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On Graded Semiprime Submodules

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Abstract—Let G be an arbitrary group with identity e and let R be a G-graded ring. In this paper we define graded semiprime submodules of a graded R-module M and we give a number of results concerning such submodules. Also, we extend some results of graded semiprime submoduls to graded weakly semiprime submodules.

Keywords—graded semiprime, graded weakly semiprime, graded secondary.

I. Introduction

WEAKLY prime ideals in a commutative ring with nonzero identity have been introduced and studied by D. D. Anderson and S. Smith (see [1]). Weakly primary ideals in a commutative ring with nonzero identity have been introduced and studied in [4]. Also, weakly prime submodules have been studied in [5]. Graded prime ideals in a commutative G-graded ring with nonzero identity have been introduced and studied by M. Refaei and K. Alzobi in [11]. Also, graded weakly prime ideals in a commutative graded ring with nonzero identity have been studied by S. Ebrahimi Atani (see [2]). Graded prime submodules and graded weakly prime submoduled have been studied in [6] and [3] respectively. Here we study graded semiprime and graded weakly semiprime submodules of a graded R-module. For example, we show that graded semiprime submodules of graded secondary modules are graded secondary. Throughout this work R will denote a commutative G-graded ring with nonzero identity and M a graded R-module.

Before we state some results let us introduce some notation and terminology. A ring (R, G) is called a G-graded ring if there exists a family $\{R_g : g \in G\}$ of additive subgroups of R such that $R = \bigoplus_{g\in G} R_g$ such that $R_gR_h \subseteq R_{gh}$ for each g and h in G. For simplicity, we will denote the graded ring (R, G) by R. If $a \in R$, then a can written uniquely as $\sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$. Moreover, if $R = \bigoplus_{g\in G} R_g$, is a graded ring, then R_e is a subring of R, $1_R \in R_e$ and R_g is an R_e-module for all $g \in G$. A ideal I of R, where R is G-graded, is called G-graded if $I = \bigoplus_{g\in G}(I \cap R_g)$ or if, equivalently, I is generated by homogeneous elements. Moreover, R/I becomes a G-graded ring with g-component $(R/I)_g = (R_g + I)/I$ for $g \in G$. Let I be a graded ideal of R, graded radical I of R, $Grad(R) = \{r \in R : x_g^{n_g} \in I$ for some $n_g \in N\}$. A graded ideal I of R is said to be graded prime if $I \neq R$; and whenever $ab \in I$, we have $a \in I$ or $b \in I$, where $a, b \in h(R)$. A proper graded ideal P of R is said to be graded weakly prime if $0 \neq ab \in P$ where $a, b \in h(R)$, implies $a \in P$ or $b \in P$. A graded ideal I of R is said to be graded maximal if $I \neq R$ and if J is a graded ideal of R such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$. A graded ring R is called a graded integral domain if $ab = 0$ for $a, b \in h(R)$, then $a = 0$ or $b = 0$. A graded ring R is called a graded local ring if it has a unique graded maximal ideal P, and denoted by (R, P). Let R_1 and R_2 be graded rings. Let $R = R_1 \times R_2$, clearly R is a graded ring. We write $h(R) = h(R_1) \times h(R_2)$. If R is G-graded, then an R-module M is said to be G-graded if it has a direct sum decomposition $M = \bigoplus_{g\in G} M_g$ such that for all $g, h \in G$; $R_gM_h \subseteq M_{gh}$. An element of some R_g or M_g is said to be homogeneous element. A submodule $N \subseteq M$, where M is G-graded, is called G-graded if $N = \bigoplus_{g\in G}(N \cap M_g)$ or if, equivalently, N is generated by homogeneous elements. Moreover, M/N becomes a G-graded module with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. A proper graded submodule N of a graded module M over a commutative graded ring R is said to be graded prime if whenever $r^km \in N$, for some $r \in h(R)$, $m \in h(M)$, then $rM \subseteq N$ or $m \in N$. A proper graded submodule N of a graded R-module M is said to be graded weakly prime if $0 \neq rm \in N$ where $r \in h(R)$, $m \in h(M)$, then $m \in N$ or $rM \subseteq N$. Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (degs)^{-1}(degr)\}$. We write $h(S^{-1}R) = \bigcup_{g\in G}(S^{-1}R)_g$. Let M be a graded R-module. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ and $g = (degs)^{-1}(degm)\}$. Let P be any graded prime ideal of a graded ring R and consider the multiplicatively closed subset of $S = h(R) - P$. We denote the graded ring of fraction $S^{-1}R$ of R by R_P^g and we call it the graded localization of R. This ring is graded local with the unique graded maximal ideal $S^{-1}P$ which will be denoted by PR_P^g. Moreover, R_P^g-module $S^{-1}M$ is denoted by M_P^g (see [9]).

