On Q-Fuzzy Ideals in Γ-Semigroups

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Abstract—In this paper the concept of Q-fuzzification of ideals of Γ-semigroups has been introduced and some important properties have been investigated. A characterization of regular Γ-semigroup in terms of Q-fuzzy ideals has been obtained. Operator semigroups of a Γ-semigroup has been made to work by obtaining various relationships between Q-fuzzy ideals of a Γ-semigroup and that of its operator semigroups.

Keywords—Q-Fuzzy set, Γ-Semigroup, Regular Γ-semigroup, Q-Fuzzy left(right) ideal, Operator semigroups.

I. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation[11]. The formal study of semigroups began in the early 20th century. Semigroups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. The concept of fuzzy sets was introduced by *Lofti Zadeh*[26] in his classic paper in 1965. *Azirel Rosenfeld*[17] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. *Nobuaki Kuroki*[13], [14], [15] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by *Kuroki*[13], [15], [16]. In [14], *Kuroki* characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Others who worked on fuzzy semigroup theory, such as *X.Y. Xie*[24], [25], *Y.B. Jun*[12], are mentioned in the bibliography. *X.Y. Xie*[24] introduced the idea of extensions of fuzzy ideals in semigroups.

The notion of a Γ-semigroup was introduced by *Sen and Saha*[22] as a generalization of semigroups and ternary semigroup. Γ-semigroup have been analyzed by lot of mathematicians, for instance by *Chattopadhyay*[2], [3], *Dutta and Adhikari*[5], [6], *Hila*[9], [10], *Chinram*[4], *Saha*[20], *Sen et al.*[21], [22], [20], *Seth*[23]. *S.K. Sardar* and *S.K. Majumder*[7], [8], [18], [19] have introduced the notion of fuzzification of ideals, prime ideals, semiprime ideals and ideal extensions of Γ-semigroups and studied them via its operator semigroups. In this paper the concept of Q-fuzzy ideals of a Γ-semigroup has been introduced. It is observed here that they satisfy level subset crieterion as well as characteristic function crieterion. Finally in order to make operator semigroups of a Γ-semigroup to work in the context of Q-fuzzy sets as it worked in the study of Γ-semigroups[5], [6], we obtain various relationships between Q-fuzzy ideals of a Γ-semigroup and that of its operator semigroups. Here, among other results we obtain an inclusion preserving bijection between the set of all Q-fuzzy ideals of a Γ-semigroup and that of its operator semigroups.

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II. Preliminaries

In this section we discuss some elementary definitions that we use in the sequel.

Definition II.1. [21] Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exist a mapping $S \times \Gamma \times S \to S$(images are denoted by $a\alpha b$) satisfying (1) $x\gamma y \in S$, (2) $(x\beta y)\gamma z = x\beta(y\gamma z)$ for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

Example 1. Let $\Gamma = \{5, 7\}$. For any $x, y \in N$ and $\gamma \in \Gamma$, we define $x\gamma y = x.\gamma.y$ where . is the usual multiplication on N. Then N is a Γ-semigroup.

Remark 1. Definition II.1 is the definition of one sided Γ-semigroup. Both sided Γ-semigroup was defined by Dutta and Adhikari[5] where the operation $\Gamma \times S \times \Gamma \to \Gamma$ also taken into consideration. They defined operator semigroups for such Γ-semigroups. The following definition is the definition of both sided Γ-semigroup given by Dutta and Adhikari.

Definition II.2. [5] Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \to a\alpha b$, and from $\Gamma \times S \times \Gamma \to \Gamma$, written as $(\alpha, a, \beta) \to \alpha a \beta$ satisfying the following associative laws: $(a\alpha b)\beta c = a(\alpha b)\beta c = a\alpha(b\beta c)$ and $\alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$ for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

Example 2. [5] Let S be the set of all integers of the form $4n + 1$ and Γ be the set of all integers of the form $4n + 3$ where n is an integer. If $a\alpha b$ is $a + \alpha + b$ and $\alpha + a + \beta$(usual sum of integers) for all $a, b \in S$ and for all $\alpha, \beta \in \Gamma$. Then S is a Γ-semigroup.

