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Nonlinear Model Predictive Swing-Up and Stabilizing Sliding Mode Controllers

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Abstract**—**In this paper, a nonlinear model predictive swing-up and stabilizing sliding controller is proposed for an inverted pendulum-cart system. In the swing up phase, the nonlinear model predictive control is formulated as a nonlinear programming problem with energy based objective function. By solving this problem at each sampling instant, a sequence of control inputs that optimize the nonlinear objective function subject to various constraints over a finite horizon are obtained. Then, this control drives the pendulum to a predefined neighborhood of the upper equilibrium point, at where sliding mode based model predictive control is used to stabilize the systems with the specified constraints. It is shown by the simulations that, due to the way of formulating the problem, short horizon lengths are sufficient for attaining the swing up goal.

Keywords**—**Inverted pendulum, model predictive control, swing-up, stabilization.

I. Introduction

IN this manuscript, the inverted pendulum swing-up and stabilization problem is modeled using two different nonlinear model predictive controllers (NMPC) whose parameters are determined by using the nonlinear programming approach. The main focus of this paper is on the swing-up phase of the problem where the objective function aims to increase the energy of the system in a finite horizon. The stabilization phase of the problem is solved extending the sliding mode control philosophy [1] to finite horizon approach. Although the objective functions of the NMPC are different for the swing-up and stabilization phases, the set of constraints are the same for both phases. The structure for the objective functions and constraints are suitable for generalization to a larger class of systems.

Due to its inherent nonlinear nature, the inverted pendulum is widely used as a test-bed for designing and testing new control techniques. In [1], the authors modeled the pendulum stabilization problem with bicriteria nonlinear programming framework based on sliding mode control using various constraints. On the other hand, the inverted pendulum swing-up problem has been also studied extensively in the literature. A popular approach in designing swing-up controllers is based on controlling its energy. The logic behind this approach is injecting energy to the pendulum by applying appropriate control force to the cart. In [2], a bang-bang control is used to raise the energy of the pendulum towards a value equal to its steady state value at the upright position. This approach does not consider the cart track length constraint. In [3], a variable structure system version of energy-speed-gradient method is treated in a rigorous manner to show that attractivity of the upright equilibrium can be achieved by applying a control of arbitrary small magnitude. This approach also does not consider any system constraints related to cart track length. In [4], the sign condition in the derivative of the energy is exploited. In this paper a servo system having a low pass property is used for the swing-up. This servo system uses a sinusoidal reference input generated from the pendulum trajectory. In another significant energy-based work [5], the swing-up and stabilization of an inverted pendulum system with a restricted cart track length is achieved by using an energy-well built within the cart track. It is constructed in such a way that the cart experiences a repulsive force as it approaches the ends of the track. Although the cart track length is integrated into the controller, this is not in a systematic and a generalizable way. In the energy-based works, the stabilization phase is carried out, generally, by using controllers designed for the linearized model of the inverted pendulum. In [6], energy-based swinging strategies are compared with a fuzzy swing-up algorithm. In [7], a Lyapunov function is obtained by using the total energy of the system, and the convergence analysis carried out using the LaSalle's invariance principle. In fuzzy logic approaches, the states of the inverted pendulum system are used as inputs. For example, in [8], the fuzzy logic method is used in both swing-up and stabilization phases. Each state of the inverted pendulum is assigned with a single input rule module (SIRM) and a dynamic importance degree. Besides, a reader may also refer this paper for a very good review of other fuzzy logic works in the literature.

The swing-up strategy in this manuscript is based on moving the energy of the inverted pendulum towards the energy of the unstable equilibrium point. Even though the energy of the inverted pendulum system is exploited, different mathematical tools are used fundamentally and some constraints related to physical system are considered in order to generate the control signal. Differing from the literature, the

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problem that is mainly in the differential equation domain is transformed, into the domain of nonlinear programming: an algebraic domain by means of the NMPC philosophy. This also allows us to embed a variety of design specifications, like system constraints, in the problem in a conceptually simple way.

NMPC is a control strategy that is based on solving online a finite horizon optimization problem including constraints at every sampling instant to generate the control input. The solution is a sequence of control inputs that optimize plant's future behavior. Only the first element of the sequence is applied to the controlled plant. Then, the plant state is sampled again and the calculations for optimization are repeated from the current state, yielding a new control and new predicted state path.

