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On 6-Figures in Finite Klingenberg Planes of parameters (p^{2k-1}, p)

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***Abstract*—In this paper, we deal with finite projective Klingenberg plane $\mathbf{M}(\mathcal{A})$ coordinatized by local ring $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q\varepsilon$ (where prime power $q = p^k$, $\varepsilon \notin \mathbf{Z}_q$ and $\varepsilon^2 = 0$). So, we get some combinatorical results on 6-figures. For example, we show that there exist $p-1$ 6-figure classes in $\mathbf{M}(\mathcal{A})$.**

***Keywords*—finite Klingenberg plane, 6-figure, ratio of 6-figure, cross-ratio.**

I. Introduction

Projective Klingenberg and Hjelmslev planes (more briefly: PK-planes and PH-planes, resp.) are generalizations of ordinary projective planes. These structures were introduced by Klingenberg in [15], [16]. As for finite PK-planes, these structures introduced by Drake and Lenz in [12] have been investigated in detail by Bacon in [4].

In our previous papers [1], [9], [10] we have studied a certain class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (briefly, MK) planes coordinatized by an local alternative ring $\mathcal{A} := \mathbf{A}(\ ) = \mathbf{A} + \mathbf{A}$ of dual numbers (an alternative ring $\mathbf{A}$, $\notin \mathbf{A}$ and $^2 = 0$) introduced by Blunck in [7]. So, we have obtained many results related to 6-figures. For more detailed information about 6-figures and their properties, the reader is referred to the papers of [8] in the case of Desarguesian planes and [11] in the case of Moufang planes.

In the present paper we are interested in finite PK-plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring $\mathbf{Z}_q$ (where q is a prime power) instead of $\mathbf{A}$. So, we will get some combinatorical result related to 6-figures.

II. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$(points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are two non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are two non-neighbour lines, then there is a unique point $g \cap h$ on both g and h.

(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

PK-plane $\mathbf{M}$ is called a *projective Hjelmslev plane* (PH-plane) If $\mathbf{M}$ furthermore provides the following axioms:

(PH1) If P, Q are two neighbour points, then there are at least two lines through P and Q.

(PH2) If g, h are two neighbour lines, then there are at least two points on both g and h.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the details see [3]).

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line h such that $P \in h$ for some line $h \sim g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbf{M}$.

Now we give the definition of an n-gon, which is meaningful when $n \geq 3$: An n-tuple of pairwise non-neighbour points is called an (ordered) *n-gon* if no three of its elements are on neighbour lines [9].

An *alternative ring* (*field*) $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws $a(ab) = a^2b, (ba)a = ba^2$, $\forall a, b \in \mathbf{R}$. An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We summarize some basic concepts about the coordinatization of MK-planes from [5].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned}
\mathbf{P} &= \{(x, y, 1) : x, y \in \mathbf{R}\} \\
&\quad \cup \{(1, y, z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\
&\quad \cup \{(w, 1, z) : w, z \in \mathbf{I}\} \\
\mathbf{L} &= \{[m, 1, p] : m, p \in \mathbf{R}\} \\
&\quad \cup \{[1, n, p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\
&\quad \cup \{[q, n, 1] : q, n \in \mathbf{I}\} \\
[m, 1, p] &= \{(x, xm + p, 1) : x \in \mathbf{R}\} \\
&\quad \cup \{(1, zp + m, z) : z \in \mathbf{I}\} \\
[1, n, p] &= \{(yn + p, y, 1) : y \in \mathbf{R}\} \\
&\quad \cup \{(zp + n, 1, z) : z \in \mathbf{I}\} \\
[q, n, 1] &= \{(1, y, yn + q) : y \in \mathbf{R}\} \\
&\quad \cup \{(w, 1, wq + n) : w \in \mathbf{I}\}
\end{aligned}$$

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and

$$\begin{aligned} P &= (x_1,x_2,x_3) \sim (y_1,y_2,y_3) = Q \\ &\Leftrightarrow \quad x_i - y_i \in \mathbf{I} \ \ (i=1,2,3)), \forall P,Q \in \mathbf{P} \\ g &= [x_1,x_2,x_3] \sim [y_1,y_2,y_3] = h \\ &\Leftrightarrow \quad x_i - y_i \in \mathbf{I} \ \ (i=1,2,3)), \forall g,h \in \mathbf{L}. \end{aligned}$$