II. Graded Semiprime Submodules

In this section, we define the graded semiprime submodules of a graded R-module M and give some of their basic properties.

Definition 2.1: Let R be a graded ring and M a graded R-module. A proper graded submodule N of M is said to be graded semiprime, if $r^km \in N$ for some $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$, then $rm \in N$.

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It is clear that every graded prime submodule is a graded semiprime submodule, but the converse is not true in general. For example, let $R = Z_{30}[i] = \{a + bi : a, b \in Z_{30}\}$ that Z_{30} is the ring of integers modulo 30 and let $G = Z_2$. Then R is a G-graded ring with $R_0 = Z_{30}$, $R_1 = iZ_{30}$. Let $I =< 6 > \bigoplus < 0 >$. The graded ideal I is graded semiprime, but it is not graded prime. Because $(2, 0).(3, 0) \in I$, but $(2, 0) \notin I$ and $(3, 0) \notin I$.

Definition 2.2: Let N be a graded submodule of graded R-module M and $g \in G$. We say that N_g is a semiprime submodule of R_e-module M_g, if $r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$, then $r_e m_g \in N_g$.

Proposition 2.3: Let M be a G-graded R-module and $N = \bigoplus_{g\in G} N_g$ a graded submodule of M. If N is a graded semiprime submodule of M, then N_g is a semiprime submodule of R_e-module M_g for any $g \in G$.

Proof: Let $r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in Z^+$. So $r_e^k m_g \in N_g \subseteq N$, hence $r_e m_g \in N$ since N is a graded semiprime submodule. Since $R_e M_g \subseteq M_{eg} = M_g$, so $r_e m_g \in N_g$, as required. ■

The following Lemma is known, but we write it here for the sake of references.

Lemma 2.4: Let M be a graded module over a graded ring R. Then the following hold:
(i) If I and J are graded ideals of R, then $I + J$ and $I \bigcap J$ are graded ideals.
(ii) If N is a graded submodule, $r \in h(R)$ and $x \in h(M)$, then Rx, IN and rN are graded submodules of M.
(iii) If N and K are graded submodules of M, then $N + K$ and $N \bigcap K$ are also graded submodules of M and $(N :_R M)$ is a graded ideal of R.
(iv) Let N_λ be a collection of graded submodules of M. Then $\sum_\lambda N_\lambda$ and $\bigcap_\lambda N_\lambda$ are graded submodues of M.

Proposition 2.5: Let M be a graded R-module, N a graded semiprime submodule of M and $m \in h(M)$. Then
(i) If $m \in N$, then $(N : m) = R$.
(ii) If $m \notin N$, then $(N : m)$ is a graded semiprime submodule of M.

Proof: (i) It is clear.
(ii) Let $x^k y \in (N : m)$ where $x, y \in h(R)$ and $k \in Z^+$. Hence $x^k ym \in N$, so $xym \in N$ since N is graded semiprime. Therefore $xy \in (N : m)$, as needed. ■

Proposition 2.6: Let M be a graded R-module and I a graded ideal of R. If N is a graded semiprime submodule of M such that $I^k M \subseteq N$ for some $k \in Z^+$, then $IM \subseteq N$.

