On Bayesian Analysis of Failure Rate under Topp Leone Distribution using Complete and Censored Samples

N. Feroze and M. Aslam

Abstract—The article is concerned with analysis of failure rate (shape parameter) under the Topp Leone distribution using a Bayesian framework. Different loss functions and a couple of non-informative priors have been assumed for posterior estimation. The posterior predictive distributions have also been derived. A simulation study has been carried to compare the performance of different estimators. A real life example has been used to illustrate the applicability of the results obtained. The findings of the study suggest that the precautionary loss function based on Jeffreys prior and singly type II censored samples can effectively be employed to obtain the Bayes estimate of the failure rate under Topp Leone distribution.

Keywords—loss functions, type II censoring, posterior distribution, Bayes estimators.

I. INTRODUCTION

TOPP and Leone [1] introduced the Topp Leone distribution and derived its first four moments; they considered the distribution suitable to model failure data. After a long silence Nadarajah and Kotz [2] gave a hazard rate function motivation to the distribution and derived general formulae for its moments and central moments. Furthermore, the maximum likelihood estimation of the parameters of distribution was discussed. The hazard rate functions of life time distributions are either; constants, monotone increasing, monotone decreasing, or U-shaped. Each case is useful for real life applications. The last case is applicable to human populations where at the infant age the death rate is high due to birth defects or infant diseases, then the death rate remains constant up to the age of thirties, then it increases again. Some manufactured items also follow this pattern. Further, Vicari et al. [3] discussed some properties of the Two-Sided Generalized Topp and Leone (TS-GTL) family and described maximum likelihood estimation (MLE) procedure. A numerical example of the MLE procedure has also been provided. A comparison with a Gaussian mixture fit has been presented. Gen and Ali [4] derived explicit algebraic expressions for both of the single and product moments of order statistics from Topp Leone distribution. An identity about single moments of order statistics has also been given. These expressions are useful for computational purposes.

The Topp Leone distribution has been generated from the left triangular distribution by elevating the cumulative distribution function to a power β > 0 that becomes the parameter of the new distribution. To deal with characteristics of the newly defined distribution is always of great interest for the researchers. The insight we can get about them can be beneficial to the professionals looking to use those distributions as models.

The probability density function (pdf) of the Topp Leone distribution is:

$$f(x) = \beta(2-2x)(2x-x^2)^{\beta-1} \quad x > 0 \quad, \quad \beta > 0 \qquad (1)$$

The cumulative distribution function (CDF) of the distribution is:

$$F(x) = (2x-x^2)^{\beta} \qquad (2)$$

If the variable x represents the failure times, the Topp Leone distribution gives a distribution for which the failure rate is proportional to a power of time. The shape parameter β is that power, which can be interpreted as:

1) If the value of shape parameter is less than one, the failure rate decreases over time.
2) If the value of shape parameter is equal to one, the failure rate is constant over time.
3) If the value of shape parameter is less than one, the failure rate increases over time.

The shape parameter β is named the shape parameter as it determines which member of the Topp Leone family of distributions is most appropriate. Different values of the shape parameter can have marked effects on the behavior of the distribution. An important aspect of the distribution is how the values of the shape parameter β affect the distribution characteristics such as the shape of the probability density function curve, the reliability and the hazard rate.

The recent article explores the Bayesian estimation of the failure rate (shape parameter) under the Topp Leone distribution. The applications of the distribution to model the failure data gave further motivation to include different censoring schemes in the study. The results from the censored samples have been compared with those of complete data. The purpose of the study is to propose a suitable combination of prior distribution and loss function to estimate the failure rate under complete and censored samples.

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II. Bayes Estimation Under Complete Samples

The likelihood function for a sample of size n from the Topp Leone distribution is:

$$L\left(\beta|\underline{x}\right) \propto \beta^n e^{-\beta \sum_{i=1}^{n} \ln\left(2x_i - x_i^2\right)^{-1}} \tag{3}$$

The use of prior information is of immense importance and one of the main differences between classical and Bayesian inference. The attempts have been made to use the Bayesian approach even when no (or minimal) prior information is available. In this regard, the use of non-informative priors has received great attention of the statisticians. Among the most widely used non-informative priors are the uniform prior proposed by Laplace [5] and Jeffreys prior suggested by Jeffreys [6]. These priors have been assumed for posterior analysis.