Baker *et al.* [3] use $(O=(0,0,1), U=(1,0,0), V=(0,1,0), E=(1,1,1))$ as a coordinatization 4-gon. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [3] and [5]. Now it is time to give the following theorem from [3].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\ \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\ ) = \mathbf{A} + \mathbf{A}$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\ )(b_1 + b_2\ ) = a_1 b_1 + (a_1 b_2 + a_2 b_1)\ ,$$

where $a_i, b_i \in \mathbf{A}$, $i = 1,2$. Then $\mathcal{A}$ is an alternative ring with ideal $\mathbf{I} = \mathbf{A}$ of non-units. For more detailed information about MK-planes $\mathbf{M}(\mathcal{A})$ coordinatized by an local alternative ring $\mathcal{A} := \mathbf{A}(\ ) = \mathbf{A} + \mathbf{A}$, see the papers of [7], [9], [1].

Theorem 2.2: If $\mathbf{R}$ is a (not necessarily commutative) local ring then $\mathbf{M}(\mathbf{R})$ is a PK-plane (cf. [13, Theorem 4.1]).

Drake and Lenz [12, Proposition 2.5] observed that the following corollary is true for PK-planes. This corollary is a generalization of results which are given for PH-planes by Kleinfeld [14, Theorem 1] and Lüneburg [17, Satz 2.11].

Corollary 2.3: Let $\mathbf{M}(\mathbf{R})$ be PK-plane. Then there are natural numbers t and r which are called the parametres of $\mathbf{M}(\mathbf{R})$ and they are uniquely determined by incidence structure of a finite PK-plane [12, Proposition 2.7], with

1) *every point (line) has t^2 neighbours;*
2) *given a point P and a line l with $P \in l$, there exist exactly t points on l which are neighbours to P and exactly t lines through P which are neighbours to l;*
3) Let r be order of the projective plane $\mathbf{M}^*$. If $t \neq 1$ we have $r \le t$ (then $\mathbf{M}$ is called *proper*; we have $t = 1$ iff $\mathbf{M}$ is an ordinary projective plane)
4) *every point (line) is incident with $t(r+1)$ lines (points);*
5) $|\mathbf{P}| = |\mathbf{L}| = t^2\left(r^2 + r + 1\right)$.

Now consider ring $\mathbf{Z}_q$ where prime power $q = p^k$. We can state the elements of $\mathbf{Z}_q$ as $\mathbf{Z}_q = U' \cup I$ where U' is the set of units of $\mathbf{Z}_q$ and I is the set of non-units of $\mathbf{Z}_q$. Here it is clear that $I = \left\{0p, 1p, 2p, \cdots, \left(p^{k-1}-1\right)p\right\}$ and so $|I| = p^{k-1}$. Let $\ \notin \mathbf{Z}_q$. Then $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q$ with componentwise addition and multiplication above is a local ring with ideal $\mathbf{I} := I + \mathbf{Z}_q$ of non-units, $|\mathbf{I}| = \left(p^{k-1}\right)p^k$. Note that the set of units of $\mathcal{A}$ is $\mathbf{U} := U' + \mathbf{Z}_q$ and $|\mathbf{U}| = \left(p^k - p^{k-1}\right)p^k = (p-1)\,p^{2k-1}$. Since $\mathcal{A}$ is a proper local ring and $\mathcal{A}/\mathbf{I} = \mathbf{Z}_p$, Ψ induces an incidence structure epimorphism from finite PK-plane $\mathbf{M}(\mathcal{A})$ onto the Desarguesian projective plane (with order p) coordinatized by the field $\mathbf{Z}_p$. So, we can give the following corollary from [2].

Corollary 2.4: For finite PK-plane $\mathbf{M}(\mathcal{A})$, the parameters t and r in Corollary 2.3 are equal to p^{2k-1} and p, respectively.

A local ring $\mathbf{R}$ is called a *Hjelmslev ring* (briefly, H-ring) if it satisfies the following two conditions:

(HR1) $\mathbf{I}$ consists of two-sided zero divisor.

(HR2) For $a, b \in \mathbf{I}$, one has $a \in b\mathbf{R}$ or $b \in a\mathbf{R}$, and also $a \in \mathbf{R}b$ or $b \in \mathbf{R}a$.

By the last definition, we can say that $\mathcal{A}$ is not, in general, a H-ring [2]. From now on we assume char $\mathbf{Z}_q \neq \mathbf{2}$ and also we restrict ourselves to finite PK-plane $\mathbf{M}(\mathcal{A}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ coordinatized by the local ring $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q$, with neighbour relation defined above.

