformally flat LP-Sasakian manifolds with a coefficient $\alpha$, the characteristic vector field $\xi$ is a concircular vector field if and only if the manifold is $\eta$-Einstein. Finally, we prove that Quasi conformally flat LP-Sasakian manifolds with a coefficient $\alpha$ is a manifold of constant curvature if the scalar curvature $r$ is constant.

## II. Preliminaries

Let M be an n-dimensional differentiable manifold endowed with a (1,1) tensor field $\phi$, contravariant vector field $\xi$, a covariant vector field $\eta$, and a Lorentzian metric g of type (1,2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \longrightarrow R$ is a non degenerate inner product of signature $(-, +, +, ..., +)$ where $T_pM$ denotes the tangent vector space of M at p and $R$ is real number space, which satisfies

$$\eta(\xi) = -1, \qquad \phi^2 X = X + \eta(X)\xi \tag{2}$$

$$g(X,\xi) = \eta(X)$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

for all vector fields $X, Y$. The structures $(\phi, \xi, \eta, g)$ are said to be Lorentzian almost paracontact structure and the manifold M with the structures $(\phi, \xi, \eta, g)$ is called Lorentzian almost paracontact manifold [4]. In the Lorentzian almost paracontact manifold M, the following relations hold [4]:

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{4}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{5}$$

where $\Omega(X,Y) = g(X, \phi Y)$.

In the Lorentzian almost paracontact manifold M, if the relations

$$\begin{aligned} (D_Z\Omega)(X,Y) &= \alpha[\{g(X,Z) + \eta(X)\eta(Z)\}\eta(Y) \\ &+ \{g(Y,Z) + \eta(Y)\eta(Z)\}\eta(X)], \end{aligned} \tag{6}$$

$$\Omega(X,Y) = \frac{1}{\alpha}(D_X\eta)(Y), \tag{7}$$

hold where $D$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$, then M is called LP-Sasakian manifolds with a coefficient $\alpha$ [2]. An LP-Sasakian manifolds with a coefficient $\alpha = 1$ is an LP-Sasakian manifolds [4].

If a vector field V satisfies the equation

$$D_X V = \beta X + T(X)V,$$

where $\beta$ is a non zero scalar function and T is a covariant vector field, then V is called a torse forming vector field [7]. In a Lorentzian manifold M, if we assume that $\xi$ is a unit torse forming vector field, then we have:

$$(D_X\eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \tag{8}$$

where $\alpha$ is a non zero scalar function. Hence, the manifold admitting a unit torse forming vector field satisfying $(8)$ is an

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LP-Sasakian manifolds with a coefficient $\alpha$. Especially, if $\eta$ satisfies

$$(D_X\eta)(Y) = \epsilon[g(X,Y)+\eta(X)\eta(Y)], \quad \epsilon^2=1 \quad (9)$$

then M is called an LSP-Sasakian Manifold [4]. In particular, if $\alpha$ satisfies (8) and the following equation

$$\alpha(X) = p\ \eta(X), \quad \alpha(X)+D_X\alpha, \quad (10)$$

where p is a scalar function, then $\xi$ is called a concircular vector field. Let us consider an LP-Sasakian manifolds M $(\phi,\xi,\eta,g)$ with a coefficient $\alpha$. Then we have the following relations [4]

$$\begin{aligned}\eta(R(X,Y)Z) &= -\alpha(X)\Omega(Y,Z)+\alpha(Y)\Omega(X,Z)\\ &+ \alpha^2\{g(Y,Z)\eta(X)\\ &- g(X,Z)\eta(Y)\}, \quad (11)\end{aligned}$$

$$S(X,\xi) = -\Psi\alpha(X)+(n-1)\alpha^2\eta(X)+\alpha(\phi X), \quad (12)$$

where $\Psi$=Trace($\phi$).

We now state the following results which will be needed in the later section.

**Lemma 1.** [2] In an LP-Sasakian manifolds with a coefficient $\alpha$, one of the following cases occur;

$$i) \quad \Psi^2 = (n-1)^2$$
$$ii) \quad \alpha(Y) = -p\ \ \eta(Y), \quad where \quad p=\alpha(\xi).$$

**Lemma 2.** [2] In a Lorentzian almost paracontact manifold M with its structure $(\phi,\xi,\eta,g)$ satisfying $\Omega(X,Y) = \frac{1}{\alpha}(D_X\eta)(Y)$, where $\alpha$ is a non-zero scalar function, the vector field $\xi$ is a torse forming if and only if the relation $\Psi^2=(n-1)^2$ holds good.