MPC for linear systems with linear quadratic performance criteria is a fairly mature field of control systems. The books [9] and [10] are significant references on this subject. Due to the nonlinear nature of the inverted pendulum system and hard constraints imposed on its inputs, nonlinear programming modeling naturally fits to this problem. For the nonlinear plants, keeping the constraints linear does not provide any advantage [11]; therefore, imposing hard constraints on the inputs is not avoided. In [12], commercially available model predictive control technology is presented. In this work, the empirical nature of the nonlinear MPC is highlighted. Due to lacking mature mathematical tools for the analysis of nonlinear MPC, numerical techniques gain popularity [13]. For the same reason, in [14], genetic algorithms are proposed for obtaining an optimal control sequence in the nonlinear MPC framework.

In this paper, the swing up and stabilization problems are modeled as two NMPC formulations with different objective functions and the same set of constraints for some finite horizon. In the following section the dynamics of a one degree of freedom inverted pendulum system are presented. The swing-up and stabilization control problems are modeled in NMPC framework in sections three and four respectively. The last section contains illustrative simulations for various experimental conditions.

II. Inverted Pendulum System Modeling

In this section inverted pendulum dynamics are described and presented in a discrete time framework. As shown in Fig. 1, inverted pendulum is, basically, composed of a pendulum attached to a moving cart from a pivot point.

The inverted pendulum system is an underactuated system that has two outputs (*x*: the position of the cart and θ: the pendulum angle) which are controlled by a single input (*u*: the force applied to the cart).

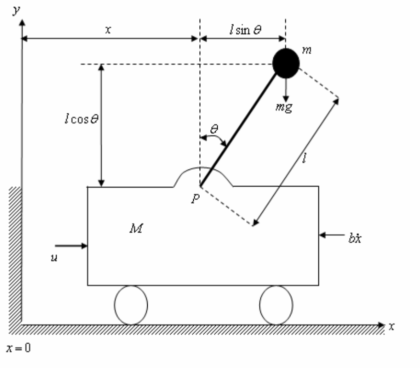

Fig. 1 Inverted pendulum system model

The inverted pendulum system has two equilibrium points. One of them is the upward equilibrium point of the pendulum, $(\theta, \dot{\theta}) = (0, 0)$, and it is named as unstable equilibrium point, that is, the pendulum may fall over at any time in any direction within the plane of motion. Hence, an appropriate force input *u* has to be generated to keep the pendulum up-straight. The other one is the downward equilibrium point of the pendulum, $(\theta, \dot{\theta}) = (\pi, 0)$, and it is named as stable equilibrium point, which means that if the pendulum is exposed to a disturbing force changing its state, it tends to restore its previous position.

The mass of the rod is assumed be concentrated at its upper end. The parameters, their definitions and typical values used in simulations are presented at Table I.

TABLE I
DEFINITIONS OF PARAMETERS AND TYPICAL VALUES FOR THE INVERTED PENDULUM

Symbol	Parameter	Value	Unit
M	Mass of the cart	3	*kg*
m	Mass of the pendulum	0.5	*kg*
l	Length of the pendulum	0.5	*m*
b	Friction constant	2	*kg/s*
g	Gravitational force	9.81	kgm/s^2

The mathematical model representing the dynamics of an inverted pendulum shown in Fig. 1 can be obtained by using the principles of the Newtonian mechanics [15]:

$$
\begin{aligned}
&(M + m)\ddot{x} - ml\dot{\theta}^2 \sin\theta + ml\ddot{\theta}\cos\theta + b\dot{x} = u \\
&m\ddot{x}\cos\theta + ml\ddot{\theta} = mg\sin\theta
\end{aligned} \tag{1}
$$

The state space representation consisting of the nonlinear differential equations is given below, where u denotes the control input, and the components $x_1, \dots, x_4$ of the state vector X are defined as $x_1 := x$, $x_2 := \dot{x}$, $x_3 := \theta$, $x_4 := \dot{\theta}$.

$$\dot{X} = \begin{bmatrix} x_2 \\ \dfrac{-bx_2 + ml\sin(x_3)x_4^2 - mg\sin(x_3)\cos(x_3)}{M + m - m\cos(x_3)^2} \\ x_4 \\ \dfrac{(bx_2 - ml\sin(x_3)x_4^2)\cos(x_3) + (M+m)g\sin(x_3)}{l(M + m - m\cos(x_3)^2)} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M + m - m\cos(x_3)^2} \\ 0 \\ \dfrac{-\cos(x_3)}{l(M + m - m\cos(x_3)^2)} \end{bmatrix} u \quad (2)$$

Equation (2) can be written compactly as,

$$\dot{X} = f(X) + h(X)u \quad (3)$$

for suitable vector functions f and h. Its Euler discretization with sampling period T is,

$$X(k+1) = F(X(k)) + H(X(k))u(k) \quad (4)$$

where

$$F(X(k)) := X(k) + Tf(X(k)) \quad (5)$$

$$H(X(k)) := Th(X(k)). \quad (6)$$

These discretized system equations will take place in the NMPC model as a set of constraints.

