Nonlinear Equations with N-dimensional Telegraph Operator Iterated K-times

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Abstract—In this article, using distribution kernel, we study the nonlinear equations with n-dimensional telegraph operator iterated k-times.

Keywords—Telegraph operator, Elementary solution, Distribution kernel.

I. Introduction

THE telegraph equation arises in the study of propagation of electrical signals in a cable of transmission line and wave phenomena. The interaction of convection and diffusion or reciprocal action of reaction and diffusion describes a number of nonlinear phenomena in physics, chemistry and biology. Further, the telegraph equation is more suitable than ordinary diffusion in modeling reaction-diffusion for such branches of applied sciences. We refer the reader to [1]-[4] and the references therein.

Kananthai [5]-[6] has studied some properties and results of the distribution $e^{\alpha x}\Box^k\delta$ and solved the convolution equation

$$e^{\alpha x}\Box^k\delta * u(x) = e^{\alpha x}\sum_{r=0}^{m} C_r\Box^r\delta,$$

which is related to the ultra-hyperbolic equation, where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$, C_r are given constants for $r = 1, 2, \dots, m$, $\Box^k$ is the n-dimensional ultra-hyperbolic operator iterated k times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

with $p + q = n$ and δ is the Dirac-delta distribution with $\Box^0\delta = \delta$, $\Box^1\delta = \Box\delta$.

In this work, by applying the distribution $e^{\alpha x}\Box^k\delta$, we study the elementary solution of the following n-dimensional telegraph equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \Delta\right)^k u(x,t) := T^k u(x,t) = \delta(x,t), \tag{1}$$

where Δ is the n-dimensional Laplacian operator iterated k times defined by

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k,$$

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and β is a positive constant. As an application, we solve the nonlinear equation with n-dimensional telegraph operator iterated k-times of the form

$$\left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \Delta\right)^k u(x,t) = f(x,t), \tag{2}$$

where $f(t,x)$ is a generalized function.

II. Some definitions and lemmas

Definition 1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$ and write

$$v = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,\ p + q = n.$$

Define by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } v > 0\}$ designating the interior of forward cone and $\overline{\Gamma}_+$ designating its closure.

For any complex number γ, we define the function

$$R_\gamma^H(v) = \begin{cases} \frac{v^{(\gamma-n)/2}}{K_n(\gamma)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{3}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\gamma) = \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2+\gamma-n}{2}\right)\Gamma\left(\frac{1-\gamma}{2}\right)\Gamma(\gamma)}{\Gamma\left(\frac{2+\gamma-p}{2}\right)\Gamma\left(\frac{p-\gamma}{2}\right)}. \tag{4}$$

Let supp $R_\gamma^H(v) \subset \overline{\Gamma}_+$ where supp $R_\gamma^H(v)$ denotes the support of $R_\gamma^H(v)$. The function R_γ^H is first introduced by Nozaki [7] and is called the ultra-hyperbolic kernel of Marcel Riesz. Moreover, $R_\gamma^H(v)$ is an ordinary function if $\mathrm{Re}(\gamma) \geq n$ and is a distribution of γ if $\mathrm{Re}(\gamma) < n$.

Definition 2. Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$s = x_1^2 + x_2^2 + \cdots + x_n^2.$$

For any complex number β, define the function

$$R_\beta^e(s) = 2^{-\beta}\pi^{-n/2}\Gamma\left(\frac{n-\beta}{2}\right)\frac{s^{(\beta-n)/2}}{\Gamma\left(\frac{\beta}{2}\right)} \tag{5}$$

The function $R_\beta^e(s)$ is called the elliptic kernel of Marcel Riesz and is ordinary function if $\mathrm{Re}(\beta) \geq n$ and is a distribution of β if $\mathrm{Re}(\beta) < n$.

Lemma 1. *[5] Let L be the partial differential operator defined by*

$$L = \Box - 2\left(\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial x_i} - \sum_{j=p+1}^{p+q}\alpha_j\frac{\partial}{\partial x_j}\right) + \left(\sum_{i=1}^{p}\alpha_i^2 - \sum_{j=p+1}^{p+q}\alpha_j^2\right). \tag{6}$$

Then

$$(e^{\alpha x}\Box^k\delta) * u(x) = L^k u(x) = \delta \tag{7}$$

In addition, the unique elementary solution of (7) *is given by* $u(x) = e^{\alpha x} R^H_{2k}(x)$*, where* $R^H_{2k}(x)$ *is defined by* (3) *with* $\gamma = 2k$.

Lemma 2. *[8]* $e^{\alpha x}\delta^{(k)} = (D-\alpha)^k\delta$ *where* $D \equiv \frac{d}{dx}$ *and* $e^{\alpha t}\delta^{(k)}$ *is a Tempered distribution of order* k *with support* 0.

