International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:6, No:9, 2012

Note on the necessity of the patch test

Rado Flajs and Miran Saje

Abstract—We present a simple nonconforming approximation of the linear two–point boundary value problem which violates patch test requirements. Nevertheless the solutions, obtained from these type of approximations, converge to the exact solution.

Keywords—generalized patch test, Irons' patch test, nonconforming finite element, convergence

I. Introduction

The patch test was introduced in mid-1960s by Irons and co–workers in articles [1] and [2]. Since then the test has been recognized as a condition that assures the convergence of a nonconforming finite element method. The theoretical importance of the patch test was recognized in the pioneering work by Strang and Fix [3]. Since then the nonconforming method has received a considerable attention in engineering literature and practice.

Sander and Beckers [4] and Oliveira in [5] were the first who questioned the validity of the classical patch test, and in particular, its necessity for convergence.

In 1979 Stummel has in [6] derived a generalized patch test and questioned the validity of the classical patch test [7], especially its sufficiency for convergence. His works started a series of pro et contra discussions, see, e. g. [8], [9]–[12], [13]–[14].

The aim of our paper is to construct a simple nonconforming approximations which do not satisfy patch test requirements, yet the solutions obtained converge to the exact solution. The necessity of the patch test for convergence is thus questioned. Although the simple nonconforming finite element constructed here is of an academic value only, the proof of its convergence and its violation of the patch test allow a general conclusion concerning the necessity of the patch test.

II. Nonconforming approximation

Let us consider the variational equation of the form

$$\int_I (a_1\, u'\, v' + a_0\, u\, v)\; dx = \int_I (f_1\, v' + f_0\, v)\; dx$$

for $u \in V$, $\forall v \in V$, where I is an open interval $(0, 1)$ and

$$V = \{v \in H^1(I), \quad v(0) = 0\}$$

is a closed subspace of the Sobolev space $H^1(I)$ equipped with the norm

$$\|\cdot\|_I : v \mapsto \sqrt{\int_I ((v')^2 + v^2)\, dx}.$$

R. Flajs and M. Saje are with the Chair of Mechanics of Faculty of Civil and Geodetic Engineering, University of Ljubljana, Slovenia, e-mails: (rado.flajs@fgg.uni-lj.si, miran.saje@fgg.uni-lj.si).

The coefficients a_1 and a_0 are measurable functions on domain I and $f_i \in L^2(I)$ for $i = 1, 0$.

We divide interval I by points $x_0, \ldots, x_n$ into n open intervals $I_i = (x_{i-1}, x_i)\,;\ 1 \le i \le n$ of length h_i. Then we approximate the solution u with

$$u_h = \sum_{i=1}^{n} z_i\, w_i \in W_h,$$

where we have chosen special nonconforming base functions $w_1, \ldots, w_n$, as illustrated in Figs. 1-3. The meaning of amplitudes z_i is shown in Fig. 4.

We rewrite the above given task into the weak form: "Find the solution $u_h \in W_h$ of equation

$$\sum_{i=1}^{n} \int_{I_i} (a_1\, u_h'\, v_h' + a_0\, u_h\, v_h)\; dx = \sum_{i=1}^{n} \int_{I_i} (f_1\, v_h' + f_0\, v_h)\; dx$$

for $\forall v_h \in W_h$", where W_h denotes a nonconforming approximation space obtained from base functions $w_1, \ldots, w_n$.

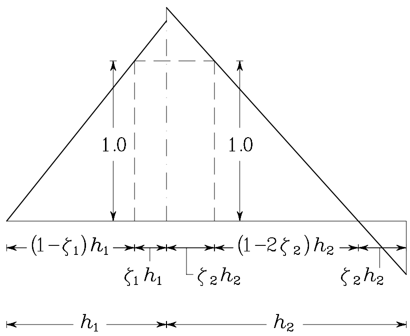

Fig. 1. Base function w_1.

According to Ciarlet [15, p. 94], we denote the finite element in R^1 with triple (I, P, Σ) where:

(i) I is an open interval in R^1 of the length h;
(ii) P is finite–dimensional space of real–valued functions over interval I. We let $\dim(P) = 2$;
(iii) Σ is a set of two linear forms (degrees of freedom) $\Sigma = \{\phi_1, \phi_2\}$ where:

$$\phi_1 : v_h \mapsto v_h(t_0), \tag{1a}$$
$$\phi_2 : v_h \mapsto v_h(t_1), \tag{1b}$$

where we have denoted two points $t_0, t_1 \in I$ with distance $\zeta\, h = \min(\frac{h}{3}, o(h))$ measured from the starting and the ending point of the interval I, respectively. We have also used well known Landau symbol o[1].