Definition II.3. [5] Let S be a Γ-semigroup. By a left(right) ideal of S we mean a non-empty subset A of S such that $S\Gamma A \subseteq A(A\Gamma S \subseteq A)$. By a two sided ideal or simply an ideal, we mean a non-empty subset of S which is both a left and right ideal of S.

Definition II.4. [5] A Γ-semigroup is called regular if, for each element $x \in S$, there exist $\beta \in \Gamma$ such that $x = x\beta x$.

Definition II.5. [26] A fuzzy subset ρ of a non-empty set X is a function $\rho : X \to [0, 1]$.

Definition II.6. Let ρ be a fuzzy subset of a non-empty set X. Then the set $\rho_t = \{x \in X : \rho(x) \geq t\}$ for $t \in [0, 1]$, is called the level subset or t-level subset of ρ.

Definition II.7. Let Q and X be two non-empty sets. A mapping $\mu : X \times Q \to [0, 1]$ is called the Q-fuzzy subset of X.

Definition II.8. Let μ be a Q-fuzzy subset of a non-empty

set X. Then the set $\mu_t = \{x \in X : \mu(x,q) \geq t \forall q \in Q\}$ for $t \in [0,1]$, is called the level subset or t-level subset of μ.

EXAMPLE 3. Let $S = \{a,b,c\}$ and $\Gamma = \{\gamma, \delta\}$, where γ, δ is defined on S with the following caley table:

γ	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

and

δ	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then S is a Γ-semigroup. Let $Q = \{p\}$. Let us consider a Q-fuzzy subset $\mu : S \times Q \to [0,1]$, by $\mu(a,p) = 0.8, \mu(b,p) = 0.7, \mu(c,p) = 0.6$. For $t = 0.7$, $\mu_t = \{a,b\}$.

III. Q-FUZZY IDEALS

Definition III.1. A non-empty Q-fuzzy subset μ of a Γ-semigroup S is called a Q-fuzzy left ideal of S if $\mu(x\gamma y, q) \geq \mu(y,q) \forall x, y \in S, \forall \gamma \in \Gamma$ and $\forall q \in Q$.

Definition III.2. A non-empty Q-fuzzy subset μ of a Γ-semigroup S is called a Q-fuzzy right ideal of S if $\mu(x\gamma y, q) \geq \mu(x,q) \forall x, y \in S, \forall \gamma \in \Gamma$ and $\forall q \in Q$.

Definition III.3. A non-empty Q-fuzzy subset of a Γ-semigroup S is called a Q-fuzzy ideal of S if it is both a Q-fuzzy left ideal and a Q-fuzzy right ideal of S.

EXAMPLE 4. Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ-semigroup if $a\gamma b$ and $\alpha a \beta$ denote the usual multiplication of integers a, γ, b and α, a, β respectively where $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Let $Q = \{p\}$. Let μ be a Q-fuzzy subset of S defined as follows:

$$\mu(x,p) = \begin{cases} 1 & \text{if } x = 0 \\ 0.1 & \text{if } x = -1, -2 \\ 0.2 & \text{if } x < -2 \end{cases}.$$

Then μ is a Q-fuzzy ideal of S.

Theorem III.4. *Let I be a non-empty subset of a Γ-semigroup S and $\chi_{I \times Q}$ be the characteristic function of $I \times Q$ then I is a left ideal(right ideal, ideal) of S if and only if $\chi_{I \times Q}$ is a Q-fuzzy left ideal(resp. Q-fuzzy right ideal, fuzzy ideal) of S.*

Proof: Let I be a left ideal of a Γ-semigroup S. Let $x, y \in S, q \in Q$ and $\gamma \in \Gamma$, then $x\gamma y \in I$ if $y \in I$. It follows that $\chi_{I \times Q}(x\gamma y, q) = 1 = \chi_{I \times Q}(y,q)$. If $y \notin I$, then $\chi_{I \times Q}(y,q) = 0$. In this case $\chi_{I \times Q}(x\gamma y, q) \geq 0 = \chi_{I \times Q}(y,q)$. Therefore $\chi_{I \times Q}$ is a Q-fuzzy left ideal of S.