III. 6-figures in $\mathbf{M}(\mathcal{A})$

Now we carry over some concepts related to 6-figures to the $\mathbf{M}(\mathcal{A})$, in view of the papers of [9], [1]. So, we will get some combinatoric results on 6-figures in $\mathbf{M}(\mathcal{A})$.

A *6-figure* is a sequence of six non-neighbour points $(ABC, A_1B_1C_1)$ such that (A, B, C) is 3-gon, and $A_1 \in BC, B_1 \in CA, C_1 \in AB$. The points A, B, C, A_1, B_1, C_1 are called vertices of this 6-figure. The 6-figures $(ABC, A_1B_1C_1)$ and $(DEF, D_1E_1F_1)$ are *equivalent* if there exists a collineation of $\mathbf{M}(\mathcal{A})$ which transforms A, B, C, A_1, B_1, C_1 to D, E, F, D_1, E_1, F_1 respectively.

Now we need the following theorem from [9].

Theorem 3.1: Let $\mu = (ABC, A_1B_1C_1)$ be a 6-figure in $\mathbf{M}(\mathcal{A})$. Then, there is an $m \in \mathbf{U}$ such that μ is equivalent to $(UVO, (0,1,1)(1,0,1)(1,m,0))$ where $U = (1,0,0), V = (0,1,0), O = (0,0,1)$ are elements of the coordinatization basis of $\mathbf{M}(\mathcal{A})$.

6-figures $\mu = (ABC, A_1B_1C_1)$ and $\ = (DEF, D_1E_1F_1)$ are *neighbour* if the points A, B, C, A_1, B_1, C_1 are neighbour to the points D, E, F, D_1, E_1, F_1; respectively.

Now, by the last definition and Theorem 3.1, we can give the following corollary without proof.

Corollary 3.2: 6-figures $\mu = (ABC, A_1B_1C_1)$ and $\ = (DEF, D_1E_1F_1)$ are neighbour if $m_1 \in \mathbf{U}$ corresponding to μ and $m_2 \in \mathbf{U}$ corresponding to $\ $ are neighbour.

So, we have the following

Corollary 3.3: There are $p-1$ 6-figures class in $\mathbf{M}(\mathcal{A})$. The classes are those: $m = 1$, $m = 2$, ... , $m = p-1$ where the elements in neighbour of any m are $m + \mathbf{Z}_q$, $1p + m + \mathbf{Z}_q$, $2p + m + \mathbf{Z}_q$,..., $\left(p^{k-1}-1\right)p + m + \mathbf{Z}_q$.

Proof: We can classify 6 figures in $\mathbf{M}(\mathcal{A})$ by the number of the elements of $\mathbf{U}$. But, when it is considered the neighbours of the elements in $\mathbf{U}$ this number becomes $p-1$. Hence, we obtain $p-1$ 6-figure classes in $\mathbf{M}(\mathcal{A})$. We can show the classes as $m=1$, $m=2$, ... , $m=p-1$ where the elements in neighbour of any m are $m+\mathbf{Z}_q$, $1p+m+\mathbf{Z}_q$, $2p+m+\mathbf{Z}_q$, $\cdots$, $\left(p^{k-1}-1\right)p+m+\mathbf{Z}_q$. ■

Theorem 3.4: There are totally

$$\left(\left(p^2+p+1\right)\left(p^{2k-1}\right)^2\right)\left(\left(p^2+p\right)\left(p^{2k-1}\right)^2\right)\left(p^2\left(p^{2k-1}\right)^2\right)\left((p-1)\,p^{2k-1}\right)^3$$

6-figures in $\mathbf{M}(\mathcal{A})$.

Proof: First if we calculate the total number of 6-figures in projective plane of order p, we have differently $\left(p^2+p+1\right)\left(p^2+p\right)p^2\left(p-1\right)^3$ 6-figures by depending on the choices of the points of a 6-figure. Finally if we consider the neighbour relation in $\mathbf{M}(\mathcal{A})$, that is, we consider Corollary 2.3 and 2.4 then the proof is clear. ■

Then, as a result of Corollary 3.3 and Theorem 3.4 we have immediately the following

Corollary 3.5: The number of 6-figures corresponding to an $m\in\mathbf{U}$ is

$$\left(p^2+p+1\right)\left(p^2+p\right)p^2\left(p-1\right)^2\left(p^{2k-1}\right)^8 .$$

Proof: Since there are totally

$$\left(\left(p^2+p+1\right)\left(p^{2k-1}\right)^2\right)\left(\left(p^2+p\right)\left(p^{2k-1}\right)^2\right)\left(p^2\left(p^{2k-1}\right)^2\right)\left((p-1)\,p^{2k-1}\right)^3$$

6-figures in $\mathbf{M}(\mathcal{A})$ and $|\mathbf{U}|=(p-1)\,p^{2k-1}$ then the proof is clear. ■

Now we need the following theorem, one of the main results of [2].