[1]We write $f(h) = O(g(h))$, if there exists such a constant $A \neq 0$ that $\lim_{h\to 0} \frac{f(h)}{g(h)} = A$. We also write $f(h) = o(g(h))$ when $\lim_{h\to 0} \frac{f(h)}{g(h)} = 0$.

International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:6, No:9, 2012

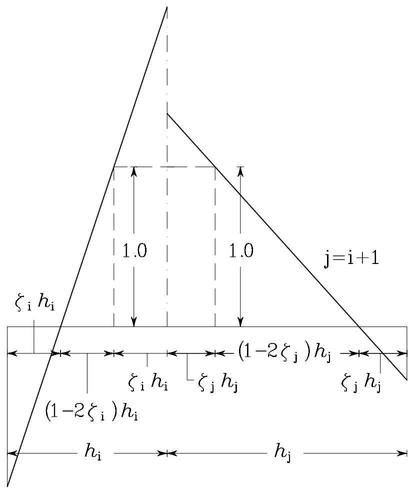

Fig. 2. Base functions $w_i;\ 2 \le i \le n-1$.

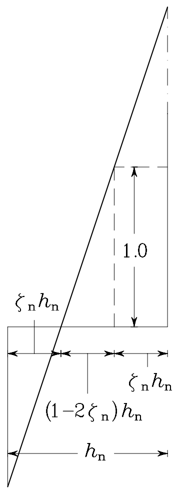

Fig. 3. Base function w_n.

The set of functionals is P–unisolvent. According to Ciarlet [15, p. 94] such finite elements are allowed.

III. Convergence

We can use F-E-M convergence test, derived by Shi [11]. According to Shi, the finite element space W_h is said to pass the F_1–test for problems of order 2, if for every function $v_h \in W_h$, the jump of v_h, denoted by $[v_h]$, across each interface F of two adjacent elements I_1 and I_2 satisfies the condition

$$\left| \int_F [v_h]\, dx \right| \le o\left(h_I^{1/2}\right) \|v_h\|_{I_1 \cup I_2}, \quad h_I = \max\left(h_{I_1}, h_{I_2}\right). \tag{2}$$

For every outer boundary $x_i \in \{x_0 \equiv 0, x_n \equiv 1\}$ with the Dirichlet boundary conditions, the jump $[v_h] \equiv v_h(x_i)$ and the above condition is understood as

$$|v_h(0)| \le o\left(h_1^{1/2}\right) \|v_h\|_{I_1} \text{ or } |v_h(1)| \le o\left(h_n^{1/2}\right) \|v_h\|_{I_n}.$$

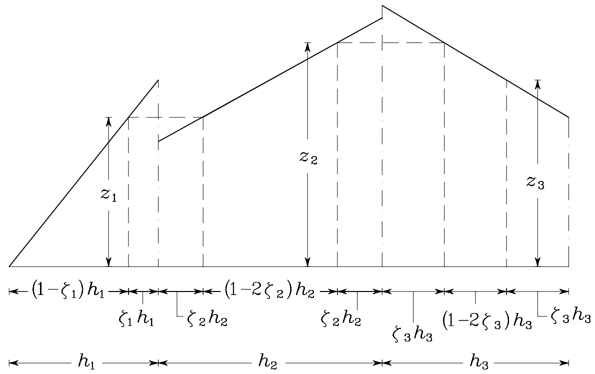

Fig. 4. Nonconforming approximation u_h ($n > 3$) (only elements 1–3 are shown).

