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Number of Parametrization of Discrete-Time Systems without Unit-Delay Element: Single-Input Single-Output Case

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Abstract**—**In this paper, we consider the parametrization of the discrete-time systems without the unit-delay element within the framework of the factorization approach. In the parametrization, we investigate the number of required parameters. We consider single-input single-output systems in this paper. By the investigation, we find, on the discrete-time systems without the unit-delay element, three cases that are (1) there exist plants which require only one parameter and (2) two parameters, and (3) the number of parameters is at most three.

Keywords**—**Linear systems, parametrization, Coprime Factorization, number of parameters.

I. Introduction

In this paper, we consider the parametrization of the parametrization of the discrete-time systems without the unit-delay element within the framework of the factorization approach.

The factorization approach to control systems has the advantage that it includes, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems, etc. [1]-[8]. In the factorization approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. This approach leads to conceptually simple and computationally tractable solutions to many important and interesting problems [9], [10]. A transfer matrix of this approach is considered as the ratio of two stable causal transfer matrices.

For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices. On the other hand, Anantharam showed in [11] a model that has plants which are stabilizable but do not admit coprime factorization. Mori and Abe also showed such a model[5].

II. Preliminaries

The stabilization problem considered in this paper follows that of [4], and [5], who consider the feedback system Σ [9, Ch.5, Fig. 5.1] as in Fig. 1. For further details the reader is referred to [9], [3]-[5].

We consider that the set of stable causal transfer functions is an integral domain, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d \neq 0\}$. This $\mathcal{F}$ is considered as the set of all possible transfer functions. Matrices over $\mathcal{F}$ are transfer matrices. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$. Define the subsets $\mathcal{P}$ and $\mathcal{P}_{\rm s}$ of $\mathcal{F}$ as follows: $\mathcal{P} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{A},\ b \in \mathcal{A}\backslash\mathcal{Z}\}$, $\mathcal{P}_{\rm s} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{Z},\ b \in \mathcal{A}\backslash\mathcal{Z}\}$. Then, every transfer function in $\mathcal{P}$ ($\mathcal{P}_{\rm s}$) is called *causal* (*strictly causal*). Analogously, if every entry of a transfer matrix is in $\mathcal{P}$ ($\mathcal{P}_{\rm s}$), the transfer matrix is called *causal* (*strictly causal*).

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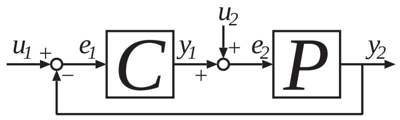

Fig. 1 Feedback system Σ

Throughout the paper, the plant we consider has single-input and single-output, and its transfer function, which is also called a *plant* itself simply, is denoted by p and belongs to $\mathcal{P}$. We can always represent p in the form of a fraction $p = nd^{-1}$, where $n \in \mathcal{A}$ and $d \in \mathcal{A}$ with nonzero d.

For $p \in \mathcal{P}$ and c, a matrix $H(p, c) \in \mathcal{F}^{2\times 2}$ is defined as

$$H(p,c) := \begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix} \tag{1}$$

provided that $1 + pc$ is a nonzero of $\mathcal{A}$. This $H(p, c)$ is the transfer matrix from $[\ u_1\ u_2\ ]^t$ to $[\ e_1\ e_2\ ]^t$ of the feedback system Σ. If $1 + pc$ is a nonzero of $\mathcal{A}$ and $H(p, c) \in \mathcal{A}^{2\times 2}$, then we say that the plant p is *stabilizable*, p is *stabilized* by c, and c is a *stabilizing controller* of p. In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [5].

It is known that $W(p, c)$ defined below is over $\mathcal{A}$ if and only if $H(p, c)$ is over $\mathcal{A}$:

$$W(p,c) := \begin{bmatrix} c(1+pc)^{-1} & -pc(1+cp)^{-1} \\ pc(1+pc)^{-1} & p(1+cp)^{-1} \end{bmatrix}. \tag{2}$$

This $W(p, c)$ is the transfer matrix from $[\ u_1\ \ u_2\ ]^t$ to $[\ y_1\ y_2\ ]^t$.

