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On One Application of Hybrid Methods For Solving Volterra Integral Equations

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Abstract**—**As is known, one of the priority directions of research works of natural sciences is introduction of applied section of contemporary mathematics as approximate and numerical methods to solving integral equation into practice. We fare with the solving of integral equation while studying many phenomena of nature to whose numerically solving by the methods of quadrature are mainly applied. Taking into account some deficiency of methods of quadrature for finding the solution of integral equation some sciences suggested of the multistep methods with constant coefficients. Unlike these papers, here we consider application of hybrid methods to the numerical solution of Volterra integral equation. The efficiency of the suggested method is proved and a concrete method with accuracy order $p = 4$ is constructed. This method in more precise than the corresponding known methods.

Keywords**—**Volterra integral equation, hybrid methods, stability and degree, methods of quadrature

I. Introduction

CONSIDER the following integral equation of Volterra-Uryson type:

$$y(x) = g(x) + \int_{x_0}^{x} K(x, s, y(s))ds. \qquad (1)$$

It is known that the research of integral equations with variable boundaries begins with the papers of Abel published in 1826 (see [1], p.18). Liouville and Laplace continued the research of integral equations began by Abel and studied integral transformations.

Thorough investigation of equation (1) in the linear case belays to the famous Italian scientist Vito Volterra that while studying some applied problems has get a model in the form of integral equations. For finding approximate solutions of equations, he used the methods of quadrature (see.[2]).

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The method suggested by Volterra more precise was revised and modified by many authors (see.[3], [4]). After application of the methods of quadrature to the equation (1), we have:

$$y_n = g_n + h\sum_{i=0}^{n} a_i K(x_n, x_i, y_i). \qquad (2)$$

As it follows from formula (2), for each value of the variable n, the sum as the right side of formula (2) is calculations while passing from the point x_n to x_{n+1}, that in one of the main shortages of the method of quadratures. The method released from the mention shortage was constructed in [5] and has the following form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{i=0}^{k}\alpha_i g_{n+i} + \sum_{i=0}^{k}\sum_{j=0}^{k}\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}), (n = 0,1,2,...,N-k). \qquad (3)$$

Usually, while using method (3) it is assumed that the kernel of the integral, the function $K(x,z,y)$ is determined in ε extended of domain G of the form: $\overline{G} = \{x_0 \le s \le x + \varepsilon \le X + \varepsilon,\ |y| \le b\}$. However, some authors investigate integral equation (1) for $\varepsilon = 0$. In this case, the method (3) is of the form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = \sum_{i=0}^{k}\alpha_i g_{n+i} + \sum_{i=0}^{k}\sum_{j=0}^{i}\beta_i^{(j)} K(x_{n+i}, x_{n+j}, y_{n+j}). \qquad (4)$$

Here unlike the mentioned papers it is suggested to apply the hybrid methods to solving equation (1) assuming that these methods are more precise than the Runge-Kutta and Adams. Note that the application of the investigation of the numerical method of Volterra nonlinear integral equation by means of hybrid methods was first studied by Makroglou (see [6])The method used by Makroglou is of the form:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h\beta_m f_{n+m}, \qquad (5)$$

where $y' = f(x,y)$.

In this paper, for finding the solution of the equation (1) we suggest to use the hybrid methods:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i+l_i},\ (|l_i| < 1,\ i = 0,1,...,k), \qquad (6)$$

that in [7] was applied to the solution of problem $y' = f(x,y),\ y(x_0) = y_0$.

II. Constructing The Hybrid Method For Solving Integral Equation With Variable Boundaries

Usually, construction of the numerical method to solving of the equation (1) begins with change of the integral by some integral sum. Here to this end we suggest the following formula:

$$\int_{x_0}^{x_m} f(s,y(s))ds =$$

$$= h\sum_{i=0}^{m} a_i f(x_0 + (i+l_i)h, y(x_0 + (i+l_i)h)) + R_m, \quad (7)$$

$$(|l_i| < 1).$$

For $l_i = 0\ (i = 0,1,...,m)$ the traditional integral sums are obtained from the suggested formula. However while using the constructed method, there arise a difficulty in determining the values of the quantities of type y_{j+l} $(|l| < 1,\ 0 \le j \le m)$, that in some sense was resolved in [7].

