

# On a New Nonlinear Sum-difference Inequality with Application

Kelong Zheng and Shouming Zhong

**Abstract**—A new nonlinear sum-difference inequality in two variables which generalize some existing results and can be used as handy tools in the analysis of certain partial difference equation is discussed. An example to show boundedness of solutions of a difference value problem is also given.

**Keywords**—Sum-Difference inequality, Nonlinear, Boundedness.

## I. INTRODUCTION

**T**HE sum-difference inequalities which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of finite difference equations. During the past few years, motivated and inspired by their applications in various branches of difference equations, many such inequalities have been established. For example, see [1], [2], [7], [9] and the references given therein. In particular, Ma and Cheung [7] considered

$$\begin{aligned} \varphi(u(m, n)) &\leq a(m, n) + b(m, n) \sum_{s=m+1}^{\beta} c(s, n) \varphi(u(s, n)) \\ &+ d(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \varphi'(u(s, t)) [f(s, t) w(u(s, t)) \\ &+ e(s, t)], \end{aligned} \quad (1)$$

where  $a(m, n)$ ,  $b(m, n)$ ,  $c(m, n)$  and  $d(m, n)$  have certain monotonicity.

In this paper, we investigate more general nonlinear sum-difference inequality as follows,

$$\begin{aligned} u(m, n) &\leq a(m, n) + b(m, n) \sum_{s=m+1}^{\beta} c(s, n) u(s, n) \\ &+ \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f_k(m, n, s, t) w_k(u(s, t)), \end{aligned} \quad (2)$$

where we do not require the monotonicity of  $a(m, n)$ ,  $b(m, n)$  and  $f_k(m, n, s, t)$ . Our result can generalize the results in [7] and be used more effectively to study the boundedness and uniqueness of the solutions of certain partial difference equation. Moreover, an example is presented to show the usefulness of our result.

Kelong Zheng and Shouming Zhong are with School of Mathematics, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, P. R. China. Kelong Zheng is also with School of Science, Southwest University of Science and Technology, Mianyang, Sichuan 621010, P. R. China. Email: zhengkelong@swust.edu.cn

## II. MAIN RESULT

Throughout this paper, we denote  $\mathbf{N}_0 = \{0, 1, 2, \dots\}$ . As usual, the empty sum is taken to be 0. Assume that

(A<sub>1</sub>)  $a(m, n)$ ,  $b(m, n)$  and  $c(m, n)$  are bounded and nonnegative for  $m, n \in \mathbf{N}_0$  and  $a(\infty, \infty) > 0$ ;

(A<sub>2</sub>)  $f_k(m, n, s, t)$  ( $k = 1, 2$ ) is nonnegative for  $m, n, s, t \in \mathbf{N}_0$ ;

(A<sub>3</sub>)  $w_k(u)$  ( $k = 1, 2$ ) is continuous and nondecreasing function on  $[0, \infty)$  and positive on  $(0, \infty)$ . They satisfy the relationship  $w_1 \propto w_2$ , that is,  $\frac{w_2}{w_1}$  is nondecreasing on  $(0, \infty)$ .

and define that

(D<sub>1</sub>)  $\tilde{a}(m, n) = \sup_{m \leq \tau, n \leq \eta, \tau, \eta \in \mathbf{N}_0} a(\tau, \eta)$  and  $\tilde{b}(m, n) = \sup_{m \leq \tau, n \leq \eta, \tau, \eta \in \mathbf{N}_0} b(\tau, \eta)$ ;

(D<sub>2</sub>)  $\tilde{f}_k(m, n, s, t) = \sup_{m \leq \tau, n \leq \eta, \tau, \eta \in \mathbf{N}_0} f_k(\tau, \eta, s, t)$ ;

(D<sub>3</sub>)  $\Delta_1 u(m, n) = u(m+1, n) - u(m, n)$  and  $\Delta_3 g(m, n, s, t) = g(m, n, s+1, t) - g(m, n, s, t)$ ;

(D<sub>4</sub>) For  $u \geq u_k > 0$ ,  $W_k(u) = \int_{u_k}^u \frac{dz}{w_k(z)}$  ( $k = 1, 2$ ). From assumption (A<sub>3</sub>),  $W_k$  is strictly increasing so its inverse  $W_k^{-1}$  is well defined, continuous and increasing in its corresponding domain.