We employ the symbols used in [12], [4] in general.

III. Parametrization without Coprime Factorizability

Here we briefly review the parameterization method of [12], which does not require coprime factorization. Let $\mathcal{H}$ be the set of $H(P,C)$'s with all stabilizing controllers C. Let H_0 be $H(P,C_0) \in \mathcal{A}^{(m+n)\times(m+n)}$, where C_0 is a fixed stabilizing controller of P with m inputs and n outputs. Let $\Omega(Q)$ be a matrix defined as

$$\begin{aligned}\Omega(Q) &= \left(H_0 - \begin{bmatrix} I_n & O_{n\times m} \\ O_{m\times n} & O_{m\times m}\end{bmatrix}\right) Q \\ &\quad \times \left(H_0 - \begin{bmatrix} O_{n\times n} & O_{n\times m} \\ O_{m\times n} & I_m\end{bmatrix}\right) \\ &\quad + H_0\end{aligned} \tag{3}$$

with a stable causal and square matrix Q in $\mathcal{A}^{(m+n)\times(m+n)}$. Then we have the identity

$$\mathcal{H} = \{\Omega(Q) \,|\, Q \text{ is stable causal and } \Omega(Q) \text{ is nonsingular}\}$$

[12, Theorems 4.2 and 4.3]. Then, from (1), any stabilizing controller has the form $\Omega_{21}\Omega_{22}^{-1}$, where Ω_{21} and Ω_{22} are the (2,1)- and (2,2)-blocks of $\Omega(Q)$, provided that Ω_{22} is nonsingular.

The parameterization above is given by a parameter matrix Q without coprime factorizability of the plant. Thus, this method can be applied to models in which some stabilizable plants do not admit doubly coprime factorizations, such as in [11], and models in which we do not yet know whether or not there always exists a doubly coprime factorization for a stabilizable plant, such as multidimensional systems with structural stability [13], [14].

IV. Discrete-Time Systems without Unit-Delay Element

The author [5] considered the case $\mathcal{A} = \mathbb{R}[z^2, z^3]$, where $\mathbb{Z}$ denotes the set of integers. This ring is an integral domain but not a unique factorization domain. In fact, $z^6 \in \mathcal{A}$ has two factorizations, $z^2 \cdot z^2 \cdot z^2$ and $z^3 \cdot z^3$. He showed that the plant

$$P := \begin{bmatrix} (1-z^3)/(1-z^2) \\ (1-8z^3)/(1-4z^2)\end{bmatrix} \in \mathcal{P}^{2\times 1} \tag{4}$$

does not admit a coprime factorization but is stabilizable and

$$C = \frac{-1}{\alpha_{I_1}\lambda_{I_1}^2 \langle (1+z)(1+2z)(1-3z)\rangle} \begin{bmatrix} \alpha_{I_1} n_1 & \alpha_{I_2} n_2 \end{bmatrix}.$$

is a stabilizing controller, where

$$\begin{aligned}
n_1 &= \langle (1+z)(1+2z)(1-3z)\rangle \\
&\quad \times (1 + \alpha_{I_1}\lambda_{I_1}\langle (1+z)(1+2z)(1-3z)\rangle), \\
n_2 &= \langle (1+z)(1+2z)(1-3z+z^2)\rangle, \\
\alpha_{I_1} &= \tfrac{-4233-23646z^2-39836z^3-201780z^4-113016z^5+75344z^6}{5852}, \\
\alpha_{I_2} &= \tfrac{10085+18418z^2+121140z^3+131852z^4+113016z^5}{5852}, \\
\lambda_{I_1} &= \alpha_{I_1}\langle (1+2z)(1-3z)(1+z+z^2)\rangle, \\
\lambda_{I_2} &= \alpha_{I_2}\langle (1+z)(1+2z+4z^2)(1-3z+z^2)\rangle.
\end{aligned}$$

V. Necessary Parameters for SISO Systems

The following result is from [15].