Consider the equations (1) at the point $x = x_{n+1}$. Then we have :

$$y(x_{n+1}) = g_{n+1} + \int_{x_0}^{x_{n+1}} K(x_{n+1}, s, y(s))ds,$$

$$(g_m = g(x_m),\ m = 0,1,2,...).$$

For defining the connections between the quantities $y(x_n)$ and $y(x_{n+1})$ consider the following difference:

$$y(x_{n+1}) - y(x_n) = g_{n+1} - g_n +$$

$$+ h\int_{x_0}^{x_n} K_n'(\xi_{n+1}, s, y(s))ds + \int_{x_n}^{x_{n+1}} K(x_{n+1}, s, y(s))ds, \quad (8)$$

where $x_n < \xi_{n+1} < x_{n+1}$.

Assume that by some method we find the solution of the equation (1), take it into account in (1) and get an identity. Then from (1) we can write:

$$y'(x) = g'(x) + K(x,x,y(x)) + \int_{x_0}^{x} K_n'(x,s,y(s))ds.$$

Assuming $x = \xi_{n+1}$ in this identity we can receive the following:

$$\int_{x_0}^{x_n} K_x'(\xi_{n+1}, s, y(s))ds = y'(\xi_{n+1}) - f'(\xi_{n+1}) +$$

$$+ K(\xi_{n+1}, \xi_{n+1}, y(\xi_{n+1})) - \int_{x_n}^{\xi_{n+1}} K_x'(\xi_{n+1}, s, y(s))ds.$$

Take into account the obtained equality in the equality (8), then we get:

$$y(x_{n+1}) - y(x_n) = g_{n+1} - g_n + h(y'(\xi_{n+1}) - f'(\xi_{n+1})) -$$

$$- hK(\xi_{n+1}, \xi_{n+1}, y(\xi_{n+1}))) + \int_{x_n}^{\xi_{n+1}} K'(x_n, s, y(s))ds + \quad (9)$$

$$+ \int_{\xi_{n+1}}^{x_{n+1}} K(x_{n+1}, s, y(s))ds.$$

It is known that using the difference relations, we can write:

$$hy'(\xi_{n+k}) = \sum_{i=0}^{k} \tilde{\alpha}_i y_{n+i}.$$

We apply formula (7) to the integral in the right side of the equality (9), apply the Lagrange interpolation formula to the represented quantities $hK(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k}))$ and assume $n := n+k-1$. Then after rejecting the remainder terms, from the obtained equalities we get:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k} \alpha_i g_{n+i} +$$

$$+ \sum_{i=0}^{k}\sum_{j=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i+l_i}, y_{n+i+l_i}), \quad (|l_i| < 1). \quad (10)$$

Here the coefficients $\alpha_i, \beta_i^{(j)}, l_i\ (i,j = 0,1,2,...,k)$ - are some real members. It is very difficult to determine them by the above described scheme.

Therefore, for determining the values of the quantities $\alpha_i, \beta_i^{(j)}, l_i\ (i,j = 0,1,2,...,k)$ it is suggested to the model equation that is obtained when the kernel of the integral of the function $K(x,z,y)$ is replaced by the function $f(z,y)$.