**Theorem 1:** Under assumptions (A<sub>1</sub>)-(A<sub>3</sub>), if  $\Delta_1 p(m, n) \tilde{a}(m, n)$  is nonpositive for  $m, n$  and nondecreasing in  $n$  and  $u(m, n)$  is a nonnegative function for  $m, n \in \mathbf{N}_0$  satisfying (2), then

$$\begin{aligned} u(m, n) &\leq W_2^{-1} [W_2(\tilde{a}(\infty, n)) \\ &+ p(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_2(m, n, s, t) \\ &- \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m, n, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s+1, \infty)))}], \end{aligned} \quad (3)$$

for  $m \geq M_1, n \geq N_1$ , where

$$\begin{aligned} p(m, n) &= 1 + \tilde{b}(m, n) \sum_{s=m+1}^{\beta} c(s, n) \prod_{i=n+1}^{s-1} (1 + c(i, n) \tilde{b}(i, n)), \\ g_1(m, n, s, t) &= p(s, t) \tilde{a}(s, t), \\ g_2(m, n, s, t) &= W_1(g_1(m, n, \infty, t)) \\ &+ p(m, n) \sum_{\tau=m+1}^{\infty} \sum_{\eta=n+1}^{\infty} \tilde{f}_1(m, n, \tau, \eta) \\ &- \sum_{\tau=s}^{\infty} \frac{\Delta_3 g_1(m, n, \tau, t)}{w_1(g_1(\infty, \infty, \tau+1, \infty))}, \end{aligned} \quad (4)$$

$\phi_2(u) = \frac{w_2(u)}{w_1(u)}$ ,  $\phi_1(u) = w_1(u)$ ,  $W_0 = I$  (Identity), and  $M_1$  and  $N_1$  are positive integers satisfying

$$W_k(\tilde{a}(\infty, N_1)) + p(M_1, N_1) \sum_{s=M_1+1}^{\infty} \sum_{t=N_1+1}^{\infty} \tilde{f}_k(M_1, N_1, s, t) + \sum_{s=M_1}^{\infty} \frac{\Delta_3 g_k(M_1, N_1, s, N_1)}{\phi_k(W_{k-1}^{-1}(g_k(\infty, \infty, s+1, \infty)))} \leq \int_{u_k}^{\infty} \frac{dz}{w_k(z)}, \quad k = 1, 2. \quad (5)$$

**Remark 1:** If  $w_k$  ( $k = 1, 2$ ) satisfies  $\int_{u_k}^{\infty} \frac{dz}{w_k(z)} = \infty$ , then we may choose  $N_1 = 0$  and  $M_1 = 0$ . As explained in [9], different choices of  $u_k$  in  $W_k$  do not affect our result.

**Remark 2:** If  $w_k(u)$  ( $k = 1, 2$ ) is continuous positive function on  $(0, \infty)$  but the sequence of  $\{w_k(u)\}$  does not satisfy  $w_1 \propto w_2$ , we can take a technique of monotoneization of the sequence of functions  $w_k(u)$  that calculated by

$$\tilde{w}_1(u) := \max_{\theta \in [0, u]} w_1(\theta), \quad \tilde{w}_2(u) := \max_{\theta \in [0, u]} \left\{ \frac{w_2(\theta)}{\tilde{w}_1(\theta)} \right\} \tilde{w}_1(u).$$

Clearly,  $\tilde{w}_k(u) \geq w_k(u)$  ( $k = 1, 2$ ) and the new function sequence  $\{\tilde{w}_k(u)\}$  satisfies the assumption  $(A_3)$ .

