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On Properties of Generalized Bi-Γ-Ideals of Γ-Semirings

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Abstract—The notion of Γ-semirings was introduced by Murali Krishna Rao as a generalization of the notion of Γ-rings as well as of semirings. We have known that the notion of Γ-semirings is a generalization of the notion of semirings. In this paper, extending Kaushik, Moin and Khan's work, we generalize the notion of generalized bi-Γ-ideals of Γ-semirings and investigate some related properties of generalized bi-Γ-ideals.

Keywords—Γ-semiring, bi-Γ-ideal, generalized bi-Γ-ideal.

I. Introduction and Preliminaries

The notion of Γ-semirings was introduced and studied in 1995 by Murali Krishna Rao [10] as a generalization of the notion of Γ-rings as well as of semiring, and the notion of generalized bi-ideals was first introduced for rings in 1970 by Szász [12], [13] and then for semigroups by Lajos [8]. Many types of ideals on the algebraic structures were characterized by several authors such as: In 2000, Dutta and Sardar [3] studied the characterization of semiprime ideals and irreducible ideals of Γ-semirings. In 2004, Sardar and Dasgupta [11] introduced the notions of primitive Γ-semirings and primitive ideals of Γ-semirings. In 2008, Kaushik, Moin and Khan [7] introduced and studied bi-Γ-ideals in Γ-semirings, Pianskool, Sangwirotjanapat and Tipyota [9] introduced and studied valuation Γ-semirings and valuation Γ-ideals of a Γ-semiring, and Chinram [1] gave some properties of quasi-ideals in Γ-semirings. In 2009, Jagatap and Pawar [6] introduced the concept of minimal quasi-ideal in Γ-semirings. Some properties of minimal quasi-ideals in Γ-semirings are provided. In 2010, Ghosh and Samanta [5] studied the relation between the fuzzy left (respectively, right) ideals of Γ-semirings and that of operator semiring. In 2011, Dutta, Sardar and Goswami [4] introduced different types of operations on fuzzy ideals of Γ-semirings and proved subsequently that these operations give rise to different structures such as complete lattice, modular lattice on some restricted class of fuzzy ideals of Γ-semirings. In 2012, Bektaş, Bayrak and Ersoy [2] introduced and studied the characterization of soft Γ-semirings and soft sub-Γ-semiring.

The concept of ideals for many types of Γ-semirings is the really interested and important thing in Γ-semirings. Therefore, we will introduce and study generalized bi-Γ-ideals of Γ-semirings in the same way as of bi-Γ-ideals of Γ-semirings which was studied by Kaushik, Moin and Khan [7].

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To present the main results we first recall the definition of a Γ-semiring which is important here and discuss some elementary definitions that we use later.

Definition I.1. [10] Let M and Γ be two additive commutative semigroups. Then M is called a Γ-*semiring* if there exists a mapping $\cdot : M \times \Gamma \times M \to M$ (the image $\cdot(a, \alpha, b)$ to be denoted by $a\alpha b$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$) satisfying the following conditions:

(1) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(2) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(3) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Let M be a Γ-semiring, A and B nonempty subsets of M, and Λ a nonempty subset of Γ. Then we define

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}$$

and

$$A\Lambda B := \left\{ \sum_{i=1}^{n} a_i \lambda_i b_i \mid n \in \mathbb{Z}^+, a_i \in A, b_i \in B \text{ and } \lambda_i \in \Lambda \text{ for all } i \right\}.$$

If $A = \{a\}$, then we also write $\{a\} + B$ as $a + B$, and $\{a\}\Lambda B$ as $a\Lambda B$, and similarly if $B = \{b\}$ or $\Lambda = \{\lambda\}$.

Example I.2. [6] Let $\mathbb{Q}$ be set of rational numbers. Let $(S, +)$ be the commutative semigroup of all 2×3 matrices over $\mathbb{Q}$ and $(\Gamma, +)$ commutative semigroup of all 3×2 matrices over $\mathbb{Q}$. Define $W\alpha Y$ usual matrix product of W, α and Y for all $W, Y \in S$ and for all $\alpha \in \Gamma$. Then S is a Γ-semiring but not a semiring.