## III. Quasi Conformally Flat LP-Sasakian Manifolds with a Coefficient $\alpha$

Let us consider a Quasi conformally flat LP-Sasakian manifolds M $(n>3)$ with a coefficient $\alpha$. Then, since the quasi conformal curvature tensor W vanishes, (1) reduces to

$$\begin{aligned}R(X,Y,Z,U) &= -\frac{b}{a}\big[S(Y,Z)g(X,U)-S(X,Z)g(Y,U)\\ &+ S(X,U)g(Y,Z)-g(X,Z)S(Y,U)\big]\\ &+ \frac{r}{n}(\frac{1}{n-1}+\frac{2b}{a})\{g(Y,Z)g(X,U)\\ &- g(X,Z)g(Y,U)\}. \quad (13)\end{aligned}$$

Putting $U=\xi$ in (13) and using (11) and (12), we get

$$\begin{aligned}-\alpha(X)\Omega(Y,Z) &+ \alpha(Y)\Omega(X,Z)\\ &+ \alpha^2\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\\ &= -\frac{b}{a}\Big[\{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\}\\ &+ g(Y,Z)\{-\Psi\alpha(X)+(n-1)\alpha^2\eta(X)\\ &+ \alpha(\phi X)\}-g(X,Z)\{-\Psi\alpha(Y)\\ &+ (n-1)\alpha^2\eta(Y)+\alpha(\phi Y)\}\Big]\\ &+ \frac{r}{n}(\frac{1}{n-1}+\frac{2b}{a})\{(g(Y,Z)\eta(X)\\ &- g(X,Z)\eta(Y))\}. \quad (14)\end{aligned}$$

Again putting $X=\xi$ in (14) and using (4) and (12), we obtain by straightforward calculations

$$\begin{aligned}S(Y,Z) &= \{\frac{ar}{n(n-1)b}+\frac{2r}{n}-p\Psi-(n-1)\alpha^2\\ &- \frac{a}{b}\alpha^2\}g(Y,Z)+\{\frac{ar}{n(n-1)b}\\ &+ \frac{2r}{n}-2(n-1)\alpha^2-\frac{a}{b}\alpha^2\}\eta(Y)\eta(Z)\\ &+ \{\Psi\alpha(Z)-\alpha(\phi Z)\}\eta(Y)\\ &+ \{\Psi\alpha(Y)-\alpha(\phi Y)\}\eta(Z)\\ &- \frac{a}{b}p\ \ \Omega(Y,Z), \quad (15)\end{aligned}$$

where $p=\alpha(\xi)$. If an LP-Sasakian manifolds M with a coefficient $\alpha$ satisfies the relation

$$S(X,Y) = cg(X,Y)+d\eta(X)\eta(Y),$$

where $c,d$ are associated functions on the manifold, then the manifold M is said to be an $\eta$-Einstein manifold. Now we have [2]

$$\begin{aligned}S(Y,Z) &= [\frac{r}{(n-1)}-\alpha^2+\frac{p\Psi}{n-1}]g(X,Y)\\ &+ [\frac{r}{(n-1)}-\alpha^2+\frac{np\Psi}{n-1}]\eta(X)\eta(Y). \quad (16)\end{aligned}$$

Contracting (16), we obtain

$$r = n(n-1)\alpha^2+n\ \ p\Psi. \quad (17)$$

By virtue of (15) and (16), we get

$$\begin{aligned}&\Big[\frac{\{a+(n-2)b\}r}{n(n-1)b}-\{a+(n-2)b\}\frac{\alpha^2}{b}\\ &+ \frac{(2-n)p\Psi}{n-1}\Big]g(Y,Z)+\Big[\frac{\{a+(n-2)b\}r}{n(n-1)b}\\ &- \{a+(n-2)b\}\frac{\alpha^2}{b}+\frac{np\Psi}{n-1}\Big]\eta(Y)\eta(Z)\\ &+ \{\Psi\alpha(Z)-\alpha(\phi Z)\}\eta(Y)\\ &+ \{\Psi\alpha(Y)-\alpha(\phi Y)\}\eta(Z)\\ &- p\frac{a}{b}\Omega(Y,Z)=0. \quad (18)\end{aligned}$$

Putting $Z=\xi$ in (18), we obtain

$$\Psi\alpha(Y)-\alpha(\phi Z) = -\Psi p\ \ \eta(Y), \quad (19)$$

for all vector fields Y. In consequence of (17) and (19), (18) becomes

$$\begin{aligned}\frac{a}{b}[\frac{\Psi}{n-1}\{g(Y,Z) &+ \eta(Y)\eta(Z)\}\\ &- \Omega(Y,Z)]=0. \quad (20)\end{aligned}$$