**Remark 3:** We take  $\varphi(u(m, n)) = v(m, n)$ , then  $u(m, n) = \varphi^{-1}(v(m, n))$ . Furthermore, if let  $f_1(m, n, s, t) = d(m, n) f(s, t)$ ,  $f_2(m, n, s, t) = d(m, n) e(s, t)$ ,  $\tilde{w}_1(v) = \varphi'(u)w(u) = \varphi'(\varphi^{-1}(v))w(\varphi^{-1}(v))$  and  $\tilde{w}_2(v) = \varphi'(u)\varphi(\varphi^{-1}(v))$ , (1) can be reduced to form of (2). So the result in [7] is the special case of our result.

Before we prove the theorem, the following lemma should be introduced [8].

**Lemma 1:** (Pachpatte[8]) Let  $u_n, a_n, b_n \geq 0$ ,  $q_n \geq 0$  be sequences defined for  $n \in \mathbf{N}_0$  satisfying the inequality

$$u_n \leq a_n + q_n \sum_{s=n+1}^m b_s u_s, \quad n \in \mathbf{N}_0. \quad (6)$$

Then

$$u_n \leq a_n + q_n \sum_{s=n+1}^m b_s a_s \prod_{i=n+1}^{s-1} (1 + b_i q_i), \quad (7)$$

for  $0 \leq n \leq m, m \in \mathbf{N}_0$ .

### III. PROOF OF THEOREM

**Proof.** From the definitions  $(D_1)$  and  $D_2$ ,  $\tilde{a}(m, n)$ ,  $\tilde{b}(m, n)$  and  $\tilde{f}_k(m, n, s, t)$  are nonnegative and nonincreasing in  $m$  and  $n$ . Meanwhile,  $\tilde{a}(m, n) \geq a(m, n)$ ,  $\tilde{b}(m, n) \geq b(m, n)$  and  $\tilde{f}_k(m, n, s, t) \geq f_k(m, n, s, t)$ .  $a(\infty, \infty) > 0$  in  $(A_1)$  implies that  $\tilde{a}(m, n) > 0$  for all  $m \geq M_1, n \geq N_1$ . From (2), we have

$$u(m, n) \leq \tilde{a}(m, n) + \tilde{b}(m, n) \sum_{s=m+1}^{\beta} c(s, n)u(s, n) + \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_k(m, n, s, t)w_k(u(s, t)). \quad (8)$$

$$Y(m, n) = \tilde{a}(m, n) + \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_k(m, n, s, t)w_k(u(s, t)). \quad (9)$$

Then

$$u(m, n) \leq Y(m, n) + \tilde{b}(m, n) \sum_{s=m+1}^{\beta} c(s, n)u(s, n). \quad (10)$$

Keeping  $n$  fixed, from Lemma 1, we obtain

$$u(m, n) \leq Y(m, n) + \tilde{b}(m, n) \sum_{s=m+1}^{\beta} c(s, n)Y(s, n) \prod_{i=n+1}^{s-1} (1 + c(i, n)\tilde{b}(i, n)) \leq Y(m, n) [1 + \tilde{b}(m, n) \sum_{s=m+1}^{\beta} c(s, n) \prod_{i=n+1}^{s-1} (1 + c(i, n)\tilde{b}(i, n))]. \quad (11)$$

where we apply the fact that  $Y(m, n)$  is nonincreasing in  $m$ . For convenience, let

$$p(m, n) = 1 + \tilde{b}(m, n) \sum_{s=m+1}^{\beta} c(s, n) \prod_{i=n+1}^{s-1} (1 + c(i, n)\tilde{b}(i, n)), \quad (12)$$

where it can be seen easily that  $p(m, n)$  is also increasing in  $m$  and  $n$ . So (11) can be rewritten as