Conversely, let $\chi_{I \times Q}$ be a Q-fuzzy left ideal of S. Let $x, y \in I, q \in Q$, then $\chi_{I \times Q}(x,q) = \chi_{I \times Q}(y,q) = 1$. Now let $x \in I$ and $s \in S, \gamma \in \Gamma, q \in Q$. Then $\chi_{I \times Q}(x,q) = 1$. Also $\chi_{I \times Q}(s\gamma x, q) \geq \chi_{I \times Q}(x,q) = 1$. Thus $s\gamma x \in I$. So I is a left ideal of S. Similarly we can prove that the other parts of the theorem. ∎

Proposition III.5. *Let I be a left ideal(right ideal, ideal) of a Γ-semigroup S, Q be any non-empty set and $\alpha \leq \beta \neq 0$ be any two elements in $[0,1]$, then the Q-fuzzy subset μ of S,*

defined by $\mu(x,q) = \begin{cases} \beta, \text{if } x \in I, q \in Q \\ \alpha, \text{otherwise} \end{cases}$ *is a Q-fuzzy left ideal(resp. Q-fuzzy right ideal, Q-fuzzy ideal) of S.*

Proof: Let I be a left ideal of a Γ-semigroup S and $\alpha, \beta \in [0,1]$. Let $s, x \in S, q \in Q$ and $\gamma \in \Gamma$. If $x \in I$, then $s\gamma x \in I$ and $\mu(x,q) = \beta = \mu(s\gamma x, q)$. Therefore $\mu(s\gamma x, q) = \mu(x,q)$. If $x \notin I$ then $\mu(x,q) = \alpha \leq \beta$ and then $\mu(s\gamma x, q) \geq \mu(x,q)$. Thus $\mu(s\gamma x, q) \geq \mu(x,q)$ for all $x, s \in S, q \in Q$ and for all $\gamma \in \Gamma$. Hence μ is a Q-fuzzy left ideal of S. Similarly we can prove all other cases. ∎

Theorem III.6. *Let S be a Γ-semigroup, Q be any non-empty set and μ be a non-empty Q-fuzzy subset of S, then μ is a Q-fuzzy left ideal(Q-fuzzy right ideal, Q-fuzzy ideal) of S if and only if μ_t's are left ideals(resp. right ideals, ideals) of S for all $t \in Im(\mu)$, where $\mu_t = \{x \in S : \mu(x,q) \geq t \forall q \in Q\}$.*

Proof: Let μ be a Q-fuzzy left ideal of S. Let $t \in Im\,\mu$, then there exist some $\alpha \in S$ such that $\mu(\alpha, q) = t$ and so $\alpha \in \mu_t$. Thus $\mu_t \neq \phi$. Let $x, y \in \mu_t$, then $\mu(x,q) \geq t$ and $\mu(y,q) \geq t$. Again let $s \in S, x \in \mu_t$ and $\gamma \in \Gamma$. Now $\mu(s\gamma x, q) \geq \mu(x,q) \geq t$. Therefore $s\gamma x \in \mu_t$. Thus μ_t is a left ideal of S.

Conversely, let μ_t's are left ideals of S for all $t \in Im\,\mu$. Again let $x \in S, s \in S$ and $\gamma \in \Gamma$, then $\mu(x,q) = t \forall q \in Q$. Thus $s\gamma x \in \mu_t$ (since μ_t is a left ideal of S). Therefore $\mu(s\gamma x, q) \geq t = \mu(x,q)$. Hence μ is a Q-fuzzy left ideal of S. Similarly we can prove the other cases. ∎

REMARK 2. Theorem III.4,III.6 and Proposition III.5 are true in case of semigroup also.

IV. COMPOSITION OF Q-FUZZY IDEALS

In this section we define composition of Q-fuzzy ideals of a Γ-semigroup and characterize regular Γ-semigroups in terms of Q-fuzzy ideals.