Theorem 3.6: The 6-figures $(ABC,A_1B_1C_1),(BCA,B_1C_1A_1),(CAB,C_1A_1B_1)$ are equivalent.

As a result of Corollary 3.5 and Theorem 3.6 we can state the following

Corollary 3.7: The number $(p^2+p+1)(p^2+p)p^2(p-1)^2\left(p^{2k-1}\right)^8$ is divided by 3.

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g:=[1,0,0]$ in $\mathbf{M}(\mathcal{A})$.

$$\begin{aligned}
&(A,B;C,D):=(a,b;c,d)\\
&=<\left((a-d)^{-1}(b-d)\right)\left((b-c)^{-1}(a-c)\right)>\\
&(S,B;C,D):=\left(s^{-1},b;c,d\right)\\
&=<\left((1-ds)^{-1}(b-d)\right)\left((b-c)^{-1}(1-cs)\right)>\\
&(A,S;C,D):=\left(a,s^{-1};c,d\right)\\
&=<\left((a-d)^{-1}(1-ds)\right)\left((1-cs)^{-1}(a-c)\right)>\\
&(A,B;S,D):=\left(a,b;s^{-1},d\right)\\
&=<\left((a-d)^{-1}(b-d)\right)\left((1-sb)^{-1}(1-sa)\right)>\\
&(A,B;C,S):=\left(a,b;c,s^{-1}\right)\\
&=<\left((1-sa)^{-1}(1-sb)\right)\left((b-c)^{-1}(a-c)\right)>,
\end{aligned}$$

where $A=(0,a,1)$, $B=(0,b,1)$, $C=(0,c,1)$, $D=(0,d,1)$, $Z=(0,1,s)$ are pairwise non-neighbour points of g and $<x>=\left\{y^{-1}xy\ \ y\in\mathcal{A}\right\}$.

The following theorem, the analogue of the theorem given in [1], states a simple way for the calculation of the cross-ratio of the points on any line l in $\mathbf{M}(\mathcal{A})$.

Theorem 3.8: According to types of lines, the cross-ratio of the points on the line l can be calculated as follows:

If A, B, C, D and S are the pairwise non-neighbour points

(a) *of the line* $l=[m,1,p]$ *where* $A=(a,am+p,1)$, $B=(b,bm+p,1)$, $C=(c,cm+p,1)$, $D=(d,dm+p,1)$ *are not near the line* UV *and* $S=(1,m+sp,s)\sim UV$,
(b) *of the line* $l=[1,n,p]$ *where* $A=(an+p,a,1)$, $B=(bn+p,b,1)$, $C=(cn+p,c,1)$, $D=(dn+p,d,1)$ *are not neighbour to* V *and* $S=(n+sp,1,s)\sim V$,
(c) *of the line* $l=[q,n,1]$ *where* $A=(1,a,q+an)$, $B=(1,b,q+bn)$, $C=(1,c,q+cn)$, $D=(1,d,q+dn)$ *are not near to* V *and* $S=(s,1,sq+n)\sim V$,

then

$$\begin{aligned}
(A,B;C,D)&=(a,b;c,d)\\
(S,B;C,D)&=\left(s^{-1},b;c,d\right)\\
(A,S;C,D)&=\left(a,s^{-1};c,d\right)\\
(A,B;S,D)&=\left(a,b;s^{-1},d\right)\\
(A,B;C,S)&=\left(a,b;c,s^{-1}\right).
\end{aligned}$$

Let $\mu=(ABC,A_1B_1C_1)$ be a 6-figure in $\mathbf{M}(\mathcal{A})$. Let $A^c=BC\cap B_1C_1$, $B^c=CA\cap C_1A_1$, $C^c=AB\cap A_1B_1$. The 6-figure $(ACB,A^cC^cB^c)$ is called the *first codescendant* of μ, written μ^c. μ is called a *first coancestor* of μ^c.

So we can give the following Lemma from [1].

