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Numerical Treatment of Matrix Differential Models Using Matrix Splines

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***Abstract*—**This paper considers the solution of the matrix differential models using quadratic, cubic, quartic, and quintic splines. Also using the Taylor's and Picard's matrix methods; one illustrative example is included.

***Keywords*—**Matrix Splines, Cubic Splines, Quartic Splines.

I. Introduction

IN this work, the evaluation of matrix functions is frequent in the solution of differential systems. So, the system [8]

$$\dot{Y}(t) = A(t)\,Y(t), \qquad Y(0) = Y_0, \quad \Delta = [0,1] \tag{1}$$

where $A(t)$ is matrix and Y_0 is a vector arises of the parabolic equation. The matrix differential equation [3], [4]

$$\ddot{Y}(t) = A(t)\,Y(t), \qquad Y(0) = Y_0, \quad \dot{Y}(0) = Y_1, \quad \Delta = [0,1] \tag{2}$$

where $A(t)$ is matrix, Y_0 and Y_1 are vectors arises of the hyperbolic equation. The matrix differential equation [10]

$$\dot{Y}(t) = A(t)\,Y(t) + Y(t)\,B(t), \qquad Y(0) = Y_0, \qquad \Delta = [0,1] \tag{3}$$

where $A(t)$ and $B(t)$ are matrices appears in systems stability and control.

Consider the matrix differential equation in the form [5]

$$\dot{Y}(t) = A(t)\,Y(t) + B(t), \qquad Y(0) = D, \quad \Delta = [0,1] \tag{4}$$

where $Y(t) \in C^{rxq}$, $A(t), B(t), C(t)$ and $D(t)$ are matrices. Let Δ is partition as $\Delta = \{0 = t_0 < ... < t_n = 1\}$. The set of matrix splines of order m defined as [1]

$$M\_C^{rxr}(\Delta)_{m-1}^{m} = \{Q : \Delta \to C^{rxq};\ \begin{cases} Q|_{[t_{i-1},t_i]}(t) \in P_m[t], \\ i \in \{1,...,n\} \\ Q \in C^{m-1}(\Delta) \end{cases} \tag{5}$$

If $m = 2$ the matrix splines are called matrix quadratic splines, $m = 3$ called matrix cubic splines, $m = 4$ called matrix quartic splines and $m = 5$ called matrix quintic splines.

Reference [2] deals with the construction of an approximate solution of the first order matrix linear differential equations using matrix cubic splines. The present paper extended the first order linear differential equations using different matrix splines and also approximate the solution by using Picard's method and Taylor's method which are best than all matrix splines [6], [7], [9].

II. The Matrix Spline Methods

This section gives the theoretical studies for the matrix differential equation in the form (4) using the matrix quadratic splines, matrix cubic splines, matrix quartic splines and matrix quintic splines.

A. The Matrix Quadratic Splines

Consider the interval $\Delta_0 = [0, k]$, $k = \Delta t$, suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)\,t + \frac{1}{2}\alpha_0 t^2 \tag{6}$$

where $Y(0) = D$, $\dot{Y}(0) = A(0)\,Y(0) + Y(0)\,B(0) + C(0)$, but to find α_0 we suppose that $S_0(t)$ satisfies the matrix differential equation (4) at $t = k$, so

$$\dot{S}_0(k) = A(k)\,S_0(k) + B(k) \tag{7}$$

From (6) and (7) we get

$$k\,(I - \frac{k}{2}A(k))\alpha_0 = A(k)\,(Y(0) + \dot{Y}(0)\,k) + B(k) - \dot{Y}(0) \tag{8}$$

where I is the identity matrix, from (8) we get α_0 and so $S_0(t)$ as in (6). Consider $\Delta_i = [ik, (i+1)k]$, $1 \le i \le n-1$; suppose the matrix quadratic solution in the form

$$S_i(t) = S_{i-1}(ik) + \dot{S}_{i-1}(ik)\,(t - ik) + \frac{1}{2}\alpha_i\,(t - ik)^2 \tag{9}$$

As above we determine α_i from

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$$k(I-\frac{k}{2}A((i+1)k))\alpha_i = A((i+1)k)(S_{i-1}(ik) + \dot{S}_{i-1}(ik)\ k) + B((i+1)k) - \dot{S}_{i-1}(ik), \quad (10)$$

and then $S_i(t)$ are determined for all $i=1,...,n$. Note that solubility of the suggested scheme (10) is guaranteed showing that the matrix coefficient of α_i is invertible.

If $M=\max_{0\le t\le 1}\|A(t)\|$ then $\left\|I-(I-\frac{k}{2}A((i+1)k))\right\|\le 1$, so we get $k\le\frac{2}{M}$ and then (10) has a unique solution α_i.

B. The Matrix Cubic Splines

Consider the interval $\Delta_0=[0,k]$; suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\ddot{Y}(0)t^2 + \frac{1}{6}\alpha_0 t^3, \quad (11)$$

where $Y(0)=D$, $\dot{Y}(0)=A(0)\,Y(0)+B(0)$ and $\ddot{Y}(0)=A(0)\dot{Y}(0)+\dot{A}(0)\,Y(0)+\dot{B}(0)$.