Definition IV.1. Let S be a Γ-semigroup, $\mu_1, \mu_2 \in QFLI(S)[QFRI(S), QFI(S)]$ and Q be a non-empty set [1]. Then the product $\mu_1 \circ \mu_2$ of μ_1 and μ_2 is defined as

$$(\mu_1 \circ \mu_2)(x,q) = \begin{cases} \sup_{x = u\gamma v} [\min\{\mu_1(u,q), \mu_2(v,q)\} \\ \quad : u, v \in S; \gamma \in \Gamma; q \in Q] \\ 0, \text{ if for any } u, v \in S \text{ and} \\ \quad \text{for any } \gamma \in \Gamma, x \neq u\gamma v \end{cases}$$

Theorem IV.2. *Let S be a Γ-semigroup and Q be any non-empty set. Then following are equivalent:* (1) *μ is a Q-fuzzy left(right) ideal of S,* (2) *$\chi \circ \mu \subseteq \mu(\mu \circ \chi \subseteq \mu)$, where χ is the characteristic function of $S \times Q$.*

Proof: Let μ be a Q-fuzzy left ideal of S. Let $a \in S, q \in Q$. Suppose there exist $u, v \in S$ and $\delta \in \Gamma$ such that $a = u\delta v$. Then, since μ is a Q-fuzzy left ideal of S, we have

$$(\chi \circ \mu)(a,q) = \sup_{a = x\gamma y} [\min\{\chi(x,q), \mu(y,q)\}]$$
$$= \sup_{a = x\gamma y} [\min\{1, \mu(y,q)\}] = \sup_{a = x\gamma y} \mu(y,q).$$

Now, since μ is a Q-fuzzy left ideal, $\mu(x\gamma y, q) \geq \mu(y, q)$ for all $x, y \in S, q \in Q$ and for all $\gamma \in \Gamma$. So in particular, $\mu(y, q) \leq \mu(a, q)$ for all $a = x\gamma y$. Hence $\sup\limits_{a=x\gamma y} \mu(y, q) \leq \mu(a, q)$. Thus $\mu(a, q) \geq (\chi \circ \mu)(a, q)$. If there do not exist $x, y \in S, \gamma \in \Gamma$ such that $a = x\gamma y$ then $(\chi \circ \mu)(a, q) = 0 \leq \mu(a, q)$. Hence $\chi \circ \mu \subseteq \mu$. By a similar argument we can show that $\mu \circ \chi \subseteq \mu$ when μ is a Q-fuzzy right ideal of S.

Conversely, let $\chi \circ \mu \subseteq \mu$. Let $x, y \in S, q \in Q$, $\gamma \in \Gamma$ and $a := x\gamma y$. Then $\mu(x\gamma y, q) = \mu(a, q) \geq (\chi \circ \mu)(a, q)$. Now

$$(\chi \circ \mu)(a, q) = \sup\limits_{a=u\alpha v} [\min\{\chi(u, q), \mu(v, q)\}] \geq \min\{\chi(x, q),$$
$$\mu(y, q)\} = \min\{1, \mu(y, q)\} = \mu(y, q).$$

Hence $\mu(x\gamma y, q) \geq \mu(y, q)$. Hence μ is a Q-fuzzy left ideal of S. By a similar argument we can show that if $\mu \circ \chi \subseteq \mu$, then μ is a Q-fuzzy right ideal of S. ∎

Using the above theorem we can deduce the following theorem.

Theorem IV.3. *Let S be a Γ-semigroup and Q be any non-empty set. Then following are equivalent:* (1) *μ is a Q-fuzzy two-sided ideal of S,* (2) *$\chi \circ \mu \subseteq \mu$ and $\mu \circ \chi \subseteq \mu$, where χ is the characteristic function of $S \times Q$.*

Proposition IV.4. *Let Q be any non-empty set, μ_1 be a Q-fuzzy right ideal and μ_2 be a Q-fuzzy left ideal of a Γ-semigroup S. Then $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$.*