If p=0, then from (19) we have $\alpha(\phi Y)=\Psi\alpha(Y)$. Thus, since $\Psi$ is an eigenvalue of the matrix $\phi$, $\Psi$ is equal to $\pm 1$. Hence by Lemma 1, we get $\alpha(Y)=0$ for all $Y$ and hence $\alpha$ is constant which contradict to our assumption. Consequently, we have $p\neq 0$ and hence from (20) we get

$$\begin{aligned}\frac{a}{b}\big[\frac{\Psi}{n-1}\{g(Y,Z) &+ \eta(Y)\eta(Z)\}\\ &- \Omega(Y,Z)\big]=0. \quad (21)\end{aligned}$$

Replacing $Y$ by $\phi Y$ in (21) and using (4), we get

$$\frac{a}{b}\big[\Omega(Y,Z) - \frac{\Psi}{n-1}\{g(Y,Z) + \eta(Y)\eta(Z)\}\big] = 0, \quad (22)$$

Combining (21) and (22), we get

$$\{\Psi^2 - (n-1)^2\}[g(Y,Z) + \eta(Y)\eta(Z)] = 0,$$

which gives by virtue $n > 3$

$$\Psi^2 = (n-1)^2. \quad (23)$$

Hence, Lemma 2 proves that $\xi$ is torse forming. Again, we have

$$(D_X\eta)(Y) = \beta\{g(X,Y) + \eta(X)\eta(Y)\}.$$

Now from (7) we get

$$\Omega(X,Y) = \frac{\beta}{\alpha}\{g(X,Y) + \eta(X)\eta(Y)\} = g(\frac{\beta}{\alpha}(X + \eta(X)\xi, Y)),$$

and $\Omega(X,Y) = g(X,\phi Y)$.
Since g is non singular, we have

$$\phi(X) = \frac{\beta}{\alpha}(X + \eta(X)\xi)$$

and

$$\phi^2(X) = (\frac{\beta}{\alpha})^2(X + \eta(X)\xi).$$

It follows from (2) that $(\frac{\beta}{\alpha})^2 = 1$ and hence $\alpha = \pm\beta$. Thus, we have

$$\phi(X) = \pm(X + \eta(X)\xi).$$

By virtue of (19) we see that $\alpha(Y) = -p\eta(Y)$. Thus, we conclude that $\xi$ is a concircular vector field. Conversely suppose that $\xi$ is a concircular vector field. Then we have:

$$(D_X\eta)(Y) = \beta\{g(X,Y) + \eta(X)\eta(Y)\},$$

where $\beta$ is a certain function and $(D_X\beta)(Y) = q\eta(X)$ for a certain scalar function q. Hence by virtue of (7), we have $\alpha = \pm\beta$. Thus

$$\Omega(X,Y) = \epsilon\{g(X,Y) + \eta(X)\eta(Y)\}, \epsilon^2 = 1,$$
$$\Psi = \epsilon(n-1), D_X\alpha = \alpha(X) + p\eta(X), p = \epsilon q.$$

Using these relations and (19) in (15), it can be easily seen that M is $\eta$-Einstein manifold. This leads to the following theorem:

**Theorem 1.** In a Quasi conformally flat LP-Sasakian manifold $M(n > 3)$ with a non constant coefficient $\alpha$, the characteristic vector field $\eta$ is a concircular vector field if and only if M is $\eta$-Einstein manifold.

Next we consider the case when $\alpha$ is constant. In this case, the following relations hold:

$$\eta(R(X,Y)Z) = \alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \quad (24)$$

$$S(X,\xi) = (n-1)\eta(X). \quad (25)$$

Putting $U = \xi$ in (13) and then using (31) and (25), we get

$$\begin{aligned} &\alpha^2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \\ = & -\frac{b}{a}\Big[S(Y,Z)\eta(X) - S(Y,Z)\eta(Y) \\ + & (n-1)\alpha^2 g(Y,Z)\eta(X) \\ - & (n-1)\alpha^2 g(X,Z)\eta(Y)\Big] \\ + & \frac{r}{n}(\frac{1}{n-1} + \frac{2b}{a})\{g(Y,Z)\eta(X) \\ - & g(X,Z)\eta(Y)\}. \end{aligned}$$