To determine α_0 we suppose that $S_0(t)$ satisfies the matrix differential equation (4) at $t=k$, so

$$\frac{k^2}{2}(I-\frac{k}{3}A(k))\alpha_0 = A(k)(Y(0)+\dot{Y}(0)k + \frac{1}{2}\ddot{Y}(0)k^2) + B(k) - \dot{Y}(0) - \ddot{Y}(0)k \quad (12)$$

and $S_0(t)$ as in (11).

Consider $\Delta_i=[ik,(i+1)k]$, $1\le i\le n-1$, suppose the matrix cubic solution in the form

$$S_i(t) = S_{i-1}(ik) + \dot{S}_{i-1}(ik)(t-ik) + \frac{1}{2}\ddot{S}_{i-1}(ik)(t-ik)^2 + \frac{1}{6}\alpha_i(t-ik)^3, \quad (13)$$

As above we determine α_i from

$$\frac{k^2}{2}(I-\frac{k}{3}A((i+1)k))\alpha_i = A((i+1)k)(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k + \frac{1}{2}\ddot{S}_{i-1}(ik)k^2) + B((i+1)k) - \dot{S}_{i-1}(ik) - \ddot{S}_{i-1}(ik)k, \quad (14)$$

and then $S_i(t)$ are determined for all $i=1,...,n$. Note that solubility of the suggested scheme (14) is guaranteed showing that the matrix coefficient of α_i is invertible.

If $M=\max_{0\le t\le 1}\|A(t)\|$ then $\left\|I-(I-\frac{k}{3}A((i+1)k))\right\|\le 1$, so we get $k\le\frac{3}{M}$ and then (14) has a unique solution α_i.

C. The Matrix Quartic Splines

Consider the interval $\Delta_0=[0,k]$; suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\ddot{Y}(0)t^2 + \frac{1}{6}\dddot{Y}(0)t^3 + \frac{1}{24}\alpha_0 t^4, \quad (15)$$

for this case α_0 can be determined from

$$\frac{k^3}{6}(I-\frac{k}{4}A(k))\alpha_0 = A(k)(Y(0)+\dot{Y}(0)k + \frac{1}{2}\ddot{Y}(0)k^2 + \frac{1}{6}\dddot{Y}(0)k^3) + B(k) - \dot{Y}(0) - \ddot{Y}(0)k - \dddot{Y}(0)k^2, \quad (16)$$

and $S_0(t)$ as in (15). Consider $\Delta_i=[ik,(i+1)k]$, $1\le i\le n-1$; suppose the matrix quartic solution in the form

$$S_i(t) = S_{i-1}(ik) + \dot{S}_{i-1}(ik)(t-ik) + \frac{1}{2}\ddot{S}_{i-1}(ik)(t-ik)^2 + \frac{1}{6}\dddot{S}_{i-1}(ik)(t-ik)^3 + \frac{1}{24}\alpha_i(t-ik)^4, \quad (17)$$

as above we determine α_i from

$$\frac{k^3}{6}(I-\frac{k}{4}A((i+1)k))\alpha_i = A((i+1)k)(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k + \frac{1}{2}\ddot{S}_{i-1}(ik)k^2 + \frac{1}{6}\dddot{S}_{i-1}(ik)k^3) + B((i+1)k) - \dot{S}_{i-1}(ik) - \ddot{S}_{i-1}(ik)k - \frac{1}{2}\dddot{S}_{i-1}(ik)k^2, \quad (18)$$

and then $S_i(t)$ are determined for all $i=1,...,n$. Note that solubility of the suggested scheme (18) is guaranteed showing that the matrix coefficient of α_i is invertible.

If $M=\max_{0\le t\le 1}\|A(t)\|$ then $\left\|I-(I-\frac{k}{4}A((i+1)k))\right\|\le 1$, so we get $k\le\frac{4}{M}$ and then (18) has a unique solution α_i.

D. The Matrix Quintic Splines

Consider the interval $\Delta_0=[0,k]$; suppose the solution in the form

$$S_0(t) = Y(0) + \dot{Y}(0)t + \frac{1}{2}\ddot{Y}(0)t^2 + \frac{1}{6}\dddot{Y}(0)t^3 + \frac{1}{24}\ddddot{Y}(0)t^4 + \frac{1}{120}\alpha_0 t^5, \quad (19)$$

for this case α_0 can be determined from

$$\frac{k^4}{24}(I-\frac{k}{5}A(k))\alpha_0 = A(k)(Y(0)+\dot{Y}(0)k+\frac{1}{2}\ddot{Y}(0)k^2 + \frac{1}{6}\dddot{Y}(0)k^3+\frac{1}{24}\ddddot{Y}(0)k^4)+B(k)-\dot{Y}(0)-\ddot{Y}(0)k - \dddot{Y}(0)k^2-\ddddot{Y}(0)k^3, \quad (20)$$

and $S_0(t)$ as in (19).