The uniform prior is assumed to be: $p\left(\beta\right) \propto 1 \quad ; \beta > 0$

$$\tag{4}$$

The posterior distribution under the assumption of uniform prior is:

$$p\left(\beta|\underline{x}\right) = \frac{\left(T\right)^{n+1}}{\Gamma\left(n+1\right)} \beta^n e^{-\beta T} \quad ; \beta > 0$$

where $T = \sum_{i=1}^{n} \ln\left(2x_i - x_i^2\right)^{-1} \tag{5}$

Jeffreys prior is defined to be: $p_j \propto \sqrt{\left|I\left(\underline{\beta}\right)\right|}$

where $I\left(\underline{\beta}\right) = -E\left[\dfrac{\partial^2 \ln f(x)}{\partial \beta^2}\right]$

Here $I\left(\underline{\beta}\right) = -E\left[\dfrac{\partial^2 \ln f(x)}{\partial \beta^2}\right] = \dfrac{1}{\beta^2}$

Therefore $p_j \propto \sqrt{\left|\left(\underline{\beta}\right)\right|} = \dfrac{1}{\beta} \tag{6}$

The posterior distribution under Jeffreys prior is:

$$p\left(\beta|\underline{x}\right) = \frac{\left(T\right)^n}{\Gamma\left(n\right)} \beta^{n-1} e^{-\beta T} \quad ; \beta > 0 \tag{7}$$

The derivation of Bayes estimators and corresponding risks has been discussed using squared error loss function (SELF), LINEX loss function (LLF) and precautionary loss function (PLF) under the assumption of uniform and Jeffreys priors.

The Bayes estimator and risk under squared error loss function (SELF) using uniform prior are:

$$\beta_{SELF} = \frac{n+1}{T} \quad \text{and} \quad \rho\left(\beta_{SELF}\right) = \frac{n+1}{T^2}$$

The Bayes estimator and risk under precautionary loss function (PLF) using uniform prior are:

$$\beta_{PLF} = \frac{\sqrt{\left(n+2\right)\left(n+1\right)}}{T}$$

and $\rho\left(\beta_{PLF}\right) = 2\left\{\dfrac{\sqrt{\left(n+2\right)\left(n+1\right)}}{T} - \dfrac{n+1}{T}\right\}$

The Bayes estimator and risk under LINEX loss function (LLF) using uniform prior are:

$$\beta_{LLF} = -\left(n+1\right)\ln\left(\frac{T}{T+1}\right)$$

and $\rho\left(\beta_{LLF}\right) = \dfrac{n+1}{T} + \left(n+1\right)\ln\left(\dfrac{T}{T+1}\right)$

The Bayes estimator and risk under squared error loss function (SELF) using Jeffreys prior are:

$$\beta_{SELF} = \frac{n}{T} \quad \text{and} \quad \rho\left(\beta_{SELF}\right) = \frac{n}{T^2}$$

The Bayes estimator and risk under precautionary loss function (PLF) using Jeffreys prior are:

$$\beta_{PLF} = \frac{\sqrt{n\left(n+1\right)}}{T}$$

and $\rho\left(\beta_{PLF}\right) = 2\left\{\dfrac{\sqrt{n\left(n+1\right)}}{T} - \dfrac{n}{T}\right\}$

The Bayes estimator and risk under LINEX loss function (LLF) using uniform prior are:

$$\beta_{LLF} = -\left(n\right)\ln\left(\frac{T}{T+1}\right)$$

and $\rho\left(\beta_{LLF}\right) = \dfrac{n}{T} + \left(n\right)\ln\left(\dfrac{T}{T+1}\right)$

III. Bayes Estimation under Singly Type II Censored Samples

Censoring is very key feature of lifetime data. Among many types of censoring Type I and Type II censoring have received considerable attention. Under Type II censored samples the

experiment is terminated after observing some fixed percentage of observation, while in Type I censored samples, the censoring takes place at certain fixed points. So it can be said that in the former case the number of observations is a random variable; while in the latter case it is fixed in advance. Censoring can be single or double. When the sample is censored at single known termination time, it is called single censored. On the other hand when it is censored at two known termination points, it is considered as doubly censored. The authors dealing with Bayesian and classical analysis of statistical distribution under singly and doubly censored samples include: Wu and Lin [7], Fermindez et al. [8], Raqab and Madi [9]. Fauzy [10], Saleem and Aslam [11], Asgharzadeh and Valiollahi [12], Akhter and Hirai [13], Yarmohammadi and Pazira [14], AL-Hussaini and Hussein [15], and Feroze and Aslam [16]-[18]. We have considered singly and doubly type II censored samples for the Bayesian analysis. The likelihood function for the singly type II censored samples can be derived as:

Let we observe 'n' items for possible failure and only first 'm' failure times have been observed, that is, $x_1 < x_2 \ldots < x_m$ and remaining 'n – m' items are still working. Under the assumptions that the lifetimes of the items are independently and identically distributed random variables following Topp Leone distribution, the likelihood function for 'm' observations is:

$$L(\beta|\underline{x}) \propto \left\{ \prod_{i=1}^{m} f(x_i) \right\} \left\{ 1 - F(x_m) \right\}^{n-m}$$

$$L(\beta|\underline{x}) \propto \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \beta^m e^{-\beta\varphi_{1k}} \quad ; \beta > 0 \qquad (8)$$

where $\varphi_{1k} = \sum_{i=1}^{m} \ln(2x_i - x_i^2)^{-1} + k \ln(2x_m - x_m^2)^{-1}$

The posterior distribution under uniform prior is:

$$p(\beta|\underline{x}) = \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \beta^m e^{-\beta\varphi_{1k}} \quad ; \beta > 0 \qquad (9)$$

where $C_1 = \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+1)}{(\varphi_{1k})^{m+1}}$

The posterior distribution under Jeffreys prior is:

$$p(\beta|\underline{x}) = \frac{1}{C_2} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \beta^{m-1} e^{-\beta\varphi_{1k}} \quad ; \beta > 0 \quad (10)$$

where $C_2 = \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m)}{(\varphi_{1k})^m}$

The Bayes estimator and risk under squared error loss function (SELF) using uniform prior are:

$$\beta_{SELF} = \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+2)}{(\varphi_{1k})^{m+2}}$$

$$\rho(\beta_{SELF}) = \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+3)}{(\varphi_{1k})^{m+3}}$$

$$- \left\{ \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+2)}{(\varphi_{1k})^{m+2}} \right\}^2$$

The Bayes estimator and risk under precautionary loss function (PLF) using uniform prior are:

$$\beta_{PLF} = \left\{ \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+3)}{(\varphi_{1k})^{m+3}} \right\}^{\frac{1}{2}}$$

$$\rho(\beta_{PLF}) = 2 \left\{ \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+3)}{(\varphi_{1k})^{m+3}} \right\}^{\frac{1}{2}} - \frac{2}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+2)}{(\varphi_{1k})^{m+2}}$$

The Bayes estimator and risk under LINEX loss function (LLF) using uniform prior are:

$$\beta_{LLF} = -\ln \left\{ \frac{\sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{1}{(\varphi_{1k}+1)^{m+1}}}{\sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{1}{(\varphi_{1k})^{m+1}}} \right\}$$

$$\rho(\beta_{LLF}) = \frac{1}{C_1} \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{\Gamma(m+2)}{(\varphi_{1k})^{m+2}}$$

$$+ \ln \left\{ \frac{\sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{1}{(\varphi_{1k}+1)^{m+1}}}{\sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k} \frac{1}{(\varphi_{1k})^{m+1}}} \right\}$$

The expressions for Bayes estimators and corresponding posterior risks under the assumption of Jeffreys prior can be obtained in a similar manner

IV. BAYES ESTIMATION UNDER DOUBLY TYPE II CENSORED SAMPLES

The doubly type II censoring is used when the observations below and above a particular point cannot either be observed

or not feasible to be observed. The likelihood function under the doubly type II censored samples can be derived as:

Consider a random sample of size 'n' from Topp Leone distribution, and let $x_r,..., x_j$ be the ordered observations that can only be observed. The remaining 'r – 1' smallest observations and the 'n – j' largest observations have been censored. Then the likelihood function for the Type II doubly censored sample $\underline{x} = (x_r,..., x_j)$ can be written as:

$$L\left(\beta\big|\underline{x}\right) \propto \left[F\left(x_r\right)\right]^{r-1}\left[1-F\left(x_j\right)\right]^{n-j}\prod_{i=r}^{j}f\left(x_i\right)$$

$$L\left(\beta\big|\underline{x}\right) \propto \sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\beta^w e^{-\beta\varphi_{2k}} \quad ;\beta>0 \quad (11)$$