We check F_1–test in R^1, particularly equation (2) from [11]. Let us have $\zeta = \zeta_1 = \ldots = \zeta_n$ and $h = h_1 = \ldots = h_n$. With the help of Figure 4, we easily derive

$$\int_F [v_h]\, dx = \frac{\zeta}{1-2\,\zeta}\left((z_{i+1}-z_i) + (z_i - z_{i-1})\right),$$

$$\|v_h\|_{1, I_i \cup I_{i+1}} = \sqrt{\frac{(z_{i+1}-z_i)^2 + (z_i - z_{i-1})^2}{(1-2\,\zeta)^2\, h}} + O\left(\sqrt{h}\right), \tag{3}$$

where I_i and I_{i+1} for $i = 2, \ldots, n-1$ denote the intervals, and $F = I_i \cap I_{i+1}$ their intersections. We can, using the abbreviations $a = z_{i+1} - z_i$ and $b = z_i - z_{i-1}$, transform equation (2) from [11] into the form

$$\frac{\zeta^2}{\left(1-2\,\zeta\right)^2}(a+b)^2 \le \left(\frac{o\left(\sqrt{h}\right)}{\sqrt{h}}\right)^2 \frac{a^2+b^2}{\left(1-2\,\zeta\right)^2} + o(h).$$

It is easy to see, that the choice $\zeta = O(h)$ fulfills the inequality (2). Using theorem 1 from [11] the convergence of the proposed approximate solution is thus assured.

IV. Numerical examples

A. First example

For the illustration of the behavior of such an approximation, we present the numerical example. We consider the boundary value problem

$$-u''(x) = 0 \quad \forall x \in (0,1), \quad u(0) = 0,\ u'(1) = 1$$

with the exact solution $u = x$. We first rewrite this problem into a weak form: "Find the solution $u \in V$ of the equation

$$\int_0^1 u'\, v'\, dx = \int_0^1 v'\, dx \qquad \forall v \in V."$$

We divide the interval $I = (0,1)$ into n equal subintervals I_i, where $h = 1/n$, and also choose equal $\zeta_i = \zeta$ for all

International Journal of Architectural, Civil and Construction Sciences
ISSN: 2415-1734
Vol:6, No:9, 2012

$i = 1, 2, \ldots, n$. Then we search for an approximate solution $u_h \in W_h$ of the equation

$$\sum_{i=1}^{n} \int_{I_i} u_h' \, v_h' \, dx = \sum_{i=1}^{n} \int_{I_i} v_h' \, dx \qquad \forall v_h \in W_h.$$

Using the same technique as described in [9], it is easy to see that $\|u_n - x\,(1-2\zeta)\|_2 \to 0$ and $\|u_n' - 1\|_2 \to 0$ when $n \to \infty$. When choosing $\zeta = O(h)$, the approximations $\|u_n - x\|_2$ and $\|u_n' - 1\|_2$ go to zero for $n \to \infty$ in the sense of the L^2 norm $\|\cdot\|_2$.

It is easy to see that the patch test, which has in [12] the following mathematical formulation

$$d_h(u^*, v_h) = a_h(u^*, v_h) - a_h(u_h, v_h) = 0$$

$\forall u^* \in P_1, \forall v_h \in W_h$, is not fulfilled. In the last equation, we have used abbreviations: the linear solution of the problem u^*, the finite element approximation u_h, the space of polynomials of first degree P_1, and the bilinear form $a_h(u,v) = \sum_{i=1}^{n} \int_{I_i} u' \, v' \, dx$.

The behavior of numerical solutions for different ζ's is presented in Figs. 6 and 5 below.

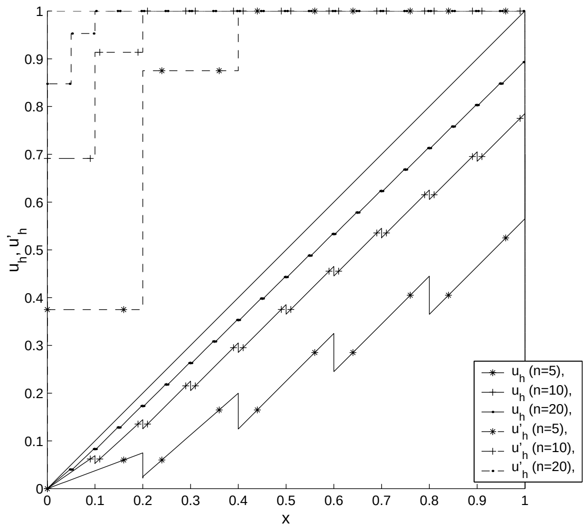

Fig. 5. Nonconforming approximations for linear solution $u = x$ and its derivative $u' = 1$ for $\zeta = h$.

