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On Hyperbolic Gompertz Growth Model

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***Abstract*—**We proposed a Hyperbolic Gompertz Growth Model (HGGM), which was developed by introducing a shape parameter (allometric). This was achieved by convoluting hyperbolic sine function on the intrinsic rate of growth in the classical gompertz growth equation. The resulting integral solution obtained deterministically was reprogrammed into a statistical model and used in modeling the height and diameter of Pines (*Pinus caribaea*). Its ability in model prediction was compared with the classical gompertz growth model, an approach which mimicked the natural variability of height/diameter increment with respect to age and therefore provides a more realistic height/diameter predictions using goodness of fit tests and model selection criteria. The Kolmogorov Smirnov test and Shapiro-Wilk test was also used to test the compliance of the error term to normality assumptions while the independence of the error term was confirmed using the runs test. The mean function of top height/Dbh over age using the two models under study predicted closely the observed values of top height/Dbh in the hyperbolic gompertz growth models better than the source model (classical gompertz growth model) while the results of R^2, Adj. R^2, MSE and AIC confirmed the predictive power of the Hyperbolic Gompertz growth models over its source model.

***Keywords*—**Height, Dbh, forest, *Pinus caribaea*, hyperbolic, gompertz.

I. Introduction

In this paper, an alternative nonlinear growth model called the hyperbolic gompertz growth model was introduced and compared with the existing classical gompertz model, which is an improvement on the richards growth model [1].

The Gompertz model was named after Benjamin Gompertz in [1825], he proposed his model for life table analysis, and was first used specifically as a growth curve by [2]. The model was later used as height-diameter model. The model has the following differential form;

$$\frac{dH}{dt} = rH\, ln\frac{K}{H}$$

Benjamin Gompertz (5 march 1779 – 14 July 1865) was a British self – educated mathematician, his model was derived from Richards model as parameter "b" tends towards zero in [3]. Gompertz model is a type of mathematical model for a time series, where growth is slowest at the start and end of a time period. It is a special case of the generalized logistic model. The Gompertz equation arises from models of self-limited growth where the rate decreases exponentially with time. The model was first introduced to describe growth in the number of tumor cells, which usually follows a sigmoidal growth pattern.

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Deterministic and stochastic models serve complementary purposes. In forestry, [4], [5] stated that deterministic models are effective for determining the expected yield, and may be used to indicate the optimum stand condition. Stochastic models may indicate the reliability of these predictions, and the risks associated with any particular regime. Both deterministic and stochastic predictions can be obtained from some models. Although stochastic models can provide some useful information not available from deterministic models, most of the information needed for forest planning and managements canbe provided efficiently also with the use of deterministic models.

A mathematical description of a real world system is often referred to as a mathematical model. A system can be formally defined as a set of elements also called components. A set of trees in a forest stand, producers and consumers in an economic system are examples of components. The elements (components) have certain characteristics or attributes and these attributes have numerical or logical values. Among the elements, relationships exist and consequently the elements are interacting. The state of a system is determined by the numerical or logical values of the attributes of the system elements. Experimenting on the state of a system with a model over time is termed simulation [5], [6]. Sustainable forest management relies to a large extent, measure on the predictions of the future conditions of individual stands which is achieved by predicting the increment from the current stand structure and updating the current values at each cycle of iteration using a functional growth model. Trees structural changes over time can be monitored and modeled under different cutting cycles, cutting intensities and optimal management policies can be arrived at based on the results of such simulation runs.