where
$$\varphi_{2k} = \sum_{i=r}^{j}\ln\left(2x_i-x_i^2\right)^{-1} + (r-1)\ln\left(2x_r-x_r^2\right)^{-1}$$
$$+ k\ln\left(2x_j-x_j^2\right)^{-1}$$

and $w = j - r + 1$

The posterior distribution under uniform prior is:

$$p\left(\beta\big|\underline{x}\right) = \frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\beta^w e^{-\beta\varphi_{2k}} \quad ;\beta>0 \quad (12)$$

where $C_3 = \sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\dfrac{\Gamma(w+1)}{\left(\varphi_{2k}\right)^{w+1}}$

The posterior distribution under Jeffreys prior is:

$$p\left(\beta\big|\underline{x}\right) = \frac{1}{C_4}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\beta^{w-1} e^{-\beta\varphi_{2k}} \quad ;\beta>0 \quad (13)$$

where $C_4 = \sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\dfrac{\Gamma(w)}{\left(\varphi_{2k}\right)^{w}}$

The Bayes estimator and risk under squared error loss function (SELF) using uniform prior are:

$$\beta_{SELF} = \frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+1)}{\left(\varphi_{2k}\right)^{w+1}}$$

$$\rho\left(\beta_{SELF}\right) = \frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+3)}{\left(\varphi_{2k}\right)^{w+3}}$$
$$-\left\{\frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+2)}{\left(\varphi_{2k}\right)^{w+2}}\right\}^2$$

The Bayes estimator and risk under precautionary loss function (PLF) using uniform prior are:

$$\beta_{PLF} = \left\{\frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+3)}{\left(\varphi_{2k}\right)^{w+3}}\right\}^{\frac{1}{2}}$$

$$\rho\left(\beta_{PLF}\right) = 2\left\{\frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+3)}{\left(\varphi_{2k}\right)^{w+3}}\right\}^{\frac{1}{2}}$$
$$-\frac{2}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+2)}{\left(\varphi_{2k}\right)^{w+2}}$$

The Bayes estimator and risk under LINEX loss function (LLF) using uniform prior are:

$$\beta_{LLF} = -\ln\left\{\frac{\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\dfrac{1}{\left(\varphi_{2k}+1\right)^{w+1}}}{\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\dfrac{1}{\left(\varphi_{2k}\right)^{w+1}}}\right\}$$

$$\rho\left(\beta_{LLF}\right) = \frac{1}{C_3}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\frac{\Gamma(w+2)}{\left(\varphi_{2k}\right)^{w+2}} + \ln\left\{\frac{\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\dfrac{1}{\left(\varphi_{2k}+1\right)^{w+1}}}{\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}\dfrac{1}{\left(\varphi_{2k}\right)^{w+1}}}\right\}$$

The expressions for Bayes estimators and corresponding posterior risks under the assumption of Jeffreys prior can be obtained in a similar manner.

V. POSTERIOR PREDICTIVE DISTRIBUTIONS

The posterior predictive distribution based on posterior distributions under uniform and Jeffreys priors for complete and censored samples are presented in the following.
The posterior predictive distribution is defined to be:

$$p\left(y\big|\underline{x}\right) = \int_0^\infty p(\beta\big|\underline{x})f\left(y;\beta\right)d\beta$$

The posterior predictive distribution under uniform prior for complete samples is:

$$p\left(y\big|\underline{x}\right) = \frac{(n+1)(2-2y)\left(2y-y^2\right)(T)^{n+1}}{\left\{T+\ln\left(2y-y^2\right)^{-1}\right\}^{n+2}} \quad ;y>0 \quad (14)$$

where T has been defined in (5).
The posterior predictive distribution under Jeffreys prior for complete samples is:

$$p\left(y|\underline{x}\right)=\frac{n\left(2-2y\right)\left(2y-y^{2}\right)\left(T\right)^{n}}{\left\{T+\ln\left(2y-y^{2}\right)^{-1}\right\}^{n+1}}\quad;y>0 \qquad (15)$$

The posterior predictive distribution under uniform prior for singly type II censored samples is:

$$p\left(y|\underline{x}\right)=\frac{\Gamma\left(m+2\right)}{C_{1}}\sum_{k=0}^{n-m}\left(-1\right)^{k}\binom{n-m}{k}\frac{\left(2-2y\right)\left(2y-y^{2}\right)}{\left\{\varphi_{1k}+\ln\left(2y-y^{2}\right)^{-1}\right\}^{m+2}}$$
$$;y>0$$
$$(16)$$

where φ_{1k} and C_{1} have been defined in (8) and (9) respectively.