The observation of Fig. 5 indicates that the approximations of u along with their first derivatives obtained by finite elements with $\zeta = O(h)$ converge to the exact solution of Eq. (4) exactly as predicted theoretically.

In contrast, the approximations obtained by finite elements with the fixed ζ do not converge to the exact solution. In fact, they converge to the solution of another boundary value problem.

B. Second example

We solve the boundary value problem

$$-u''(x) + u(x) = x^2 \quad \forall x \in (0,1),\ u(0) = 0,\ u(1) = 0 \quad (4)$$

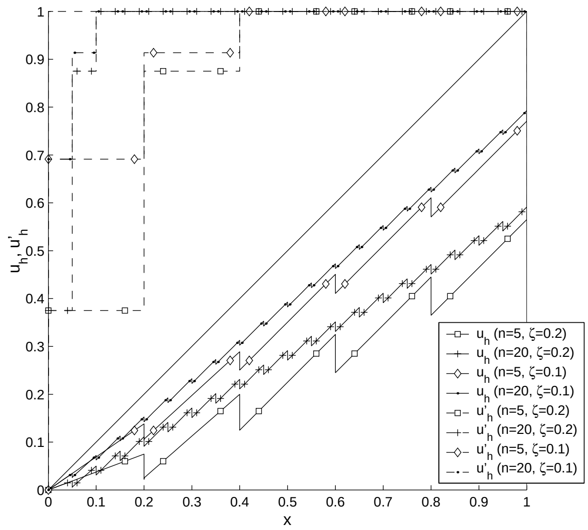

Fig. 6. Nonconforming approximations for linear solution $u = x$ and its derivative $u' = 1$ for fixed $\zeta = 0.2$ and $\zeta = 0.1$.

with the exact solution

$$u(x) = 2 + x^2 - \frac{\left(-3 + 2\,e^{-1}\right) e^{x}}{-e^{1} + e^{-1}} + \frac{\left(2\,e^{1} - 3\right) e^{-x}}{-e^{1} + e^{-1}}$$

and consider also the boundary value problem

$$-u''(x) + (1-2\,\zeta)^2 u(x) = (1-2\,\zeta)^2\, x^2 \qquad (5)$$

on interval $(0,1)$ with prescribed boundary conditions $u(0) = 0$ and $u(1) = 0$.

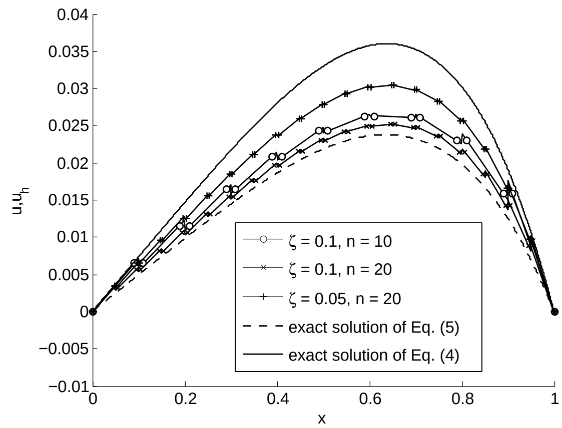

Fig. 7. Nonconforming approximations for solution u of the boundary value problem (4).

The graphs in Figs. 7 and 8 show the convergence to the exact solution of Eq. (4) for $\zeta = O(h)$ and the divergence for the fixed $\zeta = 0.1$. In fact these solutions converge to another solution – the exact solution of Eq. (5). This kind of divergence, i.e. the convergence to the solution of another boundary value problem, is quite common and could be

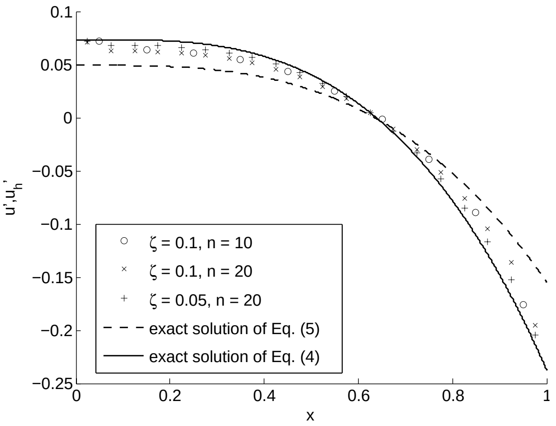

Fig. 8. Nonconforming approximations for solution derivative u' of the boundary value problem (4).

very dangerous and misleading when the exact solution of the boundary value problem is not known or we have no theoretical convergence criterion.