II. Math

Consider a nonlinear model

$$H_i = f(D_i, \boldsymbol{B}) + \mathcal{E}_i \qquad (1)$$

$$i = 1,2, \dots, n,$$

where H is the response variable, D is the independent variable, **B** is the vector of the parameters β_j to be estimated $(\beta_1, \beta_2 \dots\dots, \beta_p)$,$\mathcal{E}_i$ is a random error term , p is the number of unknown parameters, n is the number of observation. The estimator of β_j's are found by minimizing the sum of squares residual (SS_{Rss}) function

$$SS_{Rss} = \sum_{i=1}^{n}[H_i - f(D_i, B)]^2 \qquad (2)$$

Under the assumption that the $\boldsymbol{\mathcal{E}_i}$are normal and independent with mean zero and common variable σ^2. Since H_i and D_i are fixed observations, the sum of squares residual is a function of $\mathbf{B}$,these normal equations take the form of

$$\sum_{i=1}^{n}\{H_i - f(D_i, \boldsymbol{B})\}\left[\frac{\partial f(D_i,B)}{\partial \beta_j}\right] = 0 \quad (3)$$

For $j = 1,2,\ldots,p$. When the model is nonlinear in the parameters so are the normal equations consequently, for the nonlinear model, consider Table II, it is impossible to obtain the closed solution of the least squares estimate of the parameter by solving the p normal equations described in (3). Hence an iterative method must be employed to minimize the ss_{Res} [7]-[11]

The hyperbolic functions have similar names to the trigonometric functions, but they are defined in terms of the exponential function. The three main types of hyperbolic functions [12] and the sketch of their graphs are given below.

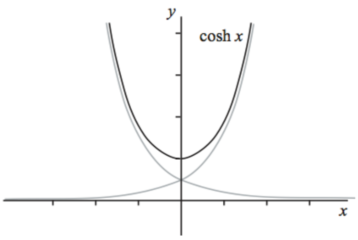

Fig. 1 Cosh Function

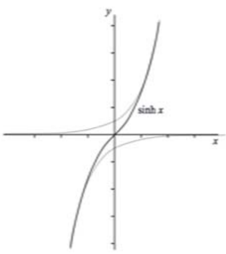

Fig. 2 Sinh Function

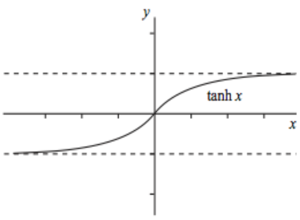

Fig. 3 Tanh Function

Hence, the hyperbolic sine function and its inverse [4], [5] provide an alternative method for evaluating;

$$\int \frac{1}{\sqrt{1+x^2}}dx$$

If we make the substitution, then;

$$\sqrt{1+x^2} = \sqrt{1+sinh^2(u)} = \sqrt{cosh^2(u)} = \cosh(u)$$

where the second equality follows from the identity $\cosh^2(u) - \sinh^2(u) = 1$ and the last equality from the fact that $\cosh(u) > 0$ for all u. Hence;

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \frac{\cosh(u)}{\cosh(u)}du = \int du = u + c = sinh^{-1}(x) + c$$

The following proposition is a consequence of the integral above i.e.

$$\frac{d}{dx}sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

Also, using the substitution x = tan (u), $-\frac{\pi}{2} < u < \frac{\pi}{2}$, that

$$\int \frac{1}{\sqrt{1+x^2}}dx = log\left|x + \sqrt{1+x^2}\right| + c$$

Since two anti-derivatives of a function can differ at most by a constant, there must exist a constant k such that

$$sinh^{-1}(x) = log\left|x + \sqrt{1+x^2}\right| + k$$

for all x. Evaluating both sides of this equality at x = 0, we have

$$0 = sinh^{-1}(0) = \log(1) + k = k$$

Thus k = 0 and

$$sinh^{-1}(x) = log\left|x + \sqrt{1+x^2}\right|$$

for all x. Since the hyperbolic sine function is defined in terms of the exponential function, we should not find it surprising that the inverse hyperbolic sine function may be expressed in terms of the natural logarithm function.