Proof: Let μ_1 be a Q-fuzzy right ideal and μ_2 be a Q-fuzzy left ideal of a Γ-semigroup S. Let $x \in S$ and $q \in Q$. Suppose there exist $u_1, v_1 \in S$ and $\gamma_1 \in \Gamma$ such that $x = u_1\gamma_1 v_1$. Then

$$(\mu_1 \circ \mu_2)(x, q) = \sup\limits_{x=u\gamma v} \min\{\mu_1(u, q), \mu_2(v, q)\}$$
$$\leq \sup\limits_{x=u\gamma v} \min\{\mu_1(u\gamma v, q), \mu_2(u\gamma v, q)\}$$
$$= \min\{\mu_1(x, q), \mu_2(x, q)\} = (\mu_1 \cap \mu_2)(x, q).$$

Suppose there do not exist $u, v \in S$ such that $x = u\gamma v$. Then $(\mu_1 \circ \mu_2)(x, q) = 0 \leq (\mu_1 \cap \mu_2)(x, q)$. Thus $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. ∎

From the above proposition and the definition of $\mu_1 \cap \mu_2$ the following proposition follows easily.

Proposition IV.5. *Let Q be a non-empty set and $\mu_1, \mu_2 \in QFI(S)$. Then $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2$.*

Proposition IV.6. *Let S be a regular Γ-semigroup, Q be any non-empty set and μ_1, μ_2 be two Q-fuzzy subsets of S. Then $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$.*

Proof: Let $c \in S$ and $q \in Q$. Since S is regular, then there exists an element $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $c = c\gamma_1 x\gamma_2 c = c\gamma c$ where $\gamma := \gamma_1 x\gamma_2 \in \Gamma$. Then

$$(\mu_1 \circ \mu_2)(c, q) = \sup\limits_{c=u\alpha v} \{\min\{\mu_1(u, q), \mu_2(v, q)\}\}$$
$$\geq \min\{\mu_1(c, q), \mu_2(c, q)\} = (\mu_1 \cap \mu_2)(c, q).$$

Therefore $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$. ∎

Theorem IV.7. *Let S be a Γ-semigroup and Q be any non-empty set. Then following are equivalent.* (1) *S is regular.* (2) *$\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$ for every Q-fuzzy right ideal μ_1 and every Q-fuzzy left ideal μ_2 of S.*

Proof: Let S be a regular Γ-semigroup. Then by Proposition IV.6, $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$. Again by Proposition IV.4, $\mu_1 \circ \mu_2 \subseteq \mu_1 \cap \mu_2$. Hence $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$.

Conversely, let S be a Γ-semigroup, $q \in Q$ and for every Q-fuzzy right ideal μ_1 and every Q-fuzzy left ideal μ_2 of S, $\mu_1 \circ \mu_2 = \mu_1 \cap \mu_2$. Let L and R be respectively a left ideal and a right ideal of S and $x \in R \cap L$. Then $x \in R$ and $x \in L$. Hence $\chi_{L \times Q}(x, q) = \chi_{R \times Q}(x, q) = 1$ (where $\chi_{L \times Q}(x)$ and $\chi_{R \times Q}(x)$ are respectively the characteristic functions of $L \times Q$ and $R \times Q$). Thus

$$(\chi_{R \times Q} \cap \chi_{L \times Q})(x) = \min\{\chi_{R \times Q}(x), \chi_{L \times Q}(x)\} = 1.$$

Now by Theorem III.4, $\chi_{L \times Q}$ and $\chi_{R \times Q}$ are respectively a Q-fuzzy left ideal and a Q-fuzzy right ideal of S. Hence by hypothesis, $\chi_{R \times Q} \circ \chi_{L \times Q} = \chi_{R \times Q} \cap \chi_{L \times Q}$. Hence

$$(\chi_{R \times Q} \circ \chi_{L \times Q})(x, q) = 1$$
$$i.e., \sup\limits_{x=y\gamma z} [\min\{\chi_{R \times Q}(y, q), \chi_{L \times Q}(z, q)\} : y, z \in S; \gamma \in \Gamma]$$
$$= 1.$$

This implies that there exist some $r, s \in S, q \in Q$ and $\gamma_1 \in \Gamma$ such that $x = r\gamma_1 s$ and $\chi_{R \times Q}(r, q) = 1 = \chi_{L \times Q}(s, q)$. Hence $r \in R$ and $s \in L$. Hence $x \in R\Gamma L$. Thus $R \cap L \subseteq R\Gamma L$. Also $R\Gamma L \subseteq R \cap L$. Hence $R\Gamma L = R \cap L$. Consequently, the Γ-semigroup S is regular. ∎