Proof: Let $am \in IM$ where $a \in I$ and $m \in M$. So $a = \sum_{g\in G} a_g$ that $a_g \in I \cap h(R)$ and $m = \sum_{g\in G} m_h$ that $m_h \in h(M)$. Hence for any $g, h \in G$, $a_g^k m_h \in I^k M \subseteq N$, so $a_g m_h \in N$ since N is a graded semiprime submodule. Therefore $am \in N$, as needed. ■

A graded R-module M is called graded multiplication if for any graded submodule N of M, $N = IM$ for some graded ideal I of R (see [9]).

Proposition 2.7: Let M be a graded multiplication R-module and K a graded submodule of M. If N is a graded semiprime submodule of M such that $K^n \subseteq N$ for some $n \in Z^+$, then $K \subseteq N$. Moreover, if $K^n = N$ for some $n \in Z^+$, then $K = N$.

Proof: Since M is a graded multiplication module, so $K = IM$ for some graded ideal I of R. Hence $K^n = (IM)^n = I^n M \subseteq N$, then $K \subseteq N$ by Proposition 2.6. Clearly, if $K^n = N$ for some $n \in Z^+$, then $K = N$. ■

Proposition 2.8: Let $R = R_1 \times R_2$ where R_i, $i = 1, 2$, is a graded commutative ring with identity for $i = 1, 2$. Let M_i be a graded R_i-module and let $M = M_1 \times M_2$ be the graded R-module with action $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ where $r_i \in R_i$ and $m_i \in M_i$. Then the following hold:
(i) N_1 is a graded semiprime submodule of M_1 if and only if $N_1 \times M_2$ is a graded semiprime submodule of M.
(ii) N_2 is a graded semiprime submodule of M_2 if and only if $M_1 \times N_2$ is a graded semiprime submodule of M.

Proof: (i) Let N_1 be a graded semiprime submodule of M_1. Suppose $(a, b)^k(m, n) \in N_1 \times M_2$ where $(a, b) \in h(R) = h(R_1) \times h(R_2)$, $(m, n) \in h(M) = h(M_1) \times h(M_2)$ and $k \in Z^+$. So $a^k m \in N_1$, and $am \in N_1$ since N_1 is a graded semiprime submodule. Hence $(a, b)(m, n) \in N_1 \times M_2$, as required. Let $N_1 \times M_2$ is a graded semiprime submodule of M. Let $a^k m \in N_1$ where $a \in h(R_1)$, $m \in h(M_1)$ and $k \in Z^+$. So $(a, 1)^k(m, 0) \in N_1 \times M_2$ where $(a, 1) \in h(R)$ and $(m, 0) \in h(M)$, thus $(a, 1)(m, 0) \in N_1 \times M_2$ since $N_1 \times M_2$ is a graded semiprime submodule. Hence $am \in N_1$, as needed.
(ii) The proof is similar to that in case (i) and we omit it. ■

A graded R-module M is called a graded secondary module provided that for every homogeneous element $r \in h(R)$, $rM = M$ or $r^n M = 0$ for some positive integer n (see [7]).

Theorem 2.9: Let M be a graded secondary R-module and N a nonzero graded semiprime R-submodule of M. Then N is graded secondary R-module.

Proof: Let $r \in h(R)$. If $r^n M = 0$ for some positive integer n, then $r^n N \subseteq r^n M = 0$, so r is nilpotent on N. Suppose that $rM = M$; we show that r divides N. Let $n \in N$. We may assume that $n = \sum_{g\in G} n_g$ where $n_g \neq 0$. So for every $g \in G$, $n_g = rm$ for some $m \in h(M)$. We have $rm' = m$ for some $m' \in h(M)$, hence $rm = r^2m' \in N$, so $m = rm' \in N$ since N is graded semiprime. Hence $n = rm \in rN$. Thus $rN = N$, as needed. ■

Corollary 2.10: Let M be a graded R-module, N a graded secondary R-submodule of M and K a graded semiprime submodule of M. Then $N \cap K$ is graded secondary.

Proof: The proof is straightforward by Theorem 2.7. ■

Proposition 2.11: Let R be a graded ring and $S \subseteq h(R)$ be a multiplication closed subset of R. If N is a graded semiprime submodule of M, then $S^{-1}N$ is a graded semiprime submodule of $S^{-1}M$.