$$u(m, n) \leq Y(m, n)p(m, n) = p(m, n)\tilde{a}(m, n) + p(m, n) \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_k(m, n, s, t)w_k(u(s, t)) = p(m, n)\tilde{a}(m, n) + \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_k(m, n, s, t)w_k(u(s, t)), \quad (13)$$

where

$$F_k(m, n, s, t) = p(m, n)\tilde{f}_k(m, n, s, t). \quad (14)$$

Obviously,  $p(m, n)\tilde{a}(m, n)$  and  $F_k(m, n, s, t)$  ( $k = 1, 2$ ) are nonincreasing in  $m$  and  $n$ .

Take any arbitrary positive integers  $M$  and  $N$  with  $M \geq M_1, N \geq N_1$ . From (13) we have an auxiliary inequality as follows

$$u(m, n) \leq g_1(M, N, m, n) + \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_k(M, N, s, t)w_k(u(s, t)) \quad (15)$$

for  $m \geq M, n \geq N$ , where  $g_1(M, N, m, n) = p(m, n)\tilde{a}(m, n)$ . Let

$$z(m, n) = \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_k(M, N, s, t)w_k(u(s, t)). \quad (16)$$

Then  $z(\infty, n) = 0$ .  $z(m, n)$  is nonnegative and nonincreasing in  $m$  and  $n$  and satisfies

$$u(m, n) \leq g_1(M, N, m, n) + z(m, n). \tag{17}$$

Moreover,

$$\begin{aligned} & -\Delta_1 z(m, n) \\ &= \sum_{k=1}^2 \sum_{t=n+1}^{\infty} F_k(M, N, m+1, t) w_k(u(m+1, t)) \\ &\leq \sum_{k=1}^2 \sum_{t=n+1}^{\infty} F_k(M, N, m+1, t) \\ &\quad w_k(g_1(M, N, m+1, t) + z(m+1, t)). \end{aligned} \tag{18}$$

Since  $\Delta_3 g_1(M, N, m, n) = \Delta_1 p(m, n) \tilde{a}(m, n)$  is nonpositive and nondecreasing in  $m$  and  $n$ , so

$$\begin{aligned} & \frac{\Delta_1 z(m, n) + \Delta_3 g_1(M, N, m, n)}{w_1(z(m+1, n) + g_1(M, N, m+1, n))} \\ &\leq \sum_{t=n+1}^{\infty} F_1(M, N, m+1, t) \\ &\quad + \sum_{t=n+1}^{\infty} [F_2(M, N, m+1, t) \\ &\quad \phi_2(g_1(M, N, m+1, t) + z(m+1, t))] \\ &\quad \frac{\Delta_3 g_1(M, N, m, n)}{w_1(z(m+1, n) + g_1(M, N, m+1, n))} \\ &\leq \sum_{t=n+1}^{\infty} F_1(M, N, m+1, t) \\ &\quad + \sum_{t=n+1}^{\infty} [F_2(M, N, m+1, t) \\ &\quad \phi_2(g_1(M, N, m+1, t) + z(m+1, t))] \\ &\quad \frac{\Delta_3 g_1(M, N, m, n)}{w_1(z(m+1, n) + g_1(\infty, \infty, m+1, \infty))} \end{aligned} \tag{19}$$