Theorem 1: ([15, Theorem 1]) Let us consider a stabilizable single-input single-output (SISO) plant. We do not assume the coprime factorizability of the plant. Then the number of parameters for the parameterization of the stabilizing controllers of the plant is up to three.

VI. One-Parameter Case

As in Section IV, we consider $\mathcal{A} = \mathbb{R}[z^2, z^3]$.

First, as a simple case, let us suppose that a plant admits a coprime factorization over $\mathcal{A}$, we can employ Youla-Kučera-parametrization [17]-[19]. In this case, the number of parameter is always one.

Let $p = 1/(z^2+1)$. Then a stabilizing controller is

$$c_0 = \frac{-z^4+2}{z^2-1}.$$

Then we have a coprime factorization $ny + dx = 1$, where

$$\begin{aligned} &n = 1, \quad d = z^2+1, \\ &y = -z^4+2, \quad x = z^2-1.\end{aligned}$$

Then all stabilizing controllers are give as

$$\frac{-z^4+2+r(z^2+1)}{z^2-1-r} \tag{5}$$

with $r \in \mathcal{A}$ and $z^2-1-r \neq 0$.

In the case of Anantharam's model [11], [16], [15], there exist an SISO plant such that [15]

(1) the plant does not admit a coprime factorization but is stabilizable,
(2) the number of parameters for all stabilizing controllers is one.

Let us consider the parametrization based on Section III. Let Q, $H(p,c_0)$, and $\Omega(Q)$ be as in Section VI. Then ω_{11}, ω_{12}, ω_{21}, and ω_{22} are

$$\begin{aligned}
\omega_{11} &= -((1+z^2)((-2+2z^2+z^4-z^6)q_{11} \\
&\quad +(-2+z^4)^2 q_{12} \\
&\quad -(-1+z^2)(q_{21}-z^2q_{21}+(-2+z^4)q_{22}))), && (6)\\
\omega_{12} &= (-2+2z^2+z^4-z^6)q_{11}+(-2+z^4)^2q_{12} \\
&\quad -(-1+z^2)(q_{21}-z^2q_{21}+(-2+z^4)q_{22}), && (7)\\
\omega_{21} &= (1+z^2)^2((-2+2z^2+z^4-z^6)q_{11} \\
&\quad +(-2+z^4)^2q_{12} \\
&\quad -(-1+z^2)(q_{21}-z^2q_{21}+(-2+z^4)q_{22})), && (8)\\
\omega_{22} &= -((1+z^2)((-2+2z^2+z^4-z^6)q_{11} \\
&\quad +(-2+z^4)^2q_{12} \\
&\quad -(-1+z^2)(q_{21}-z^2q_{21}+(-2+z^4)q_{22}))). && (9)
\end{aligned}$$

We now see that $\omega_{11} = \omega_{22}$. Let α_{ijkl} (i, j, k, $l = 1, 2$ except for $k = l = 2$) be the coefficient of q_{ij} of ω_{kl}. By using α_{ijkl}'s, we make a matrix A such as

$$A = \begin{bmatrix} \alpha_{1111} & \alpha_{1112} & \alpha_{1121} \\ \alpha_{1211} & \alpha_{1212} & \alpha_{1221} \\ \alpha_{2111} & \alpha_{2112} & \alpha_{2121} \\ \alpha_{2211} & \alpha_{2212} & \alpha_{2221} \end{bmatrix}. \tag{10}$$

