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Non-Local Behavior of a Mixed-Mode Crack in a Functionally Graded Piezoelectric Medium

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Abstract**—**In this paper, the problem of a mixed-Mode crack embedded in an infinite medium made of a functionally graded piezoelectric material (FGPM) with crack surfaces subjected to electro-mechanical loadings is investigated. Eringen's non-local theory of elasticity is adopted to formulate the governing electro-elastic equations. The properties of the piezoelectric material are assumed to vary exponentially along a perpendicular plane to the crack. Using Fourier transform, three integral equations are obtained in which the unknown variables are the jumps of mechanical displacements and electric potentials across the crack surfaces. To solve the integral equations, the unknowns are directly expanded as a series of Jacobi polynomials, and the resulting equations solved using the Schmidt method. In contrast to the classical solutions based on the local theory, it is found that no mechanical stress and electric displacement singularities are present at the crack tips when nonlocal theory is employed to investigate the problem. A direct benefit is the ability to use the calculated maximum stress as a fracture criterion. The primary objective of this study is to investigate the effects of crack length, material gradient parameter describing FGPMs, and lattice parameter on the mechanical stress and electric displacement field near crack tips.

Keywords**—**Functionally graded piezoelectric material, mixed-mode crack, non-local theory, Schmidt method.

I. INTRODUCTION

DISCOVERED as early as 1880 by Pierre and Jacques Curie, piezoelectricity can be defined as the linear electromechanical interaction between the mechanical and electrical states of crystals devoid of a center of symmetry [1]. Piezoelectric materials exhibit the piezoelectric effect; that is, electric polarization is induced in the material on application of mechanical loads and vice-versa [2]. However, a major shortcoming of piezoelectric ceramics is that they are extremely brittle with a propensity to develop cracks due to stress concentrations induced as a consequence of both mechanical and electrical loadings [3]. A class of materials which could potentially alleviate the problem of internal deboning and stress concentration in conventional piezoelectric materials are functionally graded materials (FGMs). Over the past two decades, FGMs have already shown great promise to be used as an alternative to conventional homogenous coatings [4]. These materials are generally composed of at least two-phase inhomogeneous particulate composites synthesized in such a manner that the volume fractions of the constituents vary continuously along any desired spatial direction, resulting in materials having smooth variation of mechanical properties. Therefore, any material property discontinuities that exist maybe suitably eliminated when FGMs are used as coating material. Therefore, the concept of FGMs can be extended to piezoelectric materials to synthesize FGPMs that exhibit improved reliability. For reliable service lifetime prediction of FGPM based devices, it is imperative to investigate and understand the fracture behavior of FGPMs and their effects on the electro-mechanical response. Most of the previous investigations into the fracture of FGPMs adopt classical continuum mechanics based on the local theory. Another point easily noticed is that these studies are almost exclusively devoted to the analysis of anti-plane Mode-III crack problems. Considering both electrically impermeable and permeable crack surfaces Chue and Ou [5] studied the problem of a Mode-III crack oriented perpendicular to the interface formed by bonding two functionally graded piezoelectric half-planes. Hsu and Chue [6] solved the Mode III fracture problem of an arbitrarily oriented crack in functionally graded piezoelectric strip bonded to a homogenous piezoelectric half plane. Chen and Chue [7] investigated the Mode-III fracture problem of a cracked functionally graded piezoelectric surface layer bonded to a cracked functionally graded piezoelectric substrate. A survey of the literatures shows that as opposed to Mode-III crack problems in FGPMs, studies concerning Mode-I, Mode-II and mixed-Mode crack problems are far and few between. Ueda [8] analyzed the mixed-Mode dynamic fracture problem for functionally graded piezoelectric strip containing a parallel crack under in-plane mechanical and electric impact loadings. Recently, Zhou and Chen [9] examined the interaction of two parallel Mode-I limited-permeable cracks in FGPMs. Common to all aforementioned studies, the use of classical continuum mechanics techniques is based on the local assumption to investigate crack problems in FGPMs. According to local elasticity theory, the state of stress at a specific point in the material depends only on the state of strain at the same point. Contrary to physical reasoning, the application of local elasticity theory invariably leads to stress singularities at the crack tips. A major issue here is that stress at the crack tips is indeterminate, and thus there, a fracture criterion based on maximum stress is not easy to establish. Different from classical local elasticity theory is the nonlocal elasticity theory which attempts to develop the constitutive relationships without foregoing the microstructure of the material. Nonlocal continuum mechanics initiated by Eringen