The posterior predictive distribution under Jeffreys prior for singly type II censored samples is:

$$p\left(y|\underline{x}\right)=\frac{\Gamma\left(m+1\right)}{C_{2}}\sum_{k=0}^{n-m}\left(-1\right)^{k}\binom{n-m}{k}\frac{\left(2-2y\right)\left(2y-y^{2}\right)}{\left\{\varphi_{1k}+\ln\left(2y-y^{2}\right)^{-1}\right\}^{m+1}}$$
$$;y>0$$
$$(17)$$

where, C_{2} has been defined in (10).

The posterior predictive distribution under uniform prior for doubly type II censored samples is:

$$p\left(y|\underline{x}\right)=\frac{\Gamma\left(w+2\right)}{C_{3}}\sum_{k=0}^{n-j}\left(-1\right)^{k}\binom{n-j}{k}\frac{\left(2-2y\right)\left(2y-y^{2}\right)}{\left\{\varphi_{2k}+\ln\left(2y-y^{2}\right)^{-1}\right\}^{w+2}}$$
$$;y>0$$
$$(18)$$

where φ_{2k} and C_{3} have been defined in (11) and (12) respectively.

The posterior predictive distribution under uniform prior for doubly type II censored samples is:

$$p\left(y|\underline{x}\right)=\frac{\Gamma\left(w+1\right)}{C_{4}}\sum_{k=0}^{n-j}\left(-1\right)^{k}\binom{n-j}{k}\frac{\left(2-2y\right)\left(2y-y^{2}\right)}{\left\{\varphi_{2k}+\ln\left(2y-y^{2}\right)^{-1}\right\}^{w+1}}$$
$$;y>0$$
$$(19)$$

where, C_{4} has been defined in (13).

VI. RESULTS AND DISCUSSIONS

A simulation study, via inverse transformation method, has been carried out to assess and compare the performance of different estimators under complete and 20% censored data. The results are obtained for n = 50, 100 and 150 using parametric space $\beta\in\left(0.75,1.00,1.50\right)$. This parametric space has been considered because it can be used to assess the patterns of failure rate that decreases, increases or remains constant over time. As single sample may not represent the behavior of the estimators completely, the results are replicated 1000 times and the average of the results has been presented in the following. The magnitudes of risks corresponding to each estimator have been presented in parenthesis in the tables.

TABLE I
BAYES ESTIMATORS AND RISKS UNDER COMPLETE DATA USING B = 0.75

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	0.7870	0.7947	0.7637	0.7716	0.7793	0.7487
	(0.0486)	(0.0154)	(0.0233)	(0.0476)	(0.0154)	(0.0229)
100	0.7742	0.7780	0.7626	0.7665	0.7704	0.7550
	(0.0338)	(0.0177)	(0.0216)	(0.0335)	(0.0176)	(0.0215)
150	0.7526	0.7550	0.7451	0.7476	0.7501	0.7402
	(0.0150)	(0.0050)	(0.0074)	(0.0149)	(0.0050)	(0.0074)

TABLE II
BAYES ESTIMATORS AND RISKS UNDER SINGLY TYPE II CENSORED SAMPLE
USING B = 0.75

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	0.8012	0.8090	0.7770	0.7854	0.7933	0.7617
	(0.0504)	(0.0157)	(0.0242)	(0.0494)	(0.0157)	(0.0237)
100	0.7881	0.7920	0.7758	0.7803	0.7842	0.7681
	(0.0346)	(0.0178)	(0.0221)	(0.0344)	(0.0178)	(0.0219)
150	0.7660	0.7686	0.7581	0.7610	0.7635	0.7531
	(0.0156)	(0.0051)	(0.0077)	(0.0155)	(0.0051)	(0.0076)

TABLE III
BAYES ESTIMATORS AND RISKS UNDER DOUBLY TYPE II CENSORED SAMPLE
USING B = 0.75

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	0.7932	0.8018	0.7671	0.7916	0.8003	0.7651
	(0.0547)	(0.0172)	(0.0262)	(0.0556)	(0.0175)	(0.0266)
100	0.7943	0.7990	0.7792	0.7864	0.7912	0.7715
	(0.0377)	(0.0187)	(0.0235)	(0.0375)	(0.0187)	(0.0234)
150	0.7720	0.7754	0.7614	0.7669	0.7703	0.7564
	(0.0175)	(0.0057)	(0.0086)	(0.0174)	(0.0057)	(0.0086)