Using q_{ij}'s, (10) is rewritten as

$$\begin{aligned}
\alpha_{1111} &= 2 - 3z^4 + z^8, \\
\alpha_{1112} &= -2 + 2z^2 + z^4 - z^6, \\
\alpha_{1121} &= -2 - 2z^2 + 3z^4 + 3z^6 - z^8 - z^{10}, \\
\alpha_{1211} &= -4 - 4z^2 + 4z^4 + 4z^6 - z^8 - z^{10}, \\
\alpha_{1212} &= 4 - 4z^4 + z^8, \\
\alpha_{1221} &= 4 + 8z^2 - 8z^6 - 3z^8 + 2z^{10} + z^{12}, \\
\alpha_{2111} &= -1 + z^2 + z^4 - z^6, \\
\alpha_{2112} &= 1 - 2z^2 + z^4, \\
\alpha_{2121} &= 1 - 2z^4 + z^8, \\
\alpha_{2211} &= 2 - 3z^4 + z^8, \\
\alpha_{2212} &= -2 + 2z^2 + z^4 - z^6, \\
\alpha_{2221} &= -2 - 2z^2 + 3z^4 + 3z^6 - z^8 - z^{10}.
\end{aligned}$$

We now consider the following matrix $T = (t_{ij})$:

$$\begin{aligned}
t_{11} &= 1 - 4z^2, \\
t_{12} &= -z^2, \\
t_{13} &= 1 - 4z^2 - 2z^4 + z^6, \\
t_{14} &= 0, \\
t_{21} &= 5 - 4z^2, \\
t_{22} &= 1 - z^2, \\
t_{23} &= 6 - 2z^2 - 3z^4 + z^6, \\
t_{24} &= 0, \\
t_{31} &= 1 - 6z^2 + 9z^4 - 4z^6, \\
t_{32} &= -z^2 + 2z^4 - z^6, \\
t_{33} &= 2 - 6z^2 + 7z^4 + z^6 - 4z^8 + z^{10}, \\
t_{34} &= 0, \\
t_{41} &= -1, \\
t_{42} &= 0, \\
t_{43} &= 0, \\
t_{44} &= 1.
\end{aligned}$$

The determinant of T is 1. Then TA becomes

$$TA = \begin{bmatrix} 1+z^2 & -1 & -(1+z^2)^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

The matrix T can be decomposed as

$$T = T_{10}T_9T_8T_7T_6T_5T_4T_3T_2T_1, \tag{12}$$

where

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

$$T_2 = \begin{bmatrix} 1 & 0 & z^2 & 0 \\ 0 & 1 & -z^4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2z^2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

$$T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{16}$$

$$T_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ z^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

$$T_6 = \begin{bmatrix} 1 & -z^2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -z^4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

(19)

$$T_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

$$T_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z^4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

$$T_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3z^2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

$$T_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Thus, from the matrices T_1 to T_{10}, we have the sequence of replace of parameters:

(i) $q_{11} \to (q_{11} - q_{22})$,
(ii) $q_{21} \to (q_{21} + q_{11}z^2 - q_{12}z^4)$,
(iii) $q_{21} \to (q_{21} + q_{11} - q_{12}z^2)$,
(iv) $q_{21} \to (q_{21} + q_{12})$,
(v) $q_{11} \to (q_{11} + 4q_{12} + q_{21}z^2)$,
(vi) $q_{12} \to (q_{12} - q_{11}z^2 - q_{21}z^4)$,
(vii) $q_{11} \to (q_{11} + q_{12})$,
(viii) $q_{11} \to (q_{11} + q_{21}z^4)$,
(ix) $q_{11} \to (q_{11} - 3q_{21}z^2)$,
(x) $q_{11} \to (q_{11} + q_{21})$.