where  $\phi_2(u) = \frac{w_2(u)}{w_1(u)}$ . Noticing that

$$\begin{aligned} & -\int_{z(m,n)+g_1(M,N,m,n)}^{z(m+1,n)+g_1(M,N,m+1,n)} \frac{d\tau}{w_1(\tau)} \\ &\leq -\int_{z(m,n)+g_1(M,N,m,n)}^{z(m+1,n)+g_1(M,N,m+1,n)} \frac{d\tau}{w_1(z(m+1, n) + g_1(M, N, m+1, n))} \\ &= -\frac{\Delta_1 z(m, n) + \Delta_3 g_1(M, N, m, n)}{w_1(z(m+1, n) + g_1(M, N, m+1, n))} \\ &\leq \sum_{t=n+1}^{\infty} F_1(M, N, m+1, t) \\ &\quad + \sum_{t=n+1}^{\infty} [F_2(M, N, m+1, t) \\ &\quad \phi_2(g_1(M, N, m+1, t) + z(m+1, t))] \\ &\quad \frac{\Delta_3 g_1(M, N, m, n)}{w_1(g_1(\infty, \infty, m+1, \infty))}, \end{aligned}$$

$$\begin{aligned} & -\int_{z(m,n)+g_1(M,N,m,n)}^{z(\infty,n)+g_1(M,N,\infty,n)} \frac{d\tau}{w_1(\tau)} \\ &= -\sum_{s=m}^{\infty} \int_{z(s,n)+g_1(M,N,s,n)}^{z(s+1,n)+g_1(M,N,s+1,n)} \frac{d\tau}{w_1(\tau)} \\ &\leq \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_1(M, N, s, t) \\ &\quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [F_2(M, N, s, t) \\ &\quad \phi_2(g_1(M, N, s, t) + z(s, t))] \\ &\quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_1(M, N, s, n)}{w_1(g_1(\infty, \infty, s+1, \infty))}. \end{aligned}$$

The definition of  $W_1$  and  $z(\infty, n) = 0$  show

$$\begin{aligned} & W_1(z(m, n) + g_1(M, N, m, n)) \\ &\leq W_1(g_1(M, N, \infty, n)) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_1(M, N, s, t) \\ &\quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} [F_2(M, N, s, t) \\ &\quad \phi_2(g_1(M, N, s, t) + z(s, t))] \\ &\quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_1(M, N, s, n)}{w_1(g_1(\infty, \infty, s+1, \infty))} \end{aligned} \tag{20}$$

or equivalently

$$\begin{aligned} & \Xi(m, n) \leq g_2(M, N, m, n) \\ &\quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(M, N, s, t) \phi_2(W_1^{-1}(\Xi(s, t))) \end{aligned} \tag{21}$$

where

$$\begin{aligned} & \Xi(m, n) = W_1(z(m, n) + g_1(M, N, m, n)), \\ & g_2(M, N, m, n) = W_1(g_1(M, N, \infty, n)) \\ &\quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_1(M, N, s, t) \\ &\quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_1(M, N, s, n)}{w_1(g_1(\infty, \infty, s+1, \infty))}. \end{aligned} \tag{22}$$

Letting

$$\begin{aligned} & \tilde{z}(m, n) \\ &= \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(M, N, s, t) \phi_2(W_1^{-1}(\Xi(s, t))) \end{aligned} \tag{23}$$

from (21), we get

$$\Xi(m, n) \leq g_2(M, N, m, n) + \tilde{z}(m, n). \tag{24}$$

On one hand,

$$\begin{aligned} & - \int_{\tilde{z}(m,n)+g_2(M,N,m,n)}^{\tilde{z}(m+1,n)+g_2(M,N,m+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))} \\ & \leq - \int_{\tilde{z}(m,n)+g_2(M,N,m,n)}^{\tilde{z}(m+1,n)+g_2(M,N,m+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))} \\ & \frac{\phi_2(W_1^{-1}(\tilde{z}(m+1,n) + g_2(M, N, m + 1, n)))}{\phi_2(W_1^{-1}(\tilde{z}(m+1,n) + g_2(M, N, m + 1, n)))} \\ & = - \frac{\Delta_1 \tilde{z}(m, n) + \Delta_3 g_2(M, N, m, n)}{\phi_2(W_1^{-1}(\tilde{z}(m+1,n) + g_2(M, N, m + 1, n)))} \\ & \leq \sum_{t=n+1}^{\infty} F_2(M, N, m + 1, t) \\ & \quad - \frac{\Delta_3 g_2(M, N, m, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, m + 1, \infty)))}, \end{aligned}$$