TABLE IV
BAYES ESTIMATORS AND RISKS UNDER COMPLETE DATA USING B = 1.00

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	1.1071	1.1179	1.0452	1.0976	1.1085	1.0356
	(0.1335)	(0.0216)	(0.0618)	(0.1338)	(0.0218)	(0.0619)
100	1.0978	1.1033	1.0660	1.0978	1.1033	1.0660
	(0.0663)	(0.0108)	(0.0319)	(0.0663)	(0.0108)	(0.0319)
150	1.0924	1.0961	1.0711	1.0924	1.0961	1.0711
	(0.0439)	(0.0072)	(0.0214)	(0.0439)	(0.0072)	(0.0214)

TABLE V
BAYES ESTIMATORS AND RISKS UNDER SINGLY TYPE II CENSORED SAMPLE
USING B = 1.00

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	1.1077	1.1185	1.0458	1.0860	1.0968	1.0252
	(0.1338)	(0.0216)	(0.0620)	(0.1312)	(0.0216)	(0.0607)
100	1.0985	1.1039	1.0665	1.0862	1.0916	1.0553
	(0.0664)	(0.0109)	(0.0319)	(0.0650)	(0.0107)	(0.0312)
150	1.0931	1.0967	1.0716	1.0809	1.0845	1.0603
	(0.0440)	(0.0072)	(0.0214)	(0.0430)	(0.0071)	(0.0210)

TABLE VI
BAYES ESTIMATORS AND RISKS UNDER DOUBLY TYPE II CENSORED SAMPLE
USING B = 1.00

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	1.1051	1.1170	1.0373	1.0810	1.0930	1.0147
	(0.1475)	(0.0239)	(0.0678)	(0.1443)	(0.0239)	(0.0663)
100	1.0965	1.1031	1.0584	1.0699	1.0763	1.0340
	(0.0734)	(0.0120)	(0.0350)	(0.0700)	(0.0118)	(0.0335)
150	1.0911	1.0959	1.0635	1.0646	1.0693	1.0389
	(0.0486)	(0.0080)	(0.0235)	(0.0464)	(0.0078)	(0.0224)

TABLE VII
BAYES ESTIMATORS AND RISKS UNDER COMPLETE DATA USING B = 1.50

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	1.6483	1.6644	1.5090	1.6160	1.6321	1.4794
	(0.3134)	(0.0322)	(0.1393)	(0.3072)	(0.0322)	(0.1366)
100	1.6105	1.6185	1.5394	1.6105	1.6185	1.5394
	(0.2511)	(0.0259)	(0.0811)	(0.2509)	(0.0257)	(0.0810)
150	1.5686	1.5738	1.5226	1.5686	1.5738	1.5226
	(0.0958)	(0.0104)	(0.0461)	(0.0958)	(0.0104)	(0.0461)

TABLE VII
BAYES ESTIMATORS AND RISKS UNDER SINGLY TYPE II CENSORED SAMPLE
USING B = 1.50

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	1.6479	1.6640	1.5086	1.6156	1.6317	1.4789
	(0.3136)	(0.0322)	(0.1394)	(0.3074)	(0.0322)	(0.1366)
100	1.6101	1.6181	1.5389	1.6101	1.6180	1.5389
	(0.2512)	(0.0259)	(0.0811)	(0.2508)	(0.0259)	(0.0801)
150	1.5682	1.5734	1.5221	1.5682	1.5734	1.5220
	(0.0959)	(0.0104)	(0.0461)	(0.0959)	(0.0104)	(0.0461)

TABLE IX
BAYES ESTIMATORS AND RISKS UNDER DOUBLY TYPE II CENSORED SAMPLE
USING B = 1.50

n	Uniform			Jeffreys		
	SELF	PLF	LLF	SELF	PLF	LLF
50	1.6365	1.6542	1.4859	1.6009	1.6186	1.4536
	(0.3425)	(0.0354)	(0.1506)	(0.3351)	(0.0354)	(0.1473)
100	1.5986	1.6082	1.5153	1.5950	1.6047	1.5120
	(0.2652)	(0.0275)	(0.0869)	(0.2649)	(0.0270)	(0.0868)
150	1.5570	1.5638	1.4988	1.5536	1.5604	1.4955
	(0.1049)	(0.0114)	(0.0498)	(0.1046)	(0.0114)	(0.0497)