Proof: Let $(r/s)^k.m/t \in S^{-1}N$ where $r/s \in h(S^{-1}R), m/t \in h(S^{-1}M)$ and $k \in Z^+$. So $r^k m/s^k t = n/t'$ for some $n \in N \cap h(M)$ and $t' \in S$, hence there exists $s' \in S$ such that $s't'r^k m = s's^k tn \in N$, so N graded semiprime gives $rms't' \in N$. Hence $rm/st = rms't'/sts't' \in S^{-1}N$, as needed. ■

Proposition 2.12: Let (R, P) be a graded local ring with graded maximal ideal P and $S = h(R) - P$. Then N is a graded semiprime submodule of graded R-module M if and only if N_P^g is a graded semiprime submodule of graded R_P^g-module M_P^g.

Proof: Let N be a graded semiprime submodule of M, then N_P^g is a graded semiprime submodule of M_P^g by Proposition 2.11. Let $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $r^k m/1 = (r/1)^k m/1 \in N_P^g$. Hence $rm/1 \in N_P^g$, and $rm/1 = c/s$ for some $c \in N \cap h(M)$ and $s \in S$. So there exists $t \in S$ such that $strm = tc \in N$. So $rm \in N$, because if $rm \notin N$, then $(N : rm) \neq R$, and $st \in (N : rm) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore N is a graded semiprime submodule of M. ■

Proposition 2.13: Let $K \subseteq N$ be proper graded submodules of a graded R-module M. Then N is a graded semiprime submodule of M if and only if N/K is a graded semiprime submodule of M/N.

Proof: $(\Rightarrow)$ Let $r^k(m+K) \in N/K$ where $r \in h(R), m \in h(M)$ and Z^+. So $r^k m \in N$, N graded semiprime gives $rm \in N$. Hence $r(m + K) \in N/K$.
$(\Leftarrow)$ Let $r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $r^k m + K = r^k(m + K) \in N/K$. Then $r(m+K) \in N/K$ since N/K is graded semiprime. Hence $rm \in N$, as required. ■

III. Graded weakly semiprime submodules

In this section, we define the graded weakly semiprime submodules of a graded R-module and we extend some results of graded semiprime submodules to graded weakly semiprime submodules.

Definition 3.1: Let R be a graded ring and M a graded R-module. A proper graded submodule N of M is said to be graded weakly semiprime, if $0 \neq r^k m \in N$ for some $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$, then $rm \in N$.

It is clear that every graded semiprime submodule is a graded weakly semiprime submodule. However, since 0 is always graded weakly semiprime, a graded weakly semiprime submodule need not be graded semiprime, but if R be a graded integral domain and M a faithful graded prime module, then every graded weakly semiprime is graded semiprime.

Definition 3.2: Let N be a graded submodule of a graded R-module M and $g \in G$. We say that N_g is a weakly semiprime submodule of R_e-module M_g, if $r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in Z^+$, then $r_e m_g \in N_g$.

Proposition 3.3: Let M be a graded R-module and $N = \bigoplus_{g\in G} N_g$ a graded submodule of M. If N is a graded weakly semiprime submodule of M, then N_g is a weakly semiprime submodule of R_e-module M_g for any $g \in G$.

Proof: Let $0 \neq r_e^k m_g \in N_g$ where $r_e \in R_e$, $m_g \in M_g$ and $k \in Z^+$. So $r_e^k m_g \in N_g \subseteq N$, hence $r_e m_g \in N$ since N is a graded weakly semiprime submodule. Since $R_e M_g \subseteq M_{eg} = M_g$, so $r_e m_g \in N_g$, as required. ■

Theorem 3.4: Let R be a graded ring, M a graded R-module, N a graded submodule of M and $g \in G$. Consider the following assertion.
(i) N_g is a weakly semiprime submodule of M_g.
(ii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a) \cup Rad(0 :_{R_e} a)$.
(iii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a)$ or $Rad(N_g :_{R_e} a) = Rad(0 :_{R_e} a)$.
Then $(i) \Rightarrow (ii) \Rightarrow (iii)$.