III. Hyperbolic Gompertz Growth Model

Consider a modified gompertz growth equation of the form;

$$\frac{dH}{dt} = H\,ln\left(\frac{K}{H}\right)\left[r + \frac{\theta}{\sqrt{1+t^2}}\right]$$

Separating variables gives;

$$\frac{dH}{H\,ln\left(\frac{K}{H}\right)} = \left[r + \frac{\theta}{\sqrt{1+t^2}}\right]dt$$

Let $m = ln\left(\frac{K}{H}\right)$ such that $\frac{K}{H} = e^m$

$$H = Ke^{-m}$$
$$dH = -Ke^{-m}dm$$

Substitute to obtain;

$$-\frac{dm}{m} = \left[r + \frac{\theta}{\sqrt{1+t^2}}\right]dt$$

$$-\ln m = rt + \theta \operatorname{arcsinh}(t) + C_1$$
$$\ln m^{-1} = rt + \theta \operatorname{arcsinh}(t) + C_1$$
$$m^{-1} = Ae^{rt+\theta \operatorname{arcsinh}(t)} \; where \; A = e^{C_1}$$

but $m = \ln\left(\frac{K}{H}\right)$;

Hence,

$$\left(\ln\left(\frac{K}{H}\right)\right)^{-1} = Ae^{rt+\theta \operatorname{arcsinh}(t)}$$

Finally, solving for H gives a Hyperbolastic Gompertz model

$$H = Ke^{-Ae^{-rt-\theta \operatorname{arcsinh}(t)}}$$

Therefore, we shall apply the two models below on Age-height and Age-Diameter of pines (*Pinus caribaea*) growth;
(1) $H = Ke^{-Ae^{-rt-\theta \operatorname{arcsinh}(t)}} + \varepsilon$, and $D = Ke^{-Ae^{-rt-\theta \operatorname{arcsinh}(t)}} + \varepsilon$
(2) $H = Ke^{-Ae^{-rt}} + \varepsilon$, and $D = Ke^{-Ae^{-rt}} + \varepsilon$

IV. Results and Discussion

A. Figures and Tables

Tables I-IV show the estimated parameter for gompertz and hyperbolic gompertz growth model while Table V shows their respective coefficient of determination (R^2), MSE and AIC for age-height/age-diameter models.

TABLE I
HEIGHT PARAMETER ESTIMATES USING GOMPERTZ GROWTH MODEL

Parameter	Estimate	Std. Error	95% Confidence Interval	
			Lower Bound	Upper Bound
a	1.488	.116	1.240	1.736
r	0.024	.008	.006	.042
k	28.972	5.408	17.374	40.571

TABLE II
HEIGHT PARAMETER ESTIMATES USING HYPERBOLIC GOMPERTZ GROWTH MODEL

Parameter	Estimate	Std. Error	95% Confidence Interval	
			Lower Bound	Upper Bound
a	4.626	128.536	-273.060	282.311
r	0.002	.235	-.505	.509
k	131.646	19033.894	-40988.582	41251.875
m	0.166	7.409	-15.840	16.171

TABLE III
DIAMETER PARAMETER ESTIMATES USING GOMPERTZ GROWTH MODEL

Parameter	Estimate	Std. Error	95% Confidence Interval	
			Lower Bound	Upper Bound
a	1.548	.048	1.445	1.650
r	0.026	.003	.018	.033
k	34.100	2.395	28.963	39.237

Also, the predicted and observed height and diameter were plotted to show the relationship and how best the models predicted the observed data on height and diameter of pines as shown in Figs. 4–7.

TABLE IV
DIAMETER PARAMETER ESTIMATES USING HYPERBOLIC GOMPERTZ GROWTH MODEL

Parameter	Estimate	Std. Error	95% Confidence Interval	
			Lower Bound	Upper Bound
a	0.340	.184	-.057	.738
r	0.054	.010	.032	.076
k	28.908	1.086	26.562	31.254
m	-0.525	.193	-.941	-.108

TABLE V
SUMMARY OF MODEL SELECTION CRITERIA COMPUTED FOR THE PROPOSED AND SOURCE MODELS

Models	SSE	N	K	R SQ	MSE	AIC
SOURCE(Ht)	19.091	17	3	94.10%	1.364	7.972062488
PROPOSED(Ht)	15.324	17	4	95.30%	1.179	6.235517007
SOURCE(Dbh)	4.794	17	3	99.00%	0.342	-15.51941954
PROPOSED(Dbh)	2.94	17	4	99.40%	0.226	-21.83166397