Consider $\Delta_i = [ik,(i+1)k]$, $1 \le i \le n-1$; suppose the matrix quintic solution in the form

$$S_i(t) = S_{i-1}(ik)+\dot{S}_{i-1}(ik)(t-ik)+\frac{1}{2}\ddot{S}_{i-1}(ik)(t-ik)^2 + \frac{1}{6}\dddot{S}_{i-1}(ik)(t-ik)^3+\frac{1}{24}\ddddot{S}_{i-1}(ik)(t-ik)^4 + \frac{1}{120}\alpha_i(t-ik)^5, \quad (21)$$

as above we determine α_i from

$$\frac{k^4}{24}(I-\frac{k}{5}A((i+1)k))\alpha_i = A((i+1)k)(S_{i-1}(ik) + \dot{S}_{i-1}(ik)k+\frac{1}{2}\ddot{S}_{i-1}(ik)k^2+\frac{1}{6}\dddot{S}_{i-1}(ik)k^3 + \frac{1}{24}\ddddot{S}_{i-1}(ik)k^4)+B((i+1)k)-\dot{S}_{i-1}(ik) - \ddot{S}_{i-1}(ik)k-\frac{1}{2}\dddot{S}_{i-1}(ik)k^2-\frac{1}{6}\ddddot{S}_{i-1}(ik)k^3 \quad (22)$$

and then $S_i(t)$ are determined for all $i = 1,...,n$.

Note that solubility of the suggested scheme (22) is guaranteed showing that the matrix coefficient of α_i is invertible.

If $M = \max_{0\le t\le 1}\|A(t)\|$ then $\left\|I-(I-\frac{k}{5}A((i+1)k))\right\| \le 1$, so we get $k \le \frac{5}{M}$ and then (22) has a unique solution α_i.

III. The Matrix Picard's Method

In this section we see the Picard's method for the matrix differential equation in the form (4) then the first approximation is

$$Y_{i+1}(t) = Y_0(t)+\int_0^t (A(t)Y_i(t)+B(t))\,dt \quad (23)$$

where $Y_0(t) = D$, $i = 0,1,2,\ldots$. As in ordinary differential equation we get a sequence $\{Y_i(t)\}_0^\infty$ which is convergent to the exact solution.

IV. The Matrix Taylor's Method

Suppose the approximate solution for the matrix differential equation (4) takes the form

$$Y_n(t) = Y(0)+\dot{Y}(0)t+\frac{1}{2}\ddot{Y}(0)t^2+\ldots+\frac{1}{n!}Y^{(n)}(0)t^n \quad (24)$$

where $Y(0),\dot{Y}(0),...,Y^{(n)}(0)$ all can be determined from the matrix differential equation (4).

V. Illustration of the Analysis

In this section, distinct matrix differential equations will be tested by using the proposed methods.

Example: We first consider the matrix differential equation in the form

$$\begin{pmatrix}\dot{y_1}(t)\\ y_2(t)\end{pmatrix} = \frac{1}{t^3-t-1}\begin{pmatrix}2t^2-1 & t^2-2t-1\\ -t-1 & t^3+t^2-t-1\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}, \quad 0\le t\le 1, \quad \begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}\in C^2, \quad (25)$$

this matrix differential equation has the exact solution $\begin{pmatrix}e^t\\ te^t\end{pmatrix}$, in the following table we see the matrix splines methods.

TABLE I
The Matrix Splines Methods

Quintic	Quartic	Cubic	Quadratic	$[t_i, t_{i-1}]$
1.7956E-9	1.14628E-7	6.33769E-6	3.06573E-4	[0,0.1]
5.7101E-8	8.81776E-7	6.33769E-6	7.11688E-4	[0.1,0.2]
5.46782E-7	2.2721E-6	8.32925E-6	12.397E-4	[0.2,0.3]
1.7956E-9	1.14628E-7	6.33769E-6	3.06573E-4	[0,0.1]

In the following table we see the approximation solution using quadratic matrix method in some intervals.

TABLE II
Quadratic Matrix Method

$[t_i, t_{i-1}]$	Quadratic
[0,0.1]	$\begin{pmatrix}1+t+0.527889t^2\\ t+1.0804t^2\end{pmatrix}$
[0.1,0.2]	$\begin{pmatrix}1.00056+0.988792t+0.583928t^2\\ 0.00172163+0.965567t+1.25257t^2\end{pmatrix}$
[0.2,0.3]	$\begin{pmatrix}1.00304+0.964035t+0.645822t^2\\ 0.009578+0.887004t+1.44897t^2\end{pmatrix}$

VI. Conclusion

In this work we have found the solution of the matrix differential models using quadratic, cubic, quartic, and quintic splines. Also using the Taylor's and Picard's matrix methods we reached these important numerical methods of solution through the application in the examples.

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