V. CORRESPONDING Q-FUZZY IDEALS

Many results of semigroups could be extended to Γ-semigroups directly and via operator semigroups[5] of a Γ-semigroup. In this section in order to make operator semigroups of a Γ-semigroup work in the context of Q-fuzzy sets as it worked in the study of Γ-semigroups[5], [6], we obtain various relationships between Q-fuzzy ideals of a Γ-semigroup and that of its operator semigroups. Here, among other results we obtain an inclusion preserving bijection between the set of all Q-fuzzy ideals of a Γ-semigroup and that of its operator semigroups.

Definition V.1. [5]Let S be a Γ-semigroup. Let us define a relation ρ on $S \times \Gamma$ as follows : $(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup S. Dually the right operator semigroup R of Γ-semigroup S is defined where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha a\beta, b]$.

If there exists an element $[e, \delta] \in L([\gamma, f] \in R)$ such that $e\delta s = s$(resp. $s\gamma f = s$) for all $s \in S$ then $[e, \delta]$(resp. $[\gamma, f]$) is called the left(resp. right) unity of S.

Definition V.2. Let Q be any non-empty set. For a Q-fuzzy subset μ of R we define a Q-fuzzy subset μ^* of S by

$\mu^*(a,q) = \inf\limits_{\gamma \in \Gamma} \mu([\gamma,a],q)$, where $a \in S, q \in Q$. For a Q-fuzzy subset σ of S we define a Q-fuzzy subset $\sigma^{*'}$ of R by $\sigma^{*'}([\alpha,a],q) = \inf\limits_{s \in S} \sigma(s\alpha a,q)$, where $[\alpha,a] \in R, q \in Q$. For a Q-fuzzy subset δ of L, we define a Q-fuzzy subset δ^+ of S by $\delta^+(a,q) = \inf\limits_{\gamma \in \Gamma} \delta([a,\gamma],q)$ where $a \in S, q \in Q$. For a Q-fuzzy subset η of S we define a Q-fuzzy subset $\eta^{+'}$ of L by $\eta^{+'}([a,\alpha],q) = \inf\limits_{s \in S} \eta(a\alpha s,q)$, where $[a,\alpha] \in L, q \in Q$.

Now we recall the following propositions from [5] which were proved therein for one sided ideals. But the results can be proved to be true for two sided ideals.

Proposition V.3. [5]*Let S be a Γ-semigroup with unities and L be its left operator semigroup. If A is a (right) ideal of L then A^+ is a (right)ideal of S.*

Proposition V.4. [5]*Let S be a Γ-semigroup with unities and L be its left operator semigroup. If B is a (right)ideal of S then $B^{+'}$ is a(right)ideal of L.*

Proposition V.5. [5]*Let S be a Γ-semigroup with unities and R be its right operator semigroup. If A is a (left)ideal of R then A^* is a (left)ideal of S.*

Proposition V.6. [5]*Let S be a Γ-semigroup with unities and R be its right operator semigroup. If B is a (left)ideal of S then $B^{*'}$ is a (left)ideal of R.*

For convenience of the readers, we may note that for a Γ-semigroup S and its left, right operator semigroups L, R respectively four mappings namely $()^+, ()^{+'}, ()^+, ()^*$ occur. They are defined as follows: For $I \subseteq R, I^* = \{s \in S, [\alpha,s] \in I \forall \alpha \in \Gamma\}$; for $P \subseteq S, P^{*'} = \{[\alpha,x] \in R : s\alpha x \in P \forall s \in S\}$; for $J \subseteq L, J^+ = \{s \in S, [s,\alpha] \in J \forall \alpha \in \Gamma\}$; for $Q \subseteq S, Q^{+'} = \{[x,\alpha] \in L : x\alpha s \in Q \forall s \in S\}$.