Proof: $(i) \Rightarrow (ii)$ It is clear that $(N_g :_{R_e} a) \cup Rad(0 :_{R_e} a) \subseteq Rad(N_g :_{R_e} a)$. Let $r \in Rad(N_g :_{R_e} a)$. So $r^n a \in N_g$ for some positive integer n. If $r^n a = 0$, then $r \in Rad(0 :_{R_e} a)$. If $0 \neq r^n a \in N_g$, then $ra \in N_g$ since N_g is a weakly semiprime submodule of M_g. Hence $Rad(N_g :_{R_e} a) \subseteq (N_g :_{R_e} a) \cup Rad(0 :_{R_e} a)$. Therefore the proof is complete.
$(ii) \Rightarrow (i)$ It is well known that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them. ■

An R_e-module M_g is called prime module if the zero submodule is prime.

Remark 3.5: An R_e-module M_g is prime if and only if $(0 :_{R_e} M_g) = (0 :_{R_e} m_g)$ for any $0 \neq m_g \in M_g$).

Theorem 3.6: Let R be a graded ring, M a graded R-module, N a graded submodule of M, and $g \in G$. Then the following assertion are equivalent.
(i) N_g is a weakly semiprime submodule of M_g.
(ii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a) \cup Rad(0 :_{R_e} a)$.
(iii) For $a \in M_g$, $Rad(N_g :_{R_e} a) = (N_g :_{R_e} a)$ or $Rad(N_g :_{R_e} a) = Rad(0 :_{R_e} a)$.

Proof: It is enough to show that $(iii) \Rightarrow (i)$. Let $0 \neq r^k m \in N_g$ where $r \in R_e$, $m \in M_g$ and $k \in Z^+$. So $r \in Rad(N_g :_{R_e} m)$. If $r \in Rad(0 : m)$, then $r^n m = 0$ for some $n \in Z^+$. Let t be the smallest integer such that $r^t m = 0$. If $t > k$, then $0 < t - k < t$; $r^t m = r^k(r^{t-k}m) = 0$; $r^k \in (0 :_{R_e} r^{t-k}m) = (0 :_{R_e} M_g)$ since M_g is a graded prime module. Hence $r^k M_g = 0$, so $r^k m = 0$, a contradiction. Let $k \geq t$. Thus $r^k m = r^{k-t}(r^t m) = 0$ which is a contradiction. Therefore $r \notin Rad(0 :_{R_e} m)$. So $r \in (N_g : m)$, hence $rm \in N_g$, as needed. ■

Proposition 3.7: Let $R = R_1 \times R_2$ where R_i for $i = 1, 2$, is a commutative graded ring with identity. Let M_i be a graded R_i-module and let $M = M_1 \times M_2$ be the graded R-module.

Then the following hold:
(i) If $N_1 \times M_2$ is a graded weakly semiprime submodule of M, then N_1 is a graded weakly semiprime submodule of M_1.
(ii) If $M_1 \times N_2$ is a graded weakly semiprime submodule of M, then N_2 is a graded weakly semiprime submodule of M_2.

Proof: (i) Let $N_1 \times M_2$ is a graded weakly semiprime submodule of M. Suppose $0 \neq a^k m \in N_1$ where $a \in h(R_1)$, $m \in h(M_1)$ and $k \in Z^+$. So $0 \neq (a,1)^k(m,0) \in N_1 \times M_2$, then $(a,1)(m,0) \in N_1 \times M_2$ since $N_1 \times M_2$ is a graded weakly semiprime. Hence $am \in N_1$, so N_1 is a graded weakly semiprime submodule of M_1.
(ii) The proof is similar to that in case (i). ■

Theorem 3.8: Let M be a graded secondary R-module and N a nonzero graded weakly semiprime R-submodule of M. Then N is graded secondary.