Consider the construction of the model equation and to this end fix the values of the quantity x and assume that the approximate value of $\bar{y}(x)$ at this point is known. Then in the equation (1) we replace $y(x)$ by its approximate value $\bar{y}(x)$, and then we get the equality (1) in fulfilled with some error R_x, more exactly:

$$\bar{y}(x) = g(x) + \int_{x_0}^{x} K(x,s,\bar{y}(s))ds + R_x. \quad (11)$$

Subtract (11) from (1) and have:

$$y(x) - \bar{y}(x) = \int_{x_0}^{x} K(x,s,\xi)(y(s) - \bar{y}(s))ds - R_x, \quad (12)$$

Here the ξ is between $y(x)$ and $\bar{y}(x)$. Taking into account that the function $K(x,z,y)$ is continuous in aggregate of the arguments in the domain of definition, we can assume that

$$|K_x'(x,s,z)| \le K,$$

we take this into account in (12) and have:

$$\left|y(x)-\bar{y}(x)\right| \le K\int_{x_0}^{x}\left|(y(s)-\bar{y}(s))\right|ds+\left|R_x\right|.$$

Hence it follows that in place of the model we can use the following equation:

$$y(x)=\lambda\int_{x_0}^{x}y(s)ds+R. \qquad (13)$$

Obviously if there consist a method that approximates the equation (1) with order p, then it must approximate the equation (13) with the same order. Therefore, while determining the coefficient of the method (10), we can use the model equation (13). Here, for determining the coefficients of the method (10) we suggest to use the following model equation:

$$y(x)=g(x)+\int_{x_0}^{x}f(s,y(s))ds. \qquad (14)$$

Taking into account the suggest model equation in the method (7), we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=\sum_{i=0}^{k}\alpha_i g_{n+i}+h\sum_{i=0}^{k}\gamma_i f_{n+i+l_i}, \qquad (15)$$

where $\gamma_i=\sum_{j=0}^{k}\beta_i^{(j)} \quad (i=0,1,2,...,k)$.

We assume that the solution of the equation (14) is known. Then after taking into account the solution in (14), we get the equality from which we have:

$$f(x,y)=y'(x)-g'(x).$$

Taking unto account the obtained on the method (15), we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=\sum_{i=0}^{k}\alpha_i g_{n+i}- -h\sum_{i=0}^{k}\gamma_i g'_{n+i+l_i}+h\sum_{i=0}^{k}\gamma_i y'_{n+i+l_i}. \qquad (16)$$

We want it choose the coefficients so that the method (7) (respectively the method (15)) has the degree p.Therefore, suppose that for a sufficiently smooth function $z(x)$ the method

$$\sum_{i=0}^{k}\alpha_i z_{n+i}=h\sum_{i=0}^{k}\gamma_i z'_{n+i+l_i}, \qquad (17)$$

has the degree p. Here p is a natural quantity. Then we can write the following:

$$\sum_{i=0}^{k}(\alpha_i z(x+ih)-h_i z'(x+ih)=O(h^{p+1}),\ h\to 0, \qquad (18)$$

where x - is fixed point.

Thus we obtained that if the method (17) has the degree p, then the method (15) also has the degree p if we determine the function $z(x)$ in the form: $z(x)=y(x)-g(x)$.

Now consider the selection of the values of the quantities $\alpha_i, \gamma_i, l_i \ (i,j=0,1,2,...,k)$. To this end we consider the following expansion of the function and its first derivative.

$$y(x+ih)=y(x)+ihy'(x)+ \frac{(ih)^2}{2!}y''(x)+...+\frac{(ih)^p}{p!}y^{(p)}(x)+O(h^{p+1}), \qquad (19)$$

$$y'(x+\bar{\gamma}_i h)=y'(x)+\bar{\gamma}_i hy''(x)+ \frac{(\bar{\gamma}_i h)^2}{2!}y'''(x)+...+\frac{(\bar{\gamma}_i h)^{p-1}}{(p-1)!}y^{(p)}(x)+O(h^p), \qquad (20)$$

where $x=x_0+nh$ - is a fixed point.

For determining the values of the parameters $\alpha_i,\ \beta_i,\ l_i\ (i=0,1,2,...,k)$, we take into account equalities (19) and (20) in asymptotic equality (18). Then we have:

$$\sum_{i=0}^{k}\alpha_i=0;$$

$$\sum_{i=0}^{k}\frac{i^\nu}{\nu!}\alpha_i=\sum_{i=0}^{k}\frac{\bar{\gamma}_i^{\nu-1}}{(\nu-1)!}\gamma_i \quad (\nu=1,2,...,p). \qquad (21)$$

Here $\bar{\gamma}_i=i+l_i \quad (i=0,1,2,...,k)$.