Lemma 3.1: If $\mu=(ABC,A_1B_1C_1)=(UVO,(0,1,1)(1,0,1)(1,m,0))$, then

$$\begin{aligned}
(A,B;C_1,C^c)&=(B,C;A_1,A^c)\\
&=(C,A;B_1,B^c)=<-m>.
\end{aligned}$$

We are now ready to state the definition of the ratio of a 6-figure. The conjugacy class $-(A,B;C_1,C^c)$ is called *the ratio of the 6-figure* $\mu=(ABC,A_1B_1C_1)$ and denoted by $r(\mu)$, that is, $r(\mu)=<m>$.

$(ABC,A_1B_1C_1)$ is called a *Menelaus* 6-figure if A_1, B_1 and C_1 are collinear, and $(ABC,A_1B_1C_1)$ is called a *Ceva* 6-figure if AA_1, BB_1 and CC_1 are concurrent.

Now we give the following theorem from [1].

Theorem 3.9: μ is a Menelaus or Ceva 6-figure if and only if $r(\mu) = -1$ or $r(\mu) = 1$, respectively.

We immediately have

Corollary 3.10: Menelaus and Ceva 6-figures are belong to the class $m = p - 1$ and the class $m = 1$ where $p \neq 2$, respectively.

Proof: By Theorem 3.9 if μ is a Ceva 6-figure then $r(\mu) = 1 = m$ and also if μ is a Menelaus 6-figure then $r(\mu) = -1 = m$. For the proof it is enough to say that -1 is neighbour to $p - 1$. ■

From now on we call the class $m = 1$ as Ceva class and the class $m = p - 1$ as Menelaus class. Now we need following theorem from [6].

Theorem 3.11: Every cross-ratio consists only of elements of $\mathcal{A}\backslash(\{0,1\}+\mathbf{I})$. Conversely, the conjugacy class of any such element appears as a cross-ratio; Given three pairwise non-neighbour points A, B, C and an element $r \in \mathcal{A}\backslash(\{0,1\}+\mathbf{I})$, then there is a (unique if $r \in \mathbf{Z}(\ )$) point D which is not neighbour to A, B and C with $(A, B; C, D) =< r >$.

In $\mathbf{M}(\mathcal{A})$, any pairwise non-neighbour four points A, B, C, $D \in l$ are called as *harmonic* if $(A, B; C, D) =< -1 >$ and we let $h(A, B, C, D)$ represent the statement: A, B, C, D are harmonic. Let $\mu = (ABC, A_1B_1C_1)$ be a 6-figure in $\mathbf{M}(\mathcal{A})$. By the last theorem, there exist unique points $A_2 \in BC$, $B_2 \in CA$, $C_2 \in AB$ such that $h(A, B, C_1, C_2)$, $h(B, C, A_1, A_2)$, $h(C, A, B_1, B_2)$. The 6-figure $(ABC, A_2B_2C_2)$ is called the *conjugate* of μ, having symbol $-\mu$. Likewise μ is the conjugate of $-\mu$.

Let $C^d \in AB$ be the point such that C, C^d and $AA_1 \cap BB_1$ are collinear. Let $A^d \in BC$ and $B^d \in CA$ be the points such that A, A^d and $BB_1 \cap CC_1$ are collinear and B, B^d and $AA_1 \cap CC_1$ are collinear. The 6-figure $(ACB, A^dC^dB^d)$ is called the *first descendant* of μ, written μ^d. μ is called a *first ancestor* of μ^d.

Using the definitions of $-\mu$, μ^c and μ^d the following lemmas are obtained (see [1, Lemma 20] for the first Lemma and [10, Lemma 7] for the second Lemma).

Lemma 3.2: For any 6-figure μ we have

(a) $(-\mu)^d = \mu^d$

(b) $(\mu^d)^c = (\mu^c)^c = (UVO, (0, -m^{-1}, 1)(-m, 0, 1)(1, -m^2, 0))$,

where $m \in \mathbf{U}$.

Lemma 3.3: For any 6-figure μ we have

(a) $(-\mu)^c = \mu^c = (UOV, (0, -m, 1)(1, -1, 0)(-m^{-1}, 0, 1))$

(b) $(\mu^c)^d = (\mu^d)^d = (UVO, (0, m^{-1}, 1)(m, 0, 1)(1, m^2, 0))$,

where $m \in \mathbf{U}$.

By using the results of the last two Lemmas and [10, Theorem 9] we can give the following theorem which gives the relation between the ratios of the 6-figures μ^{-1}, $-\mu$, μ^d, μ^c, $\left(\mu^d\right)^d$, $(\mu^c)^c$, $\left(\mu^d\right)^c$, $(\mu^c)^d$ and μ.