III. Formulation of the Swing-up Phase

Inverted pendulum swing-up problem is defined as swinging up the pendulum from its lower (stable) equilibrium point to the neighborhood of its upper (unstable) equilibrium point by applying appropriate control inputs. The method proposed at this paper formulates the swing up problem as a nonlinear programming based NMPC problem which rewards total energy increments and uses discretized system equations and limits on inputs and states as constraints.

The objective function is formed by using the positive definite function [16],

$$V = \frac{1}{2}(|E_\theta| + E_x)^2 \quad (7)$$

where

$$E_\theta = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(\cos\theta - 1)$$
$$E_x = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2 \quad (8)$$

In the above expression, E_θ is an energy equation of an uncontrolled inverted pendulum for a given $(\theta, \dot{\theta})$ pair [2]. The absolute value of E_θ has its maximum value $2mgl$ when $(\theta, \dot{\theta}) = (\pi, 0)$ and has its minimum value *0* when $(\theta, \dot{\theta}) = (0,0)$. E_x , the other term in (8), is a function which attains its minimum when cart position and velocity equal zero.

The positive definite function V satisfies $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. This function has its minimum value when all the states are zero, that is $(x, \dot{x}, \theta, \dot{\theta}) = (0,0,0,0)$. With the exception of this point, this function is positive valued.

By using (7) and (8), N-step horizon objective function for the problem under consideration is defined as follows:

$$V_N = \frac{1}{2}(|E_{\theta,N}| + E_{x,N})^2$$
$$V_N = \frac{1}{2}\left(\left|\frac{1}{2}ml^2(x_4(N))^2 + mgl(\cos(x_3(N)) - 1)\right| + \frac{1}{2}(x_1(N))^2 + \frac{1}{2}(x_2(N))^2\right)^2 \quad (9)$$

The upper and lower bounds imposed on the state variables and the control inputs are used as the (hard) constraints.

Now the swing up problem is formulated as the following nonlinear programming problem:

$$\begin{aligned}
&\min_{U(k,N)} V_N \\
&\text{subject to} \\
&x_1(i+1) = x_1(i) + Tx_2(i) \\
&x_2(i+1) = x_2(i) + T\left(\frac{-bx_2(i) + ml\sin(x_3(i))x_4^2(i) - mg\sin(x_3(i))\cos(x_3(i)) + u(i)}{M + m - m\cos^2(x_3(i))}\right) \\
&x_3(i+1) = x_3(i) + Tx_4(i) \\
&x_4(i+1) = x_4(i) + T\left(\frac{(bx_2(i) - u(i) - ml\sin(x_3(i))x_4^2(i))\cos(x_3(i)) + (M+m)g\sin(x_3(i))}{L(M + m - m\cos^2(x_3(i)))}\right) \\
&u_L \le u(i) \le u_U \\
&x_{1L} \le x_1(i) \le x_{1U} \\
&x_{2L} \le x_2(i) \le x_{2U} \\
&x_{4L} \le x_4(i) \le x_{4U} \\
&\text{for } \ i = 0,1,2,\dots,N-1
\end{aligned} \quad (10)$$

At the *k-th* sampling instant, this problem is solved to obtain $U(k,N)$ where

$$U(k,N) := \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+N-1) \end{bmatrix}. \quad (11)$$

Only the first component of $U(k,N)$ is applied to the input. At the *k+1*-st sampling instants the computation is repeated to obtain the control sequence $U(k+1,N+1)$, and so on.

One may notice that the model in equation (10) is nonlinear due to any one of the nonlinear objective function and the constraints. Gradient based numerical solution techniques are utilized because of the lacking analytical solutions for the equations of this form.

The swing-up phase terminates upon pendulum enters a certain neighborhood of the unstable equilibrium point. Following this, the NMPC algorithm for the stabilization phase described in the next section activates to take the control over.