Proof: Let $r \in h(R)$. If $r^n M = 0$ for some positive integer n, then $r^n N \subseteq r^n M = 0$, so r is nilpotent on N. Suppose that $rM = M$; we show that r divides N. Let $0 \neq n \in N$. We may assume that $n = \sum_{g\in G} n_g$ where $n_g \neq 0$. So for any $g \in G$, $n_g = rm$ for some $m \in h(M)$. We have $rm' = m$ for some $m' \in h(M)$, hence $0 \neq rm = r^2 m' \in N$, so $m = rm' \in N$ since N is a graded weakly semiprime submodule. Thus $n_g \in rN$, so $n \in rN$. Therefore $rN = N$, as needed. ■

Corollary 3.9: Let M be a graded R-module, N a graded secondary R-submodule of M and K a graded weakly semiprime submodule of M. Then $N \cap K$ is graded secondary.

Proof: The proof is straightforward by Theorem 3.8. ■

Proposition 3.10: Let R be a graded ring and $S \subseteq h(R)$ be a multiplication closed subset of R. If N is a graded weakly semiprime submodule of M, then $S^{-1}N$ is a graded weakly semiprime submodule of $S^{-1}M$.

Proof: Let $0/1 \neq (r/s)^k.m/t \in S^{-1}N$ where $r/s \in h(S^{-1}R), m/t \in h(S^{-1}M)$ and $k \in Z^+$. So $0/1 \neq r^k m/s^k t = n/t'$ for some $n \in N \cap h(M)$ and $t' \in S$, hence there exists $s' \in S$ such that $0 \neq s't'r^k m = s's^k tn \in N$ (because if $s't'r^k m = 0$, $r^k m/s^k t = s't'r^k m/s't's^k t = 0/1$, a contradiction), so N graded weakly semiprime gives $rms't' \in N$. Hence $rm/st = rms't'/sts't' \in S^{-1}N$, as needed. ■

Proposition 3.11: Let (R, P) be a graded local ring with graded maximal ideal P and $S = h(R) - P$. Then N is a graded weakly semiprime submodule of graded R-module M if and only if N_P^g is a graded weakly semiprime submodule of graded R_P^g-module M_P^g.

Proof: Let N be a graded weakly semiprime submodule of M, then N_P^g is a graded weakly semiprime submodule of M_P^g by Proposition 3.10. Let $0 \neq r^k m \in N$ where $r \in h(R)$, $m \in h(M)$ and $k \in Z^+$. So $0/1 \neq r^k m/1 = (r/1)^k m/1 \in N_P^g$ because if $0/1 = r^k m/1$, then $s(r^k m) = 0$ for some $s \in S$, so $s \in (0 : r^k m) \cap S \subseteq P \cap S = \emptyset$, a contradiction. Hence $rm/1 \in N_P^g$, and $rm/1 = c/s$ for some $c \in N \cap h(M)$ and $s \in S$. So there exists $t \in S$ such that $strm = tc \in N$. So $rm \in N$, because if $rm \notin N$, then $(N : rm) \neq R$, and $st \in (N : rm) \cap S \subseteq P \cap S = \emptyset$, which is a contradiction. Therefore N is a graded weakly semiprime submodule of M. ■

Proposition 3.12: Let $K \subseteq N$ be proper graded submodules of a graded R-module M. Then the following hold:
(i) If N is a graded weakly semiprime submodule of M, then N/K is a graded weakly semiprime R-submodule of M/N.
(ii) If K and N/K are graded weakly semiprime submodules of M and M/K respectively, then N is a graded weakly semiprime submodule of M.

Proof: (i) Let $0 \neq r^k(m + K) \in N/K$ where $r \in h(R)$, $m+K \in h(M/K)$ and $k \in Z^+$. So $0 \neq r^k m \in N$, N weakly semiprime gives $rm \in N$. Hence $r(m + K) \in N/K$.
(ii) Let $0 \neq r^k m \in N$ where $r \in h(R), m \in h(M)$ and $k \in Z^+$. So $r^k m+K = r^k(m+K) \in N/K$. If $0 \neq r^k m \in K$, then $rm \in K \subseteq N$ since K is graded weakly semiprime, as needed. Let $0 \neq r^k(m+K) \in N/K$, then $r(m+K) \in N/K$ since N/K is graded weakly semiprime. Hence $rm \in N$, as required. ■

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