By applying the eight replacements above to $\Omega(Q)$ of (3), we obtain

$$\Omega(Q) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned}
\phi_{11} &= (1+z^2) * (-1 + z^2 + q_{11}), \\
\phi_{12} &= 1 - z^2 - q_{11}, \\
\phi_{21} &= -((1+z^2)(-2 + z^4 + q_{11} + z^2 q_{11})), \\
\phi_{22} &= -1 + z^4 + q_{11} + z^2 q_{11},
\end{aligned}$$

which has only one parameter. Based on (24), we can obtain a stabilizing controller of p as

$$\begin{aligned}\phi_{21}\phi_{11}^{-1} &= \frac{-(-2+z^4+q_{11}+z^2q_{11})}{-1+z^2+q_{11}}\\ &= \frac{2-z^4-(z^2+1)q_{11}}{-1+z^2+q_{11}}.\end{aligned}$$

By replacing q_{11} by $-q'_{11}$, we have

$$\phi_{21}\phi_{11}^{-1} == \frac{-z^4+2+q'_{11}(z^2+1)}{z^2-1-q'_{11}},$$

which is equivalent to (5).

VII. Two-Parameter Case

In this section, we show that there exists SISO plant whose stabilizing controller is parametrized by two parameters. From now, we show that such a plant is $p=(z^2-1)/(z^3-1)$ and a its stabilizing controller $c_0=-(z^3+1)/(z^2-1)$.

Let Q, $H(p,c_0)$, and $\Omega(Q)$ be as in Section VI. Then ω_{11}, ω_{12}, ω_{21}, and ω_{22} are

$$\begin{aligned}\omega_{11} &= ((-1+z^6)q_{11}-(1+z^2+z^3+z^4\\ &\quad +z^5+z^7)q_{12}-q_{21}+z^2q_{21}+z^3q_{21}-z^5q_{21}\\ &\quad -q_{22}+z^6q_{22})/4 \qquad (25)\\ \omega_{12} &= (-((-1+z^2-z^3+z^5)q_{11})+(1+z^3)^2q_{12}\\ &\quad -(-1+z)(1+z)^2(q_{21}-zq_{21}+q_{22}\\ &\quad -zq_{22}+z^2q_{22}))/4 \qquad (26)\\ \omega_{21} &= ((1+z+z^2)^2(-((-1+2z-2z^2+z^3)q_{11})\\ &\quad +(1-z+z^2)^2q_{12}-(-1+z)(q_{21}-zq_{21}\\ &\quad +q_{22}-zq_{22}+z^2q_{22})))/4 \qquad (27)\\ \omega_{22} &= ((-1+z^6)q_{11}-(1+z^2+z^3+z^4+z^5+z^7)q_{12}\\ &\quad -q_{21}+z^2q_{21}+z^3q_{21}-z^5q_{21}\\ &\quad -q_{22}+z^6q_{22})/4 \qquad (28)\end{aligned}$$

We now see that $\omega_{11}=\omega_{22}$. Analogously to Section VI, we have matrices A $(=(\alpha....))$, T, and TA as follows:

$$\begin{aligned}\alpha_{1111} &= (-1+z^6)/4,\\ \alpha_{1112} &= (1-z^2+z^3-z^5)/4,\\ \alpha_{1121} &= (1+z^2-z^3+z^4-z^5-z^7)/4,\\ \alpha_{1211} &= (-1-z^2-z^3-z^4-z^5-z^7)/4,\\ \alpha_{1212} &= (1+2z^3+z^6)/4,\\ \alpha_{1221} &= (1+2z^2+3z^4+2z^6+z^8)/4,\\ \alpha_{2111} &= (-1+z^2+z^3-z^5)/4,\\ \alpha_{2112} &= (1-2z^2+z^4)/4,\\ \alpha_{2121} &= (1-2z^3+z^6)/4,\\ \alpha_{2211} &= (-1+z^6)/4,\\ \alpha_{2212} &= (1-z^2+z^3-z^5)/4,\\ \alpha_{2221} &= (1+z^2-z^3+z^4-z^5-z^7)/4,\end{aligned}$$

$$T=\begin{bmatrix}8 & 4z^2 & -12-8z^2-4z^4 & 0\\ -8 & 4-4z^2 & 4+4z^2+4z^4 & 0\\ 0 & 0 & 4 & 0\\ -4 & 0 & 0 & 4\end{bmatrix},$$

$$TA=\begin{bmatrix}1-2z^2-3z^3-2z^4 & 0 & -1+z^2+z^3-z^5 & 0\\ (1+z)^2(-1+2z) & 0 & (-1+z^2)^2 & 0\\ (-1+2z)(1+z+z^2)^2 & 0 & (-1+z^3)^2 & 0\end{bmatrix}^t.$$