on the other hand,

$$\begin{aligned} & \int_{h(m,n)}^{h(m+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))} \\ & = \int_{h(m,n)}^{h(m+1,n)} \frac{w_1(W_1^{-1}(\tau))d\tau}{w_2(W_1^{-1}(\tau))} \\ & = \int_{W_1^{-1}(h(m,n))}^{W_1^{-1}(h(m+1,n))} \frac{d\tau}{w_2(\tau)} \\ & = W_2(W_1^{-1}(h(m+1,n))) - W_2(W_1^{-1}(h(m,n))). \end{aligned}$$

Therefore,

$$\begin{aligned} & - \int_{\tilde{z}(m,n)+g_2(M,N,m,n)}^{\tilde{z}(\infty,n)+g_2(M,N,\infty,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))} \\ & \leq - \sum_{s=m}^{\infty} \int_{\tilde{z}(s,n)+g_2(M,N,s,n)}^{\tilde{z}(s+1,n)+g_2(M,N,s+1,n)} \frac{d\tau}{\phi_2(W_1^{-1}(\tau))} \\ & \leq \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(M, N, s, t) \\ & \quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M, N, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))}, \quad (25) \end{aligned}$$

that is,

$$\begin{aligned} & W_2(W_1^{-1}(\tilde{z}(m, n) + g_2(M, N, m, n))) \\ & \leq W_2(W_1^{-1}(\tilde{z}(\infty, n) + g_2(M, N, \infty, n))) \\ & \quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(M, N, s, t) \\ & \quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M, N, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))} \\ & = W_2(g_1(M, N, \infty, n)) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(M, N, s, t) \\ & \quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M, N, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))}, \quad (26) \end{aligned}$$

where we apply the fact that  $\tilde{z}(\infty, n) = 0$  and  $g_2(M, N, \infty, n) = W_1(g_1(M, N, \infty, n))$ . From (17) and (22),

we get

$$\begin{aligned} u(m, n) & \leq g_1(M, N, m, n) + z(m, n) = W_1^{-1}(\Xi(m, n)) \\ & \leq W_1^{-1}(g_2(M, N, m, n) + \tilde{z}(m, n)) \\ & \leq W_2^{-1}[W_2(g_1(M, N, \infty, n)) \\ & \quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(M, N, s, t) \\ & \quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(M, N, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))}] \quad (27) \end{aligned}$$

for  $m \geq M$  and  $n \geq N$ . Replacing  $m$  and  $n$  by  $M$  and  $N$  in (27) respectively, we have

$$\begin{aligned} u(M, N) & \leq W_2^{-1}[W_2(g_1(M, N, \infty, n)) \\ & \quad + \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} F_2(M, N, s, t) \\ & \quad - \sum_{s=M}^{\infty} \frac{\Delta_3 g_2(M, N, s, N)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))}]. \end{aligned}$$

Since  $M$  and  $N$  are arbitrary, we replace  $M$  and  $N$  by  $m$  and  $n$  and get

$$\begin{aligned} u(m, n) & \leq W_2^{-1}[W_2(g_1(m, n, \infty, n)) \\ & \quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} F_2(m, n, s, t) \\ & \quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m, n, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))}] \quad (28) \end{aligned}$$

for  $m \geq M_1$  and  $n \geq N_1$ . From (12) and (14),  $g_1(M, N, \infty, n) = p(\infty, n)\tilde{a}(\infty, n) = \tilde{a}(\infty, n)$ . Hence, it follows from (28) that

$$\begin{aligned} u(m, n) & \leq W_2^{-1}[W_2(\tilde{a}(\infty, n)) \\ & \quad + p(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_2(m, n, s, t) \\ & \quad - \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m, n, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s + 1, \infty)))}], \end{aligned}$$