Thus, for determining the values of the parameters $\alpha_i,\ \beta_i,\ l_i\ (i=0,1,2,...,k)$ we obtained a system of algebraic equations wherein the amount of the unknowns equal $3k+3$, the amount of equations $p+1$. Obviously, system (21) has always trivial solution. However the trivial solution of system (21) is not of interest. Therefore, consider the case when system (21) has a non trivial solution. It is known that for system (21) to have a non-zero solution, the following to be hold:

$$p<3k+2.$$

Consider the construction of concrete methods of type (17) and suppose $k=1$. Then under assumption $\alpha_1=-\alpha_0=1$ for determining the values of the quantities β_0,β_1,l_0 и l_1 we get the following system of equations:

$$\begin{aligned} &\beta_0+\beta_1=1,\\ &l\beta_0+\gamma\beta_1=1/2,\\ &l^2\beta_0+\gamma^2\beta_1=1/3,\\ &l^3\beta_0+\gamma^3\beta_1=1/4. \end{aligned} \qquad (22)$$

Here $l=l_0,$ a $\gamma=1+l_1$. Solving this nonlinear system of equations, for determining the quantity l we get the following quadratic equation:

$$l^2-l+1/6=0,$$

The quantity γ is determined from relation $\gamma + l = 1$. Note that the solution of system (22) is not unique, but the method whose coefficients satisfy system (22) is unique and have the degree $p = 4$, and the method is on the form:

$$y_{n+1} = y_n + h(f_{n+l_0} + f_{n+1+l_1})/2. \quad (23)$$

Here $l_1 = -l_0;\ \ l_0 = (3-\sqrt{3})/6,\ \ 1+l_1 = (3+\sqrt{3})/6$

For using method (23), in addition to the values y_n of the quantities y_{n+l_0} and $y_{n+\gamma}$ also should be known. Since these quantities are independent of the quantity y_{n+1} we can assume that method (23) is explicit. However while determining the quantities y_{n+l_0} and $y_{n+\gamma}$ according our algorithms, the values of the quantities y_{n+1} are used. From these values it follows that the used method (23) may be considered as implicit. Thus, we get that formally the hybrid method may be considered as an explicit method, but in their real using we fare with an implicit method. Hence it is sum that classification hybrid method (23) as the clan of explicit or implicit methods is of formal character. However, there exist implicit hybrid methods. For example, the following method (see [8]):

$$y_{n+1} = y_n + h(3f_{n+1/3} + f_{n+1})/4. \quad (24)$$

Now lets consider the construction of the methods of type (10) with maximal degree is not unique. This is connected with definition of the values of the quantity $\beta_i^{(j)}\ \ (i = 0,1,2,...,k)$ by the linear system:

$$\sum_{i=0}^{k} \beta_i^{(j)} = \gamma_i, \quad (i = 0,1,...,k).$$

Wherefrom it follows the non uniqueness of the coefficients $\beta_i^{(j)}\ \ (i = 0,1,2,...,k)$. In one variant the hybrid method with the forth degree constructed for solving equation (1) for $k = 1$ is of the form:

$$\begin{aligned} y_{n+1} &= y_n + f_{n+1} - f_n + \\ &+ h(K(x_n + h(3-\sqrt{3})/6, x_n + h(3-\sqrt{3})/6, \\ &y(x_n + h(3-\sqrt{3})/6)) + K(x_n + h(3+\sqrt{3})/6, \\ &x_n + h(3-\sqrt{3})/6, y(x_n + h(3-\sqrt{3})/6)) + \\ &+ 2K(x_n + h(3+\sqrt{3})/6, x_n + h(3+\sqrt{3})/6, \\ &y(x_n + h(3+\sqrt{3})/6)))/4. \end{aligned}$$