IV. Formulation of the Stabilization Phase

Inverted pendulum stabilization problem is defined as to stabilize the pendulum in the neighborhood of its unstable equilibrium point. This is the second phase that comes after successful completion of the swing-up phase described in the preceding section. The stabilization problem is solved using a sliding mode control based NMPC over a finite horizon. This phase of the NMPC algorithm is also a nonlinear programming problem whose objective function is a squared sliding surface function that developed in [1] and constraint equations are same as in the swing-up problem.

The constant coefficients $g_1,\dots,g_4$ in the sliding surface function $s(X)=g_1x_1+g_2x_2+g_3x_3+g_4x_4$ are chosen to yield stable sliding surface function. Sliding surface is utilized in sliding mode control strategies so that when the state trajectory is restricted to a stable sliding surface, the trajectory asymptotically approaches the origin in a robust manner [17]. Such coefficients can be obtained by using the inverted pendulum model linearized at the unstable equilibrium point.

The nonlinear programming problem that is used for NMPC which yields control inputs for the stabilization phase is as follows:

$$
\begin{aligned}
&\min_{\mathrm{U(k,N)}} \ (g_1x_1(N)+g_2x_2(N)+g_3x_3(N)+g_4x_4(N))^2\\
&\text{subject to}\\
&x_1(i+1)=x_1(i)+Tx_2(i)\\
&x_2(i+1)=x_2(i)\\
&\quad+T\left(\frac{-bx_2(i)+ml\sin(x_3(i))x_4^2(i)-mg\sin(x_3(i))\cos(x_3(i))+u(i)}{M+m-m\cos^2(x_3(i))}\right)\\
&x_3(i+1)=x_3(i)+Tx_4(i)\\
&x_4(i+1)=x_4(i)\\
&\quad+T\left(\frac{(bx_2(i)-u(i)-ml\sin(x_3(i))x_4^2(i))\cos(x_3(i))+(M+m)g\sin(x_3(i))}{l(M+m-m\cos^2(x_3(i)))}\right)\\
&u_L\le u(i)\le u_U\\
&x_{1L}\le x_1(i)\le x_{1U}\\
&x_{2L}\le x_2(i)\le x_{2U}\\
&x_{4L}\le x_4(i)\le x_{4U}\\
&\text{for}\quad i=0,1,2,\dots,N-1
\end{aligned}
\tag{12}
$$

This formulation relies on positive definite objective such that its minimum corresponds to its restriction to the sliding surface. For a stable sliding surface, clearly, this drives the system dynamics towards the unstable equilibrium point of the pendulum. The stabilization problem based on sliding mode control philosophy inherently has two sub-phases: reaching phase and sliding phase [1]. The system first reaches the sliding surface and then the system states are kept therein until it reaches the unstable equilibrium point.

V. Simulations

In this section, the simulations are performed to demonstrate the validity of NMPC approach of this paper. Throughout the experiments the sliding surface parameters $(g_1,g_2,g_3,g_4)=(2,1.5,5,1.5)$ are used for the stabilization phases. Main codes are written in MATLAB in which the physical system with a given input is simulated. The main codes contain calls for the GAMS optimization package with MINOS solver at each step to calculate the input value for the nonlinear optimization problems given by (10) and (12). Graphical outputs of two simulations for each of the cases are presented where the input is constrained to $|u|\le 10$ and $|u|\le 15$. The two simulations for different horizon lengths show the effect of the prediction horizon on the swing-up time.

A. Graphical Results

The graphical outputs for the horizon lengths 3 and 25 steps for each of the two different input constraints are shown in Figs. 2-5.

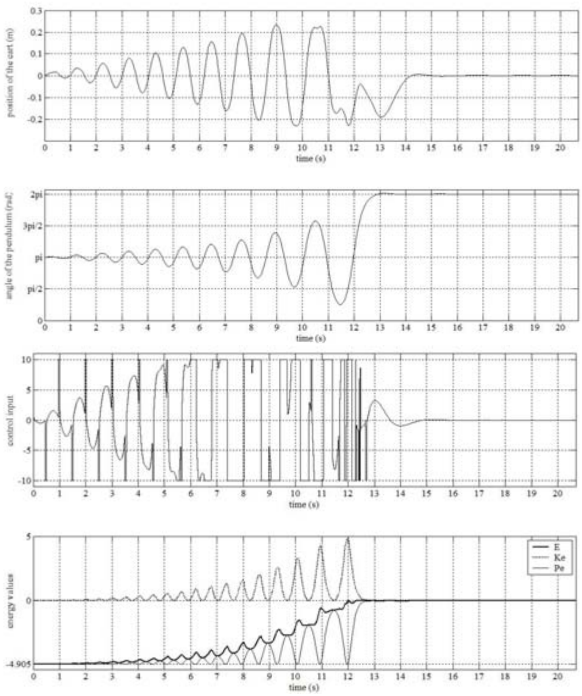

Fig. 2 Position of the cart, angle of the pendulum, control inputs and energy of the pendulum for *3*-step prediction for $u\in[-10,10]$