The simulation study indicates the convergence of estimated value of the parameter towards the true value by increasing the sample size. However, the convergence is negatively affected by the increasing values of the parameter. The failure rate (shape parameter) is over estimated for estimates under squared error loss function (SELF) and precautionary loss function (PLF); while in case of LINEX loss function (LLF) the failure rate is under estimated. The magnitude of the posterior risks is inversely proportional to sample size and is directly proportional to true parametric value and censoring rate. In comparison of priors it is found that the magnitude of risks associated with estimates under Jeffreys prior is lesser than those under uniform prior for each case. Similarly the estimates under the assumption of precautionary loss function (PLF) are having the minimum risks among all other loss functions. This property holds under each prior and for complete as well as for the censored samples. However, the performance of each loss function is decreasing with increase in true parametric value. As expected the variances of estimates under complete samples are smaller than those under censored samples. This simply happens because some of the information is lost under censored samples. It also interesting to note that the performance of the estimates under singly type II censored samples seems better than those under doubly type II censored samples. The proposed estimators can work efficiently under the moderate sample size.

A. Real Life Example

In order to discuss the practical applicability of the results obtained under above sections, the following real life data presented by Butler [19] have been used for analysis.

TABLE X
REAL LIFE SURVIVAL TIMES DATA

Survival times for 30 light bulbs in months					
0.0200	0.1860	0.2620	0.6950	0.9140	1.9950
0.0250	0.1960	0.3140	0.7400	0.9920	2.2550
0.0590	0.1970	0.5110	0.7600	1.1810	2.5090
0.0620	0.2050	0.6040	0.8460	1.1940	2.9100
0.1450	0.2100	0.6780	0.8600	1.3090	5.5430

The Bayes estimates under complete, singly and doubly type II censored samples based on uniform and Jeffreys prior using SELF, PLF and LLF have been presented in the following tables. The magnitudes of posterior risks have been given in the parenthesis.

TABLE XI
BAYES ESTIMATORS AND ASSOCIATED RISKS UNDER UNIFORM AND JEFFREYS PRIORS

Prior Distribution	Complete		
	SELF	PLF	LLF
Uniform	1.1245	1.1425	1.1046
	(0.0408)	(0.0360)	(0.0199)
Jeffreys	1.0883	1.1063	1.0690
	(0.0395)	(0.0360)	(0.0193)
	Singly Type II		
	SELF	PLF	LLF
Uniform	1.2612	1.2842	1.2274
	(0.0515)	(0.0460)	(0.0338)
Jeffreys	1.2205	1.2434	1.1878
	(0.0499)	(0.0458)	(0.0327)
	Doubly Type II		
	SELF	PLF	LLF
Uniform	1.2783	1.3016	1.2441
	(0.0526)	(0.0466)	(0.0342)
Jeffreys	1.2371	1.2603	1.2039
	(0.0509)	(0.0464)	(0.0331)

Each of the Bayes estimate indicates that the failure rate increases over time as the estimated value is greater than one in each case. This further suggests that the light bulbs are more likely to fail as time goes on. The real life data replicated the patterns of the estimators observed under simulation study. The censored sample leads to the larger estimated values of the parameter. In addition, the amounts of posterior risks associated with the estimates are bigger in case of censored sample. The performance of the estimators under PLF, Jeffreys prior and singly type II censored samples seems better than their counterparts.

VII. CONCLUSIONS AND RECOMMENDATIONS

The above analysis suggests that the exercise of SELF and PLF lead to over estimation of the failure rate (shape parameter), while it is under estimated in case of LLF. The use of PLF based on Jeffreys prior and singly type II censored samples can be preferred to obtain the Bayes estimate of the failure rate under Top Leone distribution. The real life example further strengthened these beliefs. The proposed estimators can work efficiently (even) under moderate sample sizes. The results are useful for analysts dealing with time to failure data in different fields. The study can be extended by including informative priors applying more loss functions and involving some other censoring techniques. As the failure of the items can happen due to more than one reason, the finite mixture of the components of the Topp Leone distribution can also be considered in future research.

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