Theorem 3.12: For any 6-figure μ we have

(a) $r\left(\mu^{-1}\right) = (r(\mu))^{-1} =< m^{-1} >$

(b) $r(-\mu) = -r(\mu) =< -m >$

(c) $r\left((-\mu)^d\right) = r\left(\mu^d\right) = (r(\mu))^2 =< m^2 >$

(d) $r((-\mu)^c) = r(\mu^c) = -(r(\mu))^2 =< -m^2 >$

(e) $r\left(\left(\mu^d\right)^c\right) = r((\mu^c)^c) =< -m^4 >= -(r(\mu))^4$

(f) $r\left((\mu^c)^d\right) = r\left(\left(\mu^d\right)^d\right) =< m^4 >= (r(\mu))^4$,

where $< x >^2 :=< x^2 >$ *for any* $x \in \mathbf{U}$ and $m \in \mathbf{U}$.

Proof: For the proof, it is enough to give the proof of (e) and (f). From (b) of Lemma 3.2, we know that $\left(\mu^d\right)^c = (\mu^c)^c = (UVO, U'V'O')$ where $U' = (0, -m^{-1}, 1)$, $V' = (-m, 0, 1)$, $O' = (1, -m^2, 0)$. Ratio of this 6-figure are equal to croos-ratio $-\left(U, V; (1, -m^2, 0), O^c\right)$, where

$$\begin{aligned} O^c = UV \cap U'V' &= [0,0,1] \cap \left[-m^{-2}, 1, -m^{-1}\right] \\ &= \left(1, -m^{-2}, 0\right). \end{aligned}$$

So, this cross-ratio is equal to

$$-\left((1,0,0), (0,1,0); (1,-m^2,0), \left(1,-m^{-2},0\right)\right).$$

By (c) of Theorem 3.8, this is equal to $(0, 0^{-1}; -m^2, -m^{-2}) = -m^4$. Since the proof of (f) is similar to the proof of (e) the proof is completed. ■

As a direct result of Theorem 3.9 and Theorem 3.12 we have the following result.

Corollary 3.13: a) If μ is a Menelaus 6-figure then

(i) $r(-\mu) = r\left(\mu^d\right) = r\left((\mu^c)^d\right) = r\left(\left(\mu^d\right)^d\right) =< 1 >$, that is, $-\mu, \mu^d, (\mu^c)^d$ and $\left(\mu^d\right)^d$ 6-figures are in the Ceva class.

(ii) $r\left(\mu^{-1}\right) = r(\mu^c) = r\left(\left(\mu^d\right)^c\right) = r((\mu^c)^c) =< -1 >$, that is, $\mu^{-1}, \mu^c, \left(\mu^d\right)^c$ and $(\mu^c)^c$ 6-figures are in the Menelaus class.

b) If μ is a Ceva 6-figure, then

(i) $r(-\mu) = r(\mu^c) = r\left(\left(\mu^d\right)^c\right) = r((\mu^c)^c) =< -1 >$, that is, $-\mu, \mu^c, \left(\mu^d\right)^c$ and $(\mu^c)^c$ 6-figures are in the Menelaus class.

(ii) $r\left(\mu^{-1}\right) = r\left(\mu^d\right) = r\left((\mu^c)^d\right) = r\left(\left(\mu^d\right)^d\right) =< 1 >$, that is, $\mu^{-1}, \mu^d, (\mu^c)^d$ and $\left(\mu^d\right)^d$ 6-figures are in the Ceva class.

The following theorem is the analogue of Theorem 12 given in [10] for MK-planes $\mathbf{M}(\mathcal{A})$. This theorem we give without proof, tells the relation between the solvability of the equation $x^2 = m$ (or $x^2 = -m$) in $\mathcal{A}$ where $m \in \mathbf{U}$ and the existence of the special 6-figure with ratio $< m >$ in $\mathbf{M}(\mathcal{A})$. In other words, this theorem provides a geometric property of $\mathbf{M}(\mathcal{A})$ that is equal to the condition that every element in $\mathbf{U}$ has a square root in $\mathbf{U}$.

Theorem 3.14: Let $m \in \mathbf{U}$. Then the equation $x^2 = m$ (or $x^2 = -m$) has a solution in $\mathbf{U}$ if and only if any 6-figure μ with ratio $< m >$ has ancestor (coancestor) in $\mathbf{M}(\mathcal{A})$.

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