Lemma 3. *[9] Let* z *be a complex number. Then*

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \quad z \neq 0, -1, -2, \ldots.$$

III. Main Results

Now, we shall state and prove the following main results.

Theorem 1. *Let* T^k *be the partial differential operator which iterated* k*-times defined by*

$$T^k = \left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \Delta\right)^k, \tag{8}$$

where Δ *is the* n*-dimensional Laplacian operator and* β *is a given positive constant. Then* $u(x,t) = e^{-\beta t}M_{2k}(w)$ *is a unique elementary solution of* (1)*, where* $M_\eta(w)$ *is defined by*

$$M_\eta(w) = \begin{cases} \frac{w^{(\eta-n)/2}}{H_{n+1}(\eta)} & \textit{if } t \in \Gamma_+, \\ 0 & \textit{if } t \notin \Gamma_+, \end{cases} \tag{9}$$

where $w = t^2 - x_1^2 - x_2^2 - \cdots - x_n^2$, t *is the time and*

$$H_{n+1}(\eta) = \pi^{(n-1)/2}2^{\eta-1}\Gamma\left(\frac{\eta-n+1}{2}\right)\Gamma\left(\frac{\eta}{2}\right). \tag{10}$$

Proof. Firstly, we define the $n+1$-dimensional ultra-hyperbolic operator as

$$\Box_{n+1} = \left(\frac{\partial^2}{\partial t^2} - \Delta\right).$$

Setting $\alpha_2 = \alpha_3 = \cdots \alpha_n = 0$, we have

$$e^{\alpha(t,x)}\Box^k_{n+1}\delta = e^{\alpha_1 t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)^k \delta(x,t).$$

Applying Lemma 3 for $p = 1$, $q = n$ and $p + q = n + 1$, (3) and (4) are reduced to (9) and (10), respectively.

Indeed, we have $\delta(x,t) = \delta(x)\delta(t)$ and $e^{\alpha_1 t}\delta(x) = \delta(x)$. Using Lemma 2, we get

$$\begin{aligned} e^{\alpha_1 t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\delta(x,t) &= e^{\alpha_1 t}\frac{\partial^2}{\partial t^2}\delta(x,t) - e^{\alpha_1 t}\Delta\delta(x,t) \\ &= \left(\frac{\partial}{\partial t} - \alpha_1\right)^2\delta(x,t) - \Delta e^{\alpha_1 t}\delta(x,t) \\ &= \left(\frac{\partial^2}{\partial t^2} - 2\alpha_1\frac{\partial}{\partial t} + \alpha_1^2 - \Delta\right)\delta(x,t). \end{aligned}$$

Substituting $\alpha_1 = -\beta$, it follows that

$$\begin{aligned} e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\delta(x,t) &= \left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \Delta\right)\delta(x,t) \\ &= T\delta(x,t) \end{aligned}$$

Convolving k-times for both sides of the above equation by $e^{-\beta t}(\partial^2/\partial t^2 - \Delta)\delta(x,t)$, we have

$$\begin{aligned} e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\delta(x,t) * \cdots * e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\delta(x,t) \\ = e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)^k\delta(x,t) \\ = T\delta(x,t) * \cdots * T\delta(x,t) \\ = T^k\delta(x,t). \end{aligned}$$

Then (1) can be written as

$$T^k u(x,t) = e^{-\beta t}\left(\frac{\partial^2}{\partial t^2} - \Delta\right)^k \delta(x,t) * u(x,t) = \delta(x,t).$$

Convolving both sides of the above equation by $e^{-\beta t}M_{2k}(w)$ and Applying Lemma 1, we have

$$u(x,t) = e^{-\beta t}M_{2k}(w),$$

where $M_{2k}(w)$ is defined by (9) with $\eta = 2k$. □

Theorem 2. *Given the equation*

$$\left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 - \Delta\right)^k u(x,t) = f(x,t), \tag{11}$$

where $f(x,t)$ *is a given generalized function and* $u(x,t)$ *is an unknown function. Then,*

$$u(x,t) = e^{-\beta t}M_{2k}(w) * f(x,t). \tag{12}$$

Proof. Convolving both sides of (11) by $e^{-\beta t}M_{2k}(w)$ and applying the Theorem 1, we obtain (12) as required. □

Remark 3. By using the method of proving Theorem 1 together with suitable modifications, we have $u(x,t) = e^{-\beta t}(-1)^k R^e_{2k}(s)$ is a unique elementary solution of the following equation

$$\left(\frac{\partial^2}{\partial t^2} + 2\beta\frac{\partial}{\partial t} + \beta^2 + \Delta\right)^k u(x,t) = \delta(x,t), \tag{13}$$

where $R^e_{2k}(s)$ is defined by Definition 2 with $\beta = 2k$, $s = t^2 + x_1^2 + x_2^2 + \cdots + x_n^2$ and a constant n in (5) is replaced by $n + 1$.

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