Taking into account some properties of the kernel $K(x,z,y)$ we can write the following form:

$$\begin{aligned} y_{n+1} &= y_n + f_{n+1} - f_n + \\ &+ h(-K(x_n + h(3-\sqrt{3})/6, x_n + h(3-\sqrt{3})/6, \\ &y(x_n + h(3-\sqrt{3})/6)) + 2K(x_n + h(3+\sqrt{3})/6, \\ &x_n + h(3-\sqrt{3})/6, y(x_n + h(3-\sqrt{3})/6)) + \\ &+ K(x_n + h(3+\sqrt{3})/6, x_n + h(3+\sqrt{3})/6, \\ &y(x_n + h(3+\sqrt{3})/6)))/2. \end{aligned}$$

Obviously for using these methods it is required to define the values of the quantities $y(x_n + h(3-\sqrt{3})/6)$ and $y(x_n + h(3+\sqrt{3})/6)$. To this end we can use the following formula:

$$\begin{aligned} y_{n+\frac{1}{2}+\alpha} &= y_{n+\frac{1}{2}} + \frac{\alpha^2 h}{6}(3+4\alpha)y'_{n+1} + \\ &+ \frac{\alpha h}{3}(3-4\alpha^2)y'_{n+\frac{1}{2}} - \frac{\alpha^2 h}{6}(3-4\alpha)y'_n, \end{aligned}$$

where $\alpha = \pm\sqrt{3}/6$

One can easily show that

$$y(x_n + \frac{1+2\alpha}{2}h) - y_{n+\frac{1}{2}+\alpha} = O(h^4).$$

As follows from the above proposed for determining the value of the quantity $y_{n+\frac{1}{2}\pm\alpha}$, we need to determine the value of variables y'_{n+1}, y'_{n+2}, y'_n. To this end, consider the following equation:

$$y'(x) = f'(x) + K(x,x,y(x)) + \int_{x_0}^{x} K'_x(x,s,y(s))ds.$$

If we put here $x = x_0$, then

$$y'(x_0) = f'(x_0) + K(x_0, x_0, y(x_0))$$

as $y(x_0) = f(x_0)$ known. Thus, the value y'_n can be assumed to be known. Consider the definition of data values $y'_{n+\frac{1}{2}}$ and y'_{n+1}, in assumptions known values $y_{n+\frac{1}{2}}$:

$$\begin{aligned} y'_{n+\frac{1}{2}} &= y'_n + f'_{n+\frac{1}{2}} - f'_n + K_{n+\frac{1}{2}} - K_n + \\ &+ h(2K'_x(x_{n+\frac{1}{2}}, x_n, y_n) - K'_x(x_n, x_n, y_n) + \\ &- K'_x(x_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}))/2 \end{aligned}$$

To determine the value y'_{n+1}, use the scheme:

$$\hat{y}_{n+1} = y_n + hy'_{n+\frac{1}{2}},$$

$$y'_{n+1} = y'_n + f'_{n+1} - f'_n +$$
$$+ Kh(x_{n+1}, x_{n+1}, \hat{y}_{n+1}) - K'_x(x_n, x_n, y_n) +$$
$$+ h(K'_x(x_{n+\frac{1}{2}}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + K'_x(x_{n+1}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}))/2.$$

To offer the above described scheme, it is necessary to determine the values $y_{n+\frac{3}{2}}$. In the end, one can use different methods, which is as follows:

$$y_{n+\frac{3}{2}} = y_{n+\frac{1}{2}} + h(7y'_{n+1} - 2y'_{n+\frac{1}{2}} + f'_n)/6.$$

III. Conclusion

Obviously, the well-known methods can be generalized by different ways. But here we have used the linear form of generalizations the known methods in resulting receive method (10). Note that the main deficiency of the method, which was constructed here, is to calculate the values y_{n+i+l_i}. However, as shown above, this deficiency can be corrected by the methods of predictor-corrector. Thus we received that take into consideration the high accuracy hybrid methods, they can be considered more perspective.The above mentioned variant of the predictor-corrector method for the using of hybrid methods that needs some adjustments. Consequently, the investigation of hybrid methods can be considered more perspective.

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