V. Conclusion

The convergence of such a type of finite elements with $\zeta = O(h)$ does not seem surprising, because nonconforming approximations obviously approach conforming ones. It also looks that the elegant abstract definition of the finite element proposed by Ciarlet allows finite elements different from physically based finite elements usually used in engineering practice.

According to the Stummel generalized patch test [6], roughly speaking, the limit point decides about the convergence, while according to the Irons' patch test an arbitrary point can take this role. Such a decision should be done carefully. Let us illustrate this on a simple example from the mathematical analysis. It is well known that the limit of the sequence of continuous functions need not be continuous, as well as the limit of the sequence of discontinuous functions could be continuous. Origin of the divergences met here could be due to the nature of the used approximations and possibly due to the wild nature of Sobolev spaces of the exact weak solutions. That is why one could hardly take a decision about the convergence using only particular points.

In some cases, however, the Irons' patch test can definitely serve as the theoretically useful criterion for convergence; see Wang [16].

Acknowledgment

The work was partially supported by the Slovenian Research Agency through the grant P2-0260. The support is gratefully acknowledged.

References

[1] G.P. Bazeley, Y.K. Cheung, B.M. Irons, O.C. Zienkiewicz, *Triangular elements in bending: Conforming and nonconforming solutions* in Proceedings Conference on Matrix Methods in Structural Mechanics, Wright Patterson A. F. B., Dayton, OH, 547-576.

[2] B.M. Irons, A. Razzaque, *Experience with the patch test for convergence of finite elements* in A. K. Aziz, ed., The mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, Academic Press, New York, 557-587.

[3] G. Strang, G.J. Fix, *An analysis of the finite element method*, Prentice–Hall Inc., Englewood Cliffs, N.J., 1973.

[4] G. Sander, P. Beckers *The influence of the choice of connectors in the finite element method*, Proc. conf. on the mathematical aspects of the finite element methods, Rome 75, Galligani, I., Magenes, E., eds., Lecture notes in mathematics, 606, Springer–Verlag, New York, 316–340, 1977.

[5] E.R.A. Oliveira, *The patch test and the general convergence criteria of the finite element method*, International journal of solids and structures, vol. 13, 159–178, 1977.

[6] F. Stummel, *The generalised patch test*, SIAM Journal of numerical analysis, vol. 16, no. 3, 449–471, 1979.

[7] F. Stummel, *The limitations of the patch test*, International journal for numerical methods in engineering, vol. 15, 177–188, 1980.

[8] B.M. Irons, M. Loikkanen, *An engineer's defence of the patch test*, International journal for numerical methods in engineering, vol. 19, 1391–1401, 1983.

[9] Z.C. Shi, *An explicit analysis of Stummel's patch test examples*, International journal for numerical methods in engineering, vol. 20, 1233–1246, 1984.

[10] Z.C. Shi, *A convergence condition for the quadrilateral Wilson element*, Numerische mathematik, vol. 44, 394–361, 1984.

[11] Z.C. Shi, *The F–E–M–test for convergence of nonconforming finite elements*, Mathematics of computation, vol. 49, no. 40, 391–405, 1987.

[12] Z.C. Shi, *On Stummel's examples to the patch test*, Computational mechanics, vol. 5, 81–87 (1989).

[13] R.L. Taylor, J.C. Simo, O.C. Zienkiewicz, A.C.H. Chan, *The patch test–a condition for assessing FEM convergence*, International journal for numerical methods in engineering, vol. 22, 39–62, 1986.

[14] O.C. Zienkiewicz, R.L. Taylor, *The finite element patch test revisited. A computer test for convergence, validation and error estimates*, Computer methods in applied mechanics and engineering, 149, 223–254, 1997.

[15] P.G. Ciarlet, J.L. Lions, *Handbook of numerical analysis, Volume II, Finite element methods (Part 1)*, North-Holland, Amsterdam, 1991.

[16] M. Wang, *On the necessity and sufficiency of the patch test for convergence of nonconforming finite elements*, SIAM J. Numer. Anal., 39(2), 363–384, 2001.