Example I.3. [6] Let $\mathbb{N}$ be the set of natural numbers and $\Gamma = \{1, 2, 3\}$. Then $(\mathbb{N}, \max)$ and $(\Gamma, \max)$ are commutative semigroups. Define the mapping $\mathbb{N} \times \Gamma \times \mathbb{N} \to \mathbb{N}$, by $a\alpha b = \min\{a, \alpha, b\}$ for all $a, b \in \mathbb{N}$ and $\alpha \in \Gamma$. Then $\mathbb{N}$ is a Γ-semiring.

Example I.4. [6] Let $\mathbb{Q}$ be set of rational numbers and $\Gamma = \mathbb{N}$ the set of natural numbers. Then $(\mathbb{Q}, +)$ and $(\mathbb{N}, +)$ are commutative semigroups. Define the mapping $\mathbb{Q} \times \Gamma \times \mathbb{Q} \to \mathbb{Q}$, by $a\alpha b$ usual product of a, α, b; for all $a, b \in \mathbb{Q}$ and $\alpha \in \Gamma$. Then $\mathbb{Q}$ is a Γ-semiring.

Example I.5. [2] For consider the additively abelian groups $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $\Gamma = \{2, 4, 6\}$. Let $\cdot : \mathbb{Z}_8 \times \Gamma \times \mathbb{Z}_8 \to \mathbb{Z}_8$, $(y, \alpha, s) = y\alpha s$. Then $\mathbb{Z}_8$ is a Γ-semiring.

Definition I.6. A nonempty subset A of a Γ-semiring M is called

(1) a *sub-*Γ*-semiring* of M if $(A,+)$ is a subsemigroup of $(M,+)$ and $a\gamma b \in A$ for all $a,b \in A$ and $\gamma \in \Gamma$.
(2) a Γ*-ideal* of M if $(A,+)$ is a subsemigroup of $(M,+)$, and $x\gamma a \in A$ and $a\gamma x \in A$ for all $a \in A, x \in M$ and $\gamma \in \Gamma$.
(3) a *quasi-*Γ*-ideal* of M if A is a sub-Γ-semiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(4) a *bi-*Γ*-ideal* of M if A is a sub-Γ-semiring of M and $A\Gamma M\Gamma A \subseteq A$.
(5) a *generalized bi-*Γ*-ideal* of M if $A\Gamma M\Gamma A \subseteq A$.

Remark I.7. Let M be a Γ-semiring. We have the following:

(1) Every quasi-Γ-ideal of M is a bi-Γ-ideal.
(2) Every bi-Γ-ideal of M is a generalized bi-Γ-ideal.

Definition I.8. A Γ-semiring M is called a *GB-simple* Γ*-semiring* if M is the unique generalized bi-Γ-ideal of M.

II. Main Results

Before the characterizations of generalized bi-Γ-ideals of Γ-semirings for the main results, we give some auxiliary results which are necessary in what follows. By Lemma I.7 (2) and [7], we have the following lemma.

Lemma II.1. *Let M be a Γ-semiring and $a \in M$. Then $a\Gamma M$ and $M\Gamma a$ are generalized bi-Γ-ideals of M.*

Lemma II.2. *Let M be a Γ-semiring and $\{B_i \mid i \in I\}$ a nonempty family of generalized bi-Γ-ideals of M with $\bigcap_{i\in I} B_i \neq \emptyset$. Then $\bigcap_{i\in I} B_i$ is a generalized bi-Γ-ideal of M.*

Proof: For all $i \in I$, we have

$$\left(\bigcap_{i\in I} B_i\right)\Gamma M\Gamma\left(\bigcap_{i\in I} B_i\right) \subseteq B_i\Gamma M\Gamma B_i \subseteq B_i.$$

Thus

$$\left(\bigcap_{i\in I} B_i\right)\Gamma M\Gamma\left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I} B_i.$$

Hence $\bigcap_{i\in I} B_i$ is a generalized bi-Γ-ideal of M. ■

Lemma II.3. *Let M be a Γ-semiring and $\emptyset \neq A \subseteq M$. Then*

$$A \cup A\Gamma M\Gamma A \qquad (1)$$

is the smallest generalized bi-Γ-ideal of M containing A.