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[10] is based on the nonlocal elasticity model, where the state of stress at a given point is a function of the strain states at all points in the material. The nonlocal theory was employed to great success by Eringen [11] to investigate the stress near the tip of a sharp line crack in an isotropic elastic plate. Unlike classical (or local) elasticity theory, it was shown that the stress field calculated by using nonlocal theory does not contain any singularities at the crack tips. Therefore, nonlocal elasticity models can be used to obtain the stress field in the vicinity of the crack tips and establish a fracture criterion in a natural way. A major hindrance to studying crack problems in FGPMs via nonlocal elasticity models is the inherent mathematical difficulty associated with it. To the best knowledge of the authors, the electro-elastic behavior of FGPMs with a mixed-Mode electrically impermeable crack has not been studied using the non-local theory in open literature. Therefore, in this paper, the concept of Eringen's non-local theory is adopted to solve the mixed-Mode crack problem in a FGPM. The major emphasis is to calculate the stress and electric fields in FGPMs with mixed-Mode cracks through a formulation based on the non-local theory.

This paper is organized as follows. The formulation and the boundary conditions of the problem are presented in Section II. The solution methodology is described in Section III, where the mixed boundary value problem is reduced to a system of singular integral equations. Section IV contains the validation of results, parametric study, and a discussion of the results. Finally, concluding remarks are provided in Section V.

II. Problem Description and Formulation

The problem under consideration is shown in Fig. 1. The Cartesian coordinate system (x,z) is used for all the analysis presented here. The problem domain consists of a functionally graded piezoelectric medium extending infinitely in *x* and *z* directions, with an embedded crack of length *2l* oriented along the *x*-axis.

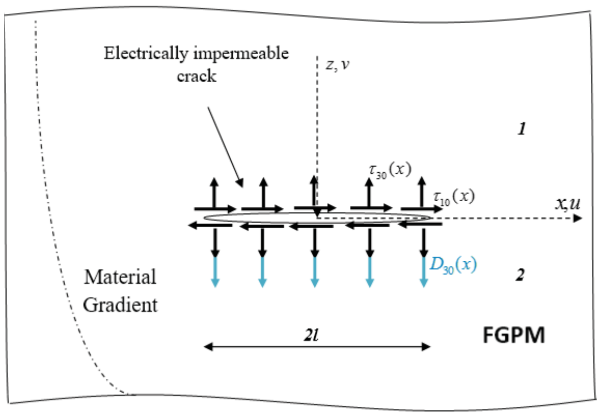

Fig. 1 Geometry and loading of the mixed-Mode crack problem

To make the problem analytically tractable and as is common in the treatment of crack problems for isotropic non-homogeneous materials [5], we assume that the material gradient is in the *z*-direction and the electro-mechanical properties depend on *z* as:

$$\left(c'_{11},c'_{13},c'_{33},c'_{44}\right)(z)=\left(c_{11},c_{13},c_{33},c_{44}\right)e^{\beta z}, \qquad \forall z, \tag{1}$$

$$\left(e'_{15},e'_{31},e'_{33}\right)(z)=\left(e_{15},e_{31},e_{33}\right)e^{\beta z}, \qquad \forall z, \tag{2}$$

$$\left(\varepsilon'_{11},\varepsilon'_{33}\right)(z)=\left(\varepsilon_{11},\varepsilon_{33}\right)e^{\beta z}, \qquad \forall z, \tag{3}$$

where c'_{ij}, e'_{ij}, and ε'_{jj} are respectively the shear moduli, the piezoelectric coefficients and the dielectric parameters, of FGPM. Their corresponding values in the FGPM medium along the crack plane are given by c_{ij}, e_{ij}, and ε_{jj}. The subscripts *i* and *j* indicate the indices used in (1)-(3). Here, β denotes the nonhomogeneity parameter that controls the variation of properties in the functionally graded piezoelectric medium. When $\beta=0$ the problem reduces to a homogenous piezoelectric material case. Mechanical loadings (τ_{30},τ_{10}) in the form of tangential and normal traction are applied on the FGPM. In addition, electrical loading (D_{30}) in the form of electric field is applied. The crack surface traction loading can be obtained by using the superposition principle from a Neumann boundary value problem consisting of the FGPM medium without cracks subject to far-field mechanical and electric loading [12]. The crack surfaces are assumed to be electrically impermeable. The *x* and *z* components of the displacement field are denoted by *u* and *v*, respectively, while the electric potential is given by ϕ. Based on the classical (or local) elasticity theory, the constitutive equations involving the strain-displacement relationships, the linear elastic stress-strain law and the general electro-elastic interaction for continuously nonhomogeneous media are given by [13]