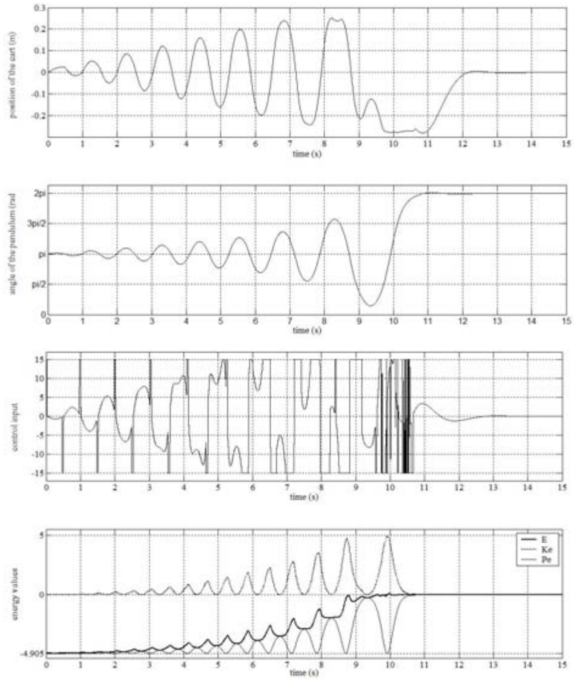

Fig. 3 Position of the cart, angle of the pendulum, control inputs and energy of the pendulum for the *3*-step prediction for $u \in [-15, 15]$

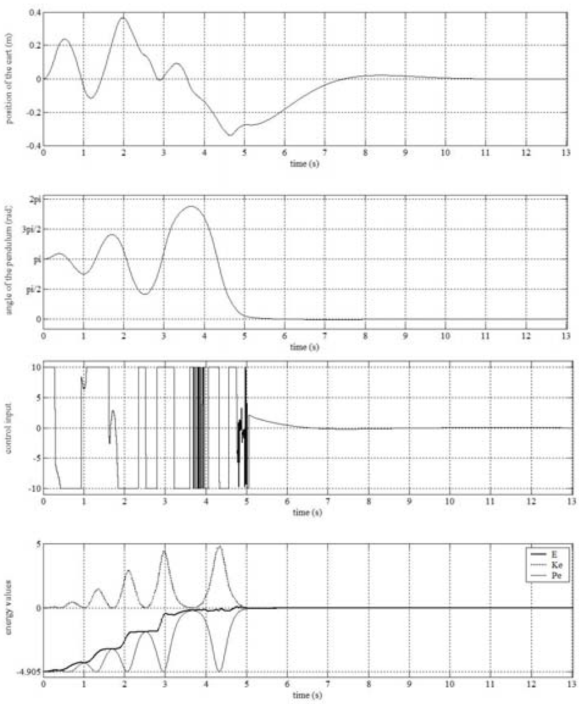

Fig. 4 Position of the cart, angle of the pendulum, control inputs and energy of the pendulum for *25*-step prediction for $u \in [-10, 10]$

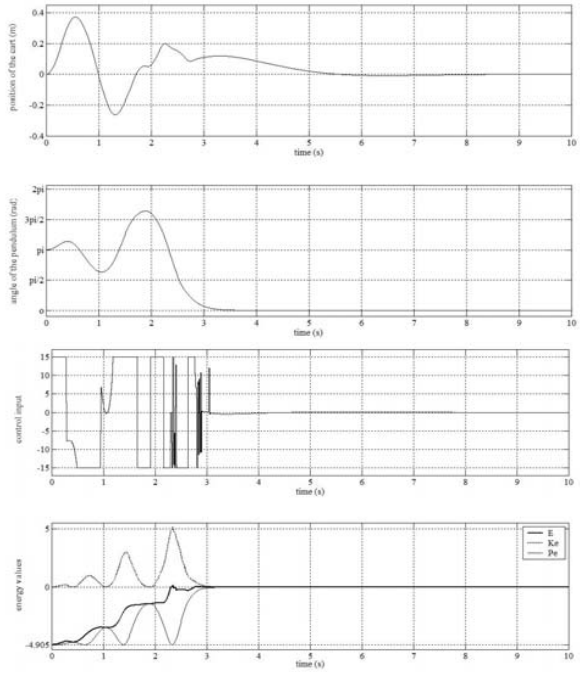

Fig. 5 Position of the cart, angle of the pendulum, control inputs and energy of the pendulum for *25*-step prediction for $u \in [-15, 15]$