Proof: Let $B = A \cup A\Gamma M\Gamma A$. Then $A \subseteq B$. Therefore

$$\begin{aligned}
& B\Gamma M\Gamma B \\
= & (A \cup A\Gamma M\Gamma A)\Gamma M\Gamma(A \cup A\Gamma M\Gamma A) \\
\subseteq & [A(\Gamma M\Gamma)(A \cup A\Gamma M\Gamma A)] \cup \\
& [A\Gamma M\Gamma A(\Gamma M\Gamma)(A \cup A\Gamma M\Gamma A)] \\
\subseteq & [A(\Gamma M\Gamma)A \cup A(\Gamma M\Gamma)A\Gamma M\Gamma A] \cup \\
& [A\Gamma M\Gamma A(\Gamma M\Gamma)A \cup A\Gamma M\Gamma A(\Gamma M\Gamma)A\Gamma M\Gamma A] \\
\subseteq & [A\Gamma M\Gamma A \cup A\Gamma M\Gamma A] \cup [A\Gamma M\Gamma A \cup A\Gamma M\Gamma A] \\
= & A\Gamma M\Gamma A \\
\subseteq & A \cup A\Gamma M\Gamma A \\
= & B.
\end{aligned}$$

Thus $B = A \cup A\Gamma M\Gamma A$ is a generalized bi-Γ-ideal of M. We shall show that B is the smallest generalized bi-Γ-ideal of M containing A. Let C be a generalized bi-Γ-ideal of M containing A. Then

$$A\Gamma M\Gamma A \subseteq C\Gamma M\Gamma C \subseteq C.$$

Thus

$$B = A \cup A\Gamma M\Gamma A \subseteq C.$$

Hence B is the smallest generalized bi-Γ-ideal of M containing A. ■

By Lemma II.3, let (A) be the smallest generalized bi-Γ-ideal of M containing A. Therefore

$$(A) = A \cup A\Gamma M\Gamma A. \qquad (2)$$

It is also very common to denote the smallest generalized bi-Γ-ideal of M containing $\{a\}$ as (a).

Lemma II.4. *Let T be a sub-Γ-semiring of a Γ-semiring M, $a \in M$ and $(a\Gamma T\Gamma a) \cap T \neq \emptyset$. Then $(a\Gamma T\Gamma a) \cap T$ is a generalized bi-Γ-ideal of T.*

Proof: Consider

$$\begin{aligned}
& (a\Gamma T\Gamma a \cap T)\Gamma T\Gamma(a\Gamma T\Gamma a \cap T) \\
\subseteq & [(a\Gamma T\Gamma a)\Gamma T \cap T\Gamma T]\Gamma(a\Gamma T\Gamma a \cap T) \\
\subseteq & [(a\Gamma T\Gamma a)\Gamma T \cap T]\Gamma(a\Gamma T\Gamma a \cap T) \\
\subseteq & [[(a\Gamma T\Gamma a\Gamma T)\Gamma(a\Gamma T\Gamma a)] \cap [T\Gamma(a\Gamma T\Gamma a \cap T)]] \\
\subseteq & [(a\Gamma T\Gamma a) \cap (T\Gamma a\Gamma T\Gamma a)] \cap T \\
\subseteq & (a\Gamma T\Gamma a) \cap T.
\end{aligned}$$

Hence $(a\Gamma T\Gamma a) \cap T$ is a generalized bi-Γ-ideal of T. ■

Lemma II.5. *Let M be a Γ-semiring and $a \in M$. Then $a\Gamma M\Gamma a$ is a generalized bi-Γ-ideal of M.*

Proof: Consider

$$(a\Gamma M\Gamma a)\Gamma M\Gamma(a\Gamma M\Gamma a) = a\Gamma(M\Gamma a\Gamma M\Gamma a\Gamma M)\Gamma a \subseteq a\Gamma M\Gamma a$$

Hence $a\Gamma M\Gamma a$ is a generalized bi-Γ-ideal of M. ■

Proposition II.6. *Let M be a Γ-semiring and T a sub-Γ-semiring of M. Then every subset of T containing $M\Gamma T$ is a sub-Γ-semiring of M.*