Proposition V.7. *Let Q be any non-empty set and μ is a Q-fuzzy subset of R(the right operator semigroup of the Γ-semigroup S). Then $(\mu_t)^* = (\mu^*)_t$ for all $t \in [0,1]$ such that the sets are non-empty.*

Proof: Let $s \in S, q \in Q$. Then

$s \in (\mu_t)^* \Leftrightarrow [\gamma,s] \in \mu_t \ \forall \gamma \in \Gamma \Leftrightarrow \mu([\gamma,s],q) \geq t \ \forall \gamma \in \Gamma$
$\Leftrightarrow \inf\limits_{\gamma \in \Gamma} \mu([\gamma,s],q) \geq t \Leftrightarrow \mu^*(s,q) \geq t \Leftrightarrow s \in (\mu^*)_t$.

Hence $(\mu_t)^* = (\mu^*)_t$.
∎

Proposition V.8. *Let Q be any non-empty set and σ is a Q-fuzzy subset of a Γ-semigroup S. Then $(\sigma_t)^{*'} = (\sigma^{*'})_t$ for all $t \in [0,1]$ such that the sets under consideration are non-empty.*

Proof: Let $[\alpha,x] \in R, q \in Q$ and t is as mentioned in the statement. Then

$[\alpha,x] \in (\sigma_t)^{*'} \Leftrightarrow s\alpha x \subseteq \sigma_t \ \forall s \in S$
$\Leftrightarrow \sigma(s\alpha x,q) \geq t \ \forall s \in S \Leftrightarrow \inf\limits_{s \in S} \sigma(s\alpha x,q) \geq t$
$\Leftrightarrow \sigma^{*'}([\alpha,x],q) \geq t \Leftrightarrow [\alpha,x] \in (\sigma^{*'})_t$.

Hence $(\sigma_t)^{*'} = (\sigma^{*'})_t$.
∎

In what follows S denotes a Γ-semigroup with unities[5], L, R be its left and right operator semigroups respectively.

Proposition V.9. *If Q be any non-empty set and $\mu \in QFI(R)(QFLI(R))$, then $\mu^* \in QFI(S)(respectively QFLI(S))$.*

Proof: Suppose $\mu \in QFI(R)$. Then μ_t is an ideal of R, $\forall t \in Im(\mu)$. Hence $(\mu_t)^*$ is an ideal of S, $\forall t \in Im(\mu)(cf.$ Proposition V.5). Let $q \in Q$. Now since μ is Q-fuzzy ideal of R, μ is a non-empty Q-fuzzy subset of R. Hence for some $[\alpha,s] \in R, \mu([\alpha,s],q) > 0$. Then $\mu_t \neq \phi$ where $t := \mu([\alpha,s],q)$. So by the same argument applied above $(\mu_t)^* \neq \phi$. Let $u \in (\mu_t)^*$. Then $[\beta,u] \in \mu_t$ for all $\beta \in \Gamma$. Hence $\mu([\beta,u],q) \geq t$. This implies that $\inf\limits_{\beta \in \Gamma} \mu([\beta,u],q) \geq t$, i.e., $\mu^*(u,q) \geq t$. Hence $u \in (\mu^*)_t$. Hence $(\mu^*)_t \neq \phi$. Consequently, $(\mu_t)^* = (\mu^*)_t (cf.$ Proposition V.7). It follows that $(\mu^*)_t$ is an ideal of S for all $t \in Im(\mu)$. Hence μ^* is a Q-fuzzy ideal of $S(cf.$ Theorem III.6). The proof for Q-fuzzy left ideal follows similarly.
∎

In a similar fashion we can deduce the following proposition.

Proposition V.10. *If Q be any non-empty set and $\sigma \in QFI(S)(QFLI(S))$, then $\sigma^{*'} \in QFI(R)(respectively QFLI(R))$.*

We can also deduce the following left operator analogues of the above propositions.