Again putting $U = \xi$ in above and making use of (25) we get

$$\begin{aligned} S(Y,Z) = & \Big[\frac{\{a + 2b(n-1)\}r}{bn(n-1)} \\ - & \frac{\alpha^2}{b}\{a + b(n-1)\}\Big]g(Y,Z) \\ + & \Big[\frac{\{a + 2b(n-1)\}r}{bn(n-1)} \\ - & \frac{\alpha^2}{b}\{a + 2b(n-1)\}\Big]\eta(Y)\eta(Z). \end{aligned} \quad (26)$$

Thus, we can state the following theorem:

**Theorem 2.** A Quasi conformally flat LP-Sasakian manifold M $(n > 3)$ with a constant coefficient $\alpha$ is an $\eta$-Einstein.

Differentiating covariantly (26) along X and making use of (7), we obtain

$$\begin{aligned} (D_X S)(Y,Z) = & \frac{dr(X)}{b(n-1)n}\{a + 2b(n-1)\} \times \\ & \{g(Y,Z) + \eta(Y)\eta(Z)\} \\ + & \frac{\alpha\{a + 2b(n-1)\}}{b}(\frac{r}{n(n-1)} - \alpha^2) \times \\ & \{\Omega(X,Y)\eta(Z) + \Omega(X,Z)\eta(Y)\}. \end{aligned} \quad (27)$$

where $dr(X) = D_X r$. This implies that

$$\begin{aligned} (D_X S)(Y,Z) - & (D_Y S)(X,Z) \\ = & \frac{dr(X)}{n}\big(2 + \frac{a}{b(n-1)}\big)\{g(Y,Z) \\ + & \eta(Y)\eta(Z)\big\} - \frac{dr(Y)}{n}\big(2 \\ + & \frac{a}{b(n-1)}\big)\{g(X,Z) + \eta(X)\eta(Z)\} \\ + & \frac{\alpha}{b}\{a + 2b(n-1)\}(\frac{r}{n(n-1)} - \alpha^2) \times \\ & \big\{\Omega(X,Z)\eta(Y) + \Omega(Y,Z)\eta(X)\big\}. \end{aligned} \quad (28)$$

On the other hand, we also have for a Quasi conformally flat curvature tensor [6]

$$\begin{aligned} & (D_X S)(Y,Z) - (D_Y S)(X,Z) \\ = & \frac{\{2a - (n-1)(n-4)b\}}{2(a+b)n(n-1)}\big[dr(X)g(Y,Z) \\ - & dr(Y)g(X,Z)\big], \end{aligned} \quad (29)$$

provided $a + 2b(n-1) \neq 0$. From $(28)$ and $(29)$, it follows that

$$\begin{aligned} & \frac{dr(X)}{n}\big(2+\frac{a}{b(n-1)}\big)\big\{g(Y,Z)+\eta(Y)\eta(Z)\big\} \\ - & \frac{dr(Y)}{n}\big(2+\frac{a}{b(n-1)}\big)\big\{g(X,Z)+\eta(X)\eta(Z)\big\} \\ + & \frac{\alpha}{b}\{a+2(n-1)b\}\big(\frac{r}{n(n-1)}-\alpha^2\big)\times \\ & \{\Omega(X,Z)\eta(Y)+\Omega(Y,Z)\eta(X)\} \\ = & \frac{\{2a-(n-1)(n-4)b\}}{2(a+b)n(n-1)}\big[dr(X)g(Y,Z) \\ - & dr(Y)g(X,Z)\big]. \end{aligned} \quad (30)$$

If r is constant then $(30)$ yields

$$r = n(n-1)\alpha^2.$$

Hence from $(13)$, it follows that

$$R(X,Y,Z,U) = \alpha^2[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)], \quad (31)$$

which shows that the manifold is of constant curvature. Thus, we can state the following:

**Theorem 3.** In a Quasi conformally flat LP-Sasakian manifold M $(n > 3)$ with a constant coefficient $\alpha$, if the scalar curvature tensor is constant then M is of constant curvature.

## IV. Conclusion

The present paper is about the study of some geometrical properties of quasi conformally flat LP-Sasakian manifolds with a coefficient $\alpha$. It is established that a quasi conformally flat LP-Sasakian manifold M $(n > 3)$ with a constant coefficient $\alpha$ is an $\eta-$ Einstein and in a quasi conformally flat LP-Sasakian manifold M $(n > 3)$ with a non coefficient $\alpha$, the characteristic vector field $\eta$ is a concircular vector field if and only if M is $\eta-$Einstein manifold.

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