Proof: Let A be a subset of T such that $M\Gamma T \subseteq A$. Then

$$A\Gamma A \subseteq M\Gamma T \subseteq A.$$

Hence A is a sub-Γ-semiring of M. ■

Proposition II.7. *Let M be a Γ-semiring and T a Γ-ideal of M. Then every subset of T containing $M\Gamma T \cup T\Gamma M$ is a Γ-ideal of M.*

Proof: Let B be a subset of T such that $M\Gamma T \cup T\Gamma M \subseteq B$. Then

$$M\Gamma B \subseteq M\Gamma T \subseteq M\Gamma T \cup T\Gamma M \subseteq B$$

and

$$B\Gamma M \subseteq T\Gamma M \subseteq T\Gamma M \cup M\Gamma T \subseteq B.$$

Hence B is a Γ-ideal of M. ■

Proposition II.8. *Let M be a Γ-semiring and T a quasi-Γ-ideal of M. Then every subset of T containing $T\Gamma M \cap M\Gamma T$ is a quasi-Γ-ideal of M.*

Proof: Let C be a subset of T such that $T\Gamma M \cap M\Gamma T \subseteq C$. Then

$$C\Gamma C \subseteq T\Gamma M \cap M\Gamma T \subseteq C$$

and

$$C\Gamma M \cap M\Gamma C \subseteq T\Gamma M \cap M\Gamma T \subseteq C.$$

Hence C is a quasi-Γ-ideal of M. ■

Proposition II.9. *Let M be a Γ-semiring and T a bi-Γ-ideal of M. Then every subset of T containing $T\Gamma M\Gamma T$ and all of its images is a bi-Γ-ideal of M.*

Proof: Let D be a subset of T such that $T\Gamma M\Gamma T \subseteq D$ and $D\Gamma D \subseteq D$. Then

$$D\Gamma M\Gamma D \subseteq T\Gamma M\Gamma T \subseteq D.$$

Hence D is a bi-Γ-ideal of M. ■

Proposition II.10. *Let M be a Γ-semiring and T a generalized bi-Γ-ideal of M. Then every subset of T containing $T\Gamma M\Gamma T$ is a generalized bi-Γ-ideal of M.*

Proof: Let E be a subset of T such that $T\Gamma M\Gamma T \subseteq E$. Then

$$E\Gamma M\Gamma E \subseteq T\Gamma M\Gamma T \subseteq E.$$

Hence E is a generalized bi-Γ-ideal of M. ■

Theorem II.11. *Let M be a Γ-semiring. Then the following statements are equivalent.*

(1) *M is a GB-simple Γ-semiring.*
(2) *$a\Gamma M\Gamma a = M$ for all $a \in M$.*
(3) *$(a) = M$ for all $a \in M$.*

Proof: (1) $\Rightarrow$ (2) Assume that M is a GB-simple Γ-semiring and $a \in M$. By Lemma II.5, we have $a\Gamma M\Gamma a$ is a generalized bi-Γ-ideal of M. Since M is a GB-simple Γ-semiring, we have $a\Gamma M\Gamma a = M$.
(2) $\Rightarrow$ (3) Assume that $a\Gamma M\Gamma a = M$ for all $a \in M$ and let $a \in M$. Then, by (2), we have

$$(a) = \{a\} \cup a\Gamma M\Gamma a = \{a\} \cup M = M.$$

(3) $\Rightarrow$ (1) Assume that $(a) = M$ for all $a \in M$, and let A be a generalized bi-Γ-ideal of M and $a \in A$. Then $(a) \subseteq A$. By assumption, we have

$$M = (a) \subseteq A \subseteq M.$$

Thus $M = A$. Therefore M is a GB-simple Γ-semiring. ■

Lemma II.12. *Let B be a generalized bi-Γ-ideal of a Γ-semiring M and T a sub-Γ-semiring of M. If T is a GB-simple Γ-semiring such that $T \cap B \neq \emptyset$, then $T \subseteq B$.*