Proposition V.11. *If Q be any non-empty set and $\delta \in QFI(L)(QFRI(L))$, then $\delta^+ \in QFI(S)(respectively QFRI(S))$.*

Proposition V.12. *If Q be any non-empty set and $\eta \in QFI(S)(QFRI(S))$, then $\eta^{+'} \in QFI(L)(respectively QFRI(L))$.*

Theorem V.13. *Let S be a Γ-semigroup with unities, Q be any non-empty set and L be its left operator semigroup. Then there exist an inclusion preserving bijection $\sigma \mapsto \sigma^{+'}$ between the set of all Q-fuzzy ideals (Q-fuzzy right ideals) of S and set of all Q-fuzzy ideals (resp. Q-fuzzy right ideals) of L, where σ is a Q-fuzzy ideal (resp. Q-fuzzy right ideal) of S.*

Proof: Let $\sigma \in QFI(S)(QFRI(S))$ and $x \in S, q \in Q$. Then $(\sigma^{+'})^+(x,q) = \inf\limits_{\gamma \in \Gamma} \sigma^{+'}([x,\gamma],q) = \inf\limits_{\gamma \in \Gamma} [\inf\limits_{s \in S} \sigma(x\gamma s,q)] \geq \sigma(x,q)$. Hence $\sigma \subseteq (\sigma^{+'})^+$. Let $[\gamma,f]$ be the right unity of S. Then $x\gamma f = x$ for all $x \in S$. Then $\sigma(x,q) = \sigma(x\gamma f,q) \geq \inf\limits_{\alpha \in \Gamma} [\inf\limits_{s \in S} \sigma(x\alpha s,q)] = \inf\limits_{\alpha \in \Gamma} \sigma^{+'}([x,\alpha],q) = (\sigma^{+'})^+(x,q)$. So $\sigma \supseteq (\sigma^{+'})^+$. Hence

$(\sigma^{+'})^{+} = \sigma$. Now let $\mu \in QFI(L)(QFRI(L))$. Then

$$(\mu^{+})^{+'}([x,\alpha],q) = \inf_{s \in S} \mu^{+}(x\alpha s, q) = \inf_{s \in S} [\inf_{\gamma \in \Gamma} \mu([x\alpha s, \gamma], q)]$$
$$= \inf_{s \in S} [\inf_{\gamma \in \Gamma} \mu([x,\alpha][s,\gamma], q)] \geq \mu([x,\alpha], q).$$

So $\mu \subseteq (\mu^{+})^{+'}$. Let $[e, \delta]$ be the left unity of L. Then

$$\mu([x,\alpha], q) = \mu([x,\alpha][e,\delta], q) \geq \inf_{s \in S} [\inf_{\gamma \in \Gamma} \mu([x,\alpha][s,\gamma], q)]$$
$$= (\mu^{+})^{+'}([x,\alpha], q).$$

So $\mu \supseteq (\mu^{+})^{+'}$ and hence $\mu = (\mu^{+})^{+'}$. Thus the correspondence $\sigma \mapsto \sigma^{+'}$ is a bijection. Now let $\sigma_1, \sigma_2 \in QFI(S)(QFRI(S))$ be such that $\sigma_1 \subseteq \sigma_2$. Then for all $[x,\alpha] \in L, q \in Q, \sigma_1^{+'}([x,\alpha], q) = \inf_{s \in S} \sigma_1(x\alpha s, q) \leq \inf_{s \in S} \sigma_2(x\alpha s, q) = \sigma_2^{+'}([x,\alpha], q)$. Thus $\sigma_1^{+'} \subseteq \sigma_2^{+'}$. Similarly we can show that if $\mu_1 \subseteq \mu_2$ where $\mu_1, \mu_2 \in QFI(L)(QFRI(L))$ then $\mu_1^{+} \subseteq \mu_2^{+}$. Hence $\sigma \mapsto \sigma^{+}$ is an inclusion preserving bijection. The rest of the proof follows from Proposition V.11 and Proposition V.12. ∎

In a similar way by using Proposition V.9 and Proposition V.10 we can deduce the following theorem.

Theorem V.14. *Let S be a Γ-semigroup with unities, Q be any non-empty set and R be its right operator semigroup. Then there exist an inclusion preserving bijection $\sigma \mapsto \sigma^{*'}$ between the set of all Q-fuzzy ideals (Q-fuzzy left ideals) of S and set of all Q-fuzzy ideals (resp. Q-fuzzy left ideals) of R, where σ is a Q-fuzzy ideal (resp. Q-fuzzy left ideal) of S.*

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