The determinant of T is 511, which is a unit of $\mathcal{A}$. The matrix T can be decomposed as

$$T=T_7T_6T_5T_4T_3T_2T_1, \qquad (29)$$

where

$$T_1 = \begin{bmatrix}4&0&0&0\\0&4&0&0\\0&0&4&0\\4&0&0&4\end{bmatrix}, \qquad (30)$$

$$T_2 = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\-1&0&0&1\end{bmatrix}, \qquad (31)$$

$$T_3 = \begin{bmatrix}1&0&0&0\\0&1&-z^2&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \qquad (32)$$

$$T_4 = \begin{bmatrix}1&0&0&0\\0&1&-2&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \qquad (33)$$

$$T_5 = \begin{bmatrix}1&z^2/2&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \qquad (34)$$

$$T_6 = \begin{bmatrix}2&0&-3&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}, \qquad (35)$$

$$T_7 = \begin{bmatrix}1&0&0&0\\-1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}. \qquad (36)$$

Thus, from the matrices T_1 to T_8, we have the sequence of replace of parameters:

(i) $q_{11}\to 4q_{11}$, $q_{12}\to 4q_{12}$, $q_{21}\to 4q_{21}$, $q_{22}\to 4q_{22}$,
(ii) $q_{11}\to(q_{11}-q_{22})$,
(iii) $q_{21}\to(q_{21}-q_{12}z^2)$,
(iv) $q_{21}\to(q_{21}-2q_{12})$,
(v) $q_{12}\to(q_{12}+1/2q_{11}z^2)$,
(vi) $q_{21}\to(q_{21}-3q_{11})$, $q_{11}\to 2q_{11}$,
(vii) $q_{11}\to(q_{11}-q_{12})$,
(viii) $q_{11}\to(q_{11}-q_{12})$.

By applying the eight replacements above to $\Omega(Q)$ of (3), we obtain

$$\Omega(Q)=\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}, \qquad (37)$$

where

$$\begin{aligned}\phi_{11} &= 1/2-z^3/2+q_{11}-2z^2q_{11}-3z^3q_{11}-2z^4q_{11}\\ &\quad -q_{21}+z^2q_{21}+z^3q_{21}-z^5q_{21},\\ \phi_{12} &= -1/2+z^2/2-q_{11}+3z^2q_{11}+2z^3q_{11}+q_{21}\\ &\quad -2z^2q_{21}+z^4q_{21},\\ \phi_{21} &= 1/2+z^2/2+z^4/2-q_{11}+z^2q_{11}+4z^3q_{11}\\ &\quad +3z^4q_{11}+2z^5q_{11}+q_{21}-2z^3q_{21}+z^6q_{21},\\ \phi_{22} &= 1/2-z^3/2+q_{11}-2z^2q_{11}-3z^3q_{11}-2z^4q_{11}\\ &\quad -q_{21}+z^2q_{21}+z^3q_{21}-z^5q_{21},\end{aligned}$$

which has two parameters, q_{11} and q_{21}. Based on (37), we can obtain a stabilizing controller of p as $\phi_{21}\phi_{11}^{-1}$. Thus, the parametrization of all stabilizing controllers of p is achieved by *two* parameters (q_{11} and q_{21}).

VIII. Conclusion and Future Works

In this paper, we have considered SISO the discrete-time systems without the unit-delay element. In the model, we have shown that the number of parameterization is depend on plants. We have shown concrete plant examples which have the parameterization of stabilizing controllers of one or two parameters.

We will investigate the relationship a plant and the number of parameters of stabilizing controllers of the plant.

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