Proof: Assume that T is a GB-simple Γ-semiring such that $T \cap B \neq \emptyset$ and let $a \in T \cap B$. By Lemma II.3, we have $\{a\} \cup a\Gamma T\Gamma a$ is a generalized bi-Γ-ideal of T. Since T is a GB-simple Γ-semiring, we have $\{a\} \cup a\Gamma T\Gamma a = T$. Thus

$$T = \{a\} \cup a\Gamma T\Gamma a \subseteq B \cup B\Gamma M\Gamma B \subseteq B \cup B \subseteq B.$$

Hence $T \subseteq B$. ■

Theorem II.13. *Let M be a Γ-semiring, B a generalized bi-Γ-ideal of M and $\emptyset \neq A \subseteq M$. Then $B\Gamma A$ and $A\Gamma B$ are generalized bi-Γ-ideals of M.*

Proof: Since B is a generalized bi-Γ-ideal of M, we have

$$(B\Gamma A)\Gamma M\Gamma(B\Gamma A) = (B\Gamma(A\Gamma M)\Gamma B)\Gamma A \subseteq (B\Gamma M\Gamma B)\Gamma A \subseteq B\Gamma A$$

and

$$(A\Gamma B)\Gamma M\Gamma(A\Gamma B) = A\Gamma(B\Gamma(M\Gamma A)\Gamma B) \subseteq A\Gamma(B\Gamma M\Gamma B) \subseteq A\Gamma B.$$

Therefore $B\Gamma A$ and $A\Gamma B$ are generalized bi-Γ-ideals of M. ■

Theorem II.14. *Let M be a Γ-semiring and B a bi-Γ-ideal of M. Then B is a minimal generalized bi-Γ-ideal of M if and only if B is a GB-simple Γ-semiring.*

Proof: Assume that B is a minimal generalized bi-Γ-ideal of M. By assumption, B is a Γ-semiring. Let C be a generalized bi-Γ-ideal of B. Then

$$C\Gamma B\Gamma C \subseteq C \subseteq B. \quad (3)$$

Since B is a generalized bi-Γ-ideal of M and by Theorem II.13, we have $C\Gamma B\Gamma C$ is a generalized bi-Γ-ideal of M. Since B is a minimal generalized bi-Γ-ideal of M, we get $C\Gamma B\Gamma C = B$. Thus, by (3), we have $B = C$. Hence B is a GB-simple Γ-semiring.

Conversely, assume that B is a GB-simple Γ-semiring. Let C be a generalized bi-Γ-ideal of M such that $C \subseteq B$. Then

$$C\Gamma B\Gamma C \subseteq C\Gamma M\Gamma C \subseteq C.$$

Thus C is a generalized bi-Γ-ideal of B. Since B is a GB-simple Γ-semiring, we have $B = C$. Hence B is a minimal generalized bi-Γ-ideal of M. ■

Theorem II.15. *Let M be a Γ-semiring having a proper generalized bi-Γ-ideal. Then every proper generalized bi-Γ-ideal of M is minimal if and only if the intersection of any two distinct proper generalized bi-Γ-ideals is empty.*

Proof: Assume that every proper generalized bi-Γ-ideal of M is minimal and let B_1 and B_2 be two distinct proper generalized bi-Γ-ideals of M. By assumption, we have B_1 and B_2 are minimal. We shall show that $B_1 \cap B_2 = \emptyset$. Suppose that $B_1 \cap B_2 \neq \emptyset$. By Lemma II.2, we have $B_1 \cap B_2$ is a proper generalized bi-Γ-ideal of M. Since $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$, we get $B_1 \cap B_2 = B_1$ and $B_1 \cap B_2 = B_2$. Thus $B_1 = B_2$ which is a contradiction. Hence $B_1 \cap B_2 = \emptyset$.

Conversely, assume that the intersection of any two distinct proper generalized bi-Γ-ideals is empty. Let B be a proper generalized bi-Γ-ideal of M and C a generalized bi-Γ-ideals of M such that $C \subseteq B$. Suppose that $C \neq B$. Then C is a proper generalized bi-Γ-ideal of M. Since $C \subset B$ and by assumption, we have $C = C \cap B = \emptyset$ which is a contradiction. Therefore $C = B$, so B is minimal. ■

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