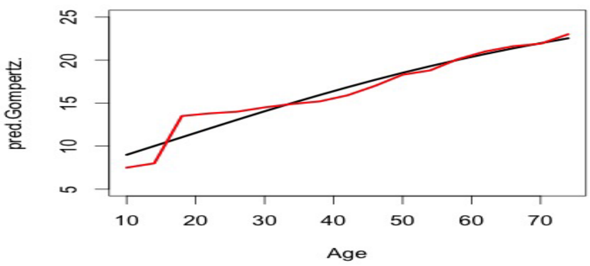

Fig. 4 Observed Height against Predicted height (Gompertz growth model)

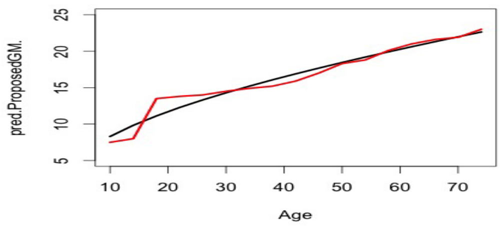

Fig. 5 Observed Height against Predicted height (Hyperbolic Gompertz growth model)

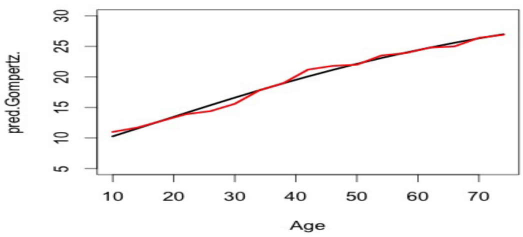

Fig. 6 Observed Diameter against Predicted diameter(Gompertz growth model)

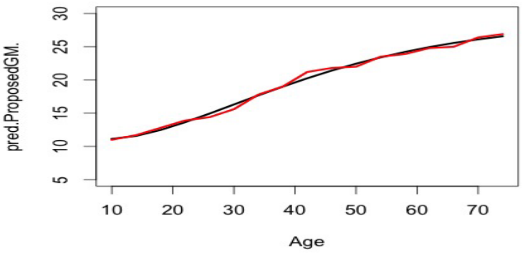

Fig. 7 Observed Diameter against Predicted diameter(Hyperbolic Gompertz growth model)

V. Residual Tests

Two assumptions made in the models are:

- Errors are independent
- Errors are normally distributed.

These assumptions were verified by examining the residuals. If the fitted models are correct, residuals should exhibit tendencies that tend to confirm or at least should not exhibit a denial of the assumptions.

Hence, we tested the following hypotheses stated below;
H0. Errors are independent (Using Runs Test)
H1. Errors are not independent
and
H0. Errors are normally distributed (Using Shapiro-Wilk test)
H1. Errors are not normally distributed

Let m be the number of pluses and n be the number of minuses in the series of residuals. The test is based on the number of runs(r), where a run is defined as a sequence of symbols of one kind separated by symbols of another kind. A good large sample approximation to the sampling distribution of the number of runs is the normal distribution with mean;

$$Mean = \frac{2mn}{m+n} + 1$$

and,

$$Variance(\sigma^2) = \frac{2mn(2mn - m - n)}{(m+n)^2(m+n-1)}$$

Therefore, for large samples like ours the required test statistic is;

$$Z = \frac{(r + h - \mu)}{\sigma} \sim N(0,1)$$

where,

$$h = \begin{cases} 0.5, if\ r < \mu \\ -0.5, r > \mu \end{cases}$$

Also, the required test statistic for the test of normality (Shapiro-Wilk test) is given by;

$$W = \frac{S^2}{b}$$

where;

$$S^2 = \sum a(k)\{x_{n+1-k} - x_{(k)}\}$$

and,

$$b = \sum (x_i - \bar{x})^2$$

In the above, the parameter k takes the values; $x_{(k)}$ is the k^{th} order statistic of the set of residuals and the values of coefficient a(k) for different values of n and k are given in the Shapiro-Wilk table. H_0 is rejected at level α i.e. W is less than the tabulated value. The results showed in table 6 and 7 below presents the Runs tests, K-S test, and S-W test results which complied with the normality and independent assumptions of the error term.