Typical values for the inverted pendulum parameters given in Table I are used for the simulations. The cart track length is constrained to [-1, 1], and simulations were performed for the cases where the input is constrained to [-10, 10] and [-15, 15] intervals. In the figures, the first and second rows show cart position and pendulum angle versus time respectively. The third row shows input versus time, while the last row shows energy versus time. In energy graphics, *Ke, Pe,* and *E* denote the kinetic energy, potential energy, and the total energy of the pendulum respectively.

As seen from the figures, the cart stays within the prespecified limits. The graphical outputs reveal that the approach of this paper works satisfactory even for the horizon length of *3*. Nonetheless, the swing-up time decreases as the horizon length increases. A more detailed analysis of the effect of the horizon length on swing-up time is given in next subsection. It is observed that there is chattering in the control input u as the pendulum approaches to the upper equilibrium point. This is due to the term that minimizes the speed of the pendulum to avoid overshooting the upper equilibrium point. One may also notice in Figs. 2-5 that the energy graphics versus time is very close to a monotone increasing behavior. This becomes more significant as the horizon length increases. Energy ripples about the upper equilibrium point is due to the braking effect resulting from the angular speed minimization term in the objective function.

B. The Effect of Prediction Horizon on Swing-Up Time

The effect of the prediction length on swing-up time is analyzed in this subsection. Fig. 6 shows swing-up time versus prediction step for the input constraint $|u| \le 10$.

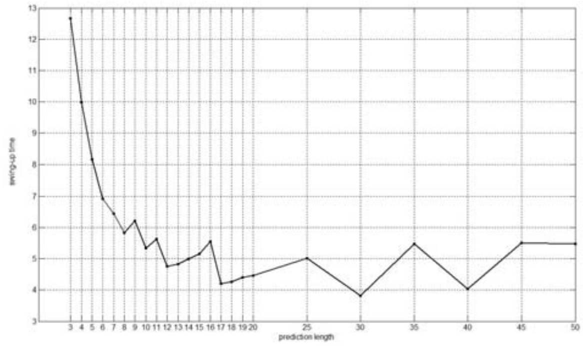

Fig. 6 Swing-up time versus prediction steps for $u \in [-10, 10]$

Fig. 7 shows swing-up time versus prediction step for the input constraint $|u| \le 15$.

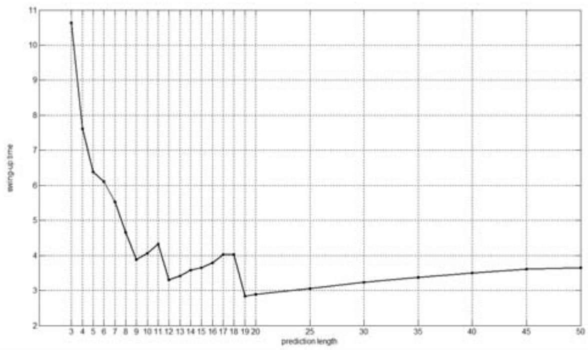

Fig. 7 Swing-up time versus prediction steps for $u \in [-15, 15]$

The Figs. 6 and 7 show that good swing-up performances are obtained for the prediction horizons larger than 10 steps. The improvement in the swing up time is significant for the horizon lengths up to 10 steps. For the horizon lengths between 10 to 50 steps, the swings up time improvements are not significant. Thus 10 steps or its neighboring number of steps is suitable for the swing up problem under consideration.

VI. Conclusion

A novel nonlinear programming based NMPC approach is used to solve the inverted pendulum swing-up and stabilization problem. The problem is formulated in two phases: The first phase considers the swing up problem while the second phase considers the stabilization problem. These two phases have the same constraints which include the cart track length. However, the first phase has energy based objective function while the second phase uses a sliding mode function in its objective function. The graphical outputs have shown that the approach used in this paper works well even for relatively small prediction horizon lengths.

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