This completes the proof.  $\square$

#### IV. APPLICATION TO A DIFFERENCE EQUATION

Consider a nonlinear difference equation

$$\begin{aligned} v(m, n) & = \alpha(m, n) + \sum_{s=m+1}^{\beta} G(s, n, v(s, n)) \\ & \quad + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} H(m, n, s, t, v(s, t)) \quad (29) \end{aligned}$$

for  $m, n \in \mathbf{N}_0$ , where  $v(m, n)$  is an unknown function for  $m, n \in \mathbf{N}_0$ . Suppose that

$$\begin{aligned} |G(m, n, v(m, n))| & \leq c(m, n)|v(m, n)|, \\ |H(m, n, s, t, v(s, t))| & \leq f_1(m, n, s, t)w_1(|v(s, t)|) \\ & \quad + f_2(m, n, s, t)w_2(|v(s, t)|), \end{aligned}$$

where  $w_1 \propto w_2$  holds. If  $v(m, n)$  is a solution of (29), then

$$u(m, n) \leq a(m, n) + \sum_{s=m+1}^{\beta} c(s, n)u(s, n) + \sum_{k=1}^2 \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} f_k(m, n, s, t)w_k(u(s, t)), \quad (30)$$

where  $u(m, n) = |v(m, n)|$  and  $a(m, n) = |\alpha(m, n)|$ . Applying Theorem 2.1, we have

$$u(m, n) \leq W_2^{-1}[W_2(\tilde{a}(\infty, n)) + p(m, n) \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \tilde{f}_2(m, n, s, t) + \sum_{s=m}^{\infty} \frac{\Delta_3 g_2(m, n, s, n)}{\phi_2(W_1^{-1}(g_2(\infty, \infty, s+1, \infty)))}] \quad (31)$$

where  $p(m, n) = 1 + \sum_{s=m+1}^{\beta} c(s, n) \prod_{i=n+1}^{s-1} (1 + c(i, n))$  and other functions are defined in Theorem 2.1. Clearly, (31) implies the boundedness of solutions of equation (29).

#### REFERENCES

- [1] W. S. Cheung, Q. H. Ma, J. Pečarić, Some discrete nonlinear inequalities and applications to difference equations, *Acta Mathematica Scientia* 2008, 28B(2) 417-430.
- [2] W. S. Cheung, J. Ren, Discrete non-linear inequalities and applications to boundary value problems, *J. Math. Anal. Appl.* 319 (2006) 708-724.
- [3] S. Deng, Nonlinear discrete inequalities with two variables and their applications, *Appl. Math. Comput.* (2010), doi:10.1016/j.amc.2010.07.022.
- [4] F. C. Jiang, F. W. Meng, Explicit bounds on some new nonlinear integral inequalities with delay, *J. Comput. Appl. Math.* 205(2007), 479-486.
- [5] Y. H. Kim, On some new integral inequalities for functions in one and two variables, *Acta Math. Sinica*, 2(2)(2005), 423-434.
- [6] O. Lipovan, A retarded integral inequality and its applications, *J. Math. Anal. Appl.* 285(2003), 436-443.
- [7] Q. H. Ma, W. S. Cheung, Some new nonlinear difference inequalities and their applications, *Journal of Computational and Applied Mathematics* 202(2007), 339-351.
- [8] B. G. Pachpatte, *Inequalities for Finite Difference Equations*, Marcel Dekker, New York, 2002.
- [9] Y. Wu, X. Li, S. Deng, Nonlinear delay discrete inequalities and their applications to Volterra type difference equations, *Advances in Difference Equations*, Volume 2010, Article ID 795145, 14 pages.
- [10] K. Zheng, Some retarded nonlinear integral inequalities in two variables and applications, *JIPAM. J. Inequal. Pure Appl. Math.* 9(2)(2008), Article 57, 11 pp.