TABLE VI
Result of the Test of Independence of Residuals Using Run Test

Residual	Test Value	No of Runs	Z	Asymp. Sig.(2 tailed
Gomp. Height	-0.0041	6	-1.494	0.135^{ns}
Gomp. Diameter	0.0018	8	-0.381	0.703^{ns}
HGomp. Height	-0.0024	6	-1.494	0.135^{ns}
HGomp. Diameter	0.0035	10	0.015	0.988^{ns}

^{ns} not significant

TABLE VII
Result of the Test of Normality of Residuals Using K-S & S-W Tests

Residual	Kolmogorov-Sminov		Shapiro-Wilk	
	Statistic	Asmp. Sig.	Statistic	Asmp. Sig.
Gomp. Height	0.156	***0.200^{ns}***	0.971	***0.831^{ns}***
Gomp. Diameter	0.173	***0.188^{ns}***	0.941	***0.325^{ns}***
HGomp. Height	0.168	***0.200^{ns}***	0.955	***0.539^{ns}***
HGomp. Diameter	0.130	***0.200^{ns}***	0.958	***0.597^{ns}***

^{ns} not significant

VI. Conclusion

We have proposed a new growth model by introducing an allometric parameter θ using the hyperbolic sine function. The mean function of top height and Dbh over age using the Hyperbolic Gompertz growth model predicted closely the observed values of top height and diameter of pines.

References

[1] Richards, F. J. 1959: A flexible growth function for empirical use. Journal of Experimental Botany 10: 290–300.
[2] Winsor, C. P 1932: The Gompertz curve as a growth curve. Proc. National Academy of Science, 18, 1-8.
[3] Gompertz B 1825: On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. Phil Trans of the Royal Soc, 182:513-585.
[4] Oyamakin, S. O.; Chukwu, U. A.; and Bamiduro, T. A. 2013: On Comparison of Exponential and Hyperbolic Exponential Growth Models in Height/Diameter Increment of PINES (Pinuscaribaea)," Journal of Modern Applied Statistical Methods: Vol. 12: Iss. 2, Article 24.Pp 381-404.
[5] Oyamakin S. O., Chukwu A. U. 2014: On the Hyperbolic Exponential Growth Model in Height/Diameter Growth of PINES (Pinuscaribaea), International Journal of Statistics and Applications, Vol. 4 No. 2, 2014, pp. 96-101. doi: 10.5923/j.statistics.20140402.03.
[6] Kansal AR, Torquato S, Harsh GR, 2000: Simulated brain tumor growth dynamics using a three dimentional cellular automaton. J Theor Biol., 203:367-82.
[7] Draper, N.R. and H. Smith, 1981. Applied Regression Analysis. John Wiley and Sons, New York
[8] Ratkowskay, D.A., 1983. Nonlinear Regression modeling. Marcel Dekker, New York. 276p

[9] Marquardt, D.W., 1963. An algorithm for least squares estimation of nonlinear parameters. Journal of the society for Industrial and Applied Mathematics, 11: 431 – 441.
[10] Seber, G. A. F. and C. J. Wild, 1989. Nonlinear Regression. John Wiley and Sons: NY
[11] Fekedulegn, D. 1996. Theoretical nonlinear mathematical models in forest growth and yield modeling. Thesis, Dept. of Crop Science, Horticulture and Forestry, University College Dublin, Ireland. 200p.
[12] Tabatabai M, Williams DK, Bursac Z. 2005: Hyperbolastic growth models: Theory and application. TheorBiol Med Model, 2(14): 1-13.