$$\varepsilon = L\,U, \tag{4}$$

where $\varepsilon=\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{33}\\ \varepsilon_{13}\end{bmatrix}, U=\begin{bmatrix}u\\ v\end{bmatrix}, L=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x}\end{bmatrix}$, in which ε_{11}, ε_{33} and ε_{13} are the components of the local strain field in the coordinate system (x,z).

$$\sigma = C_{\sigma}\begin{bmatrix}\varepsilon\\ E\end{bmatrix}, \tag{5}$$

where $\sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{33} \\ \sigma_{13} \\ D_1^C \\ D_3^C \end{bmatrix}, E = \begin{bmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial z} \end{bmatrix}$, $C_\sigma = \begin{bmatrix} c'_{11} & c'_{13} & 0 & 0 & e'_{31} \\ c'_{13} & c'_{33} & 0 & 0 & e'_{33} \\ 0 & 0 & c'_{44} & e'_{15} & 0 \\ 0 & 0 & e'_{15} & -\varepsilon'_{11} & 0 \\ e'_{31} & e'_{33} & 0 & 0 & -\varepsilon'_{33} \end{bmatrix}$, in which σ_{11}, σ_{33}, and σ_{13} are the components of the local stress field in the coordinate system (x,z). The x and z components of the local electric displacement field are given by D_1^C and D_3^C, respectively.

Here, we adopt Eringen's nonlocal elasticity theory and reformulate (5) to obtain the constitutive relations in the nonlocal framework. According to the nonlocal theory, the stress at a point X in a body depends not only on the strain at point X but also on those at all other points of the body [10]. Accordingly, the constitutive relations can be expressed in an integral form as shown as [13]:

$$\left[\tau_{ik}, D_k\right](X) = \int_V \alpha\left(\left|X - X'\right|\right)\left[\sigma_{ik}, D_k^C\right](X')\,dV(X'), \tag{6}$$

where wn, in nonlocal elasticity theory can be expressed in an integral form here τ_{ik}, D_k (with $i,k = 1,3$) are the nonlocal stress tensor and electric displacement fields, respectively, defined at a point X in the body. It is easily noted that (6) relates the nonlocal quantities to their corresponding local components σ_{ik}, D_k^C defined in (5). Here, $\alpha\left(\left|X - X'\right|\right)$ is a nonlocal kernel also called the influence function that introduces the effect of strain and electric fields at points X' to the stress and electric displacement at point X. This influence function expression is given by

$$\alpha\left(|X - X'|\right) = \alpha_0\, exp\left\{-\left(\frac{\delta}{a}\right)^2\left[\left(x - x'\right)^2 + \left(z - z'\right)^2\right]\right\}. \tag{7}$$

where a is taken as the lattice parameter of the material and $\alpha_0 = \dfrac{1}{\pi}\left(\dfrac{\delta}{a}\right)^2$. Here, δ is an external characteristic length.

Neglecting body forces and local electric charge, the following electro-elasticity equations in the graded medium can be easily derived [14]:

$$\begin{aligned} &\left(c_{11}\frac{\partial^2}{\partial x^2} + c_{44}\left(\frac{\partial^2}{\partial z^2} + \beta\frac{\partial}{\partial z}\right)\right)u(x,z) + \left(c_{13}\frac{\partial^2}{\partial x \partial z} + c_{44}\left(\frac{\partial^2}{\partial x \partial z} + \beta\frac{\partial}{\partial x}\right)\right)v(x,z) \\ &+\left(e_{31}\frac{\partial^2}{\partial x \partial z} + e_{15}\left(\frac{\partial^2}{\partial x \partial z} + \beta\frac{\partial}{\partial x}\right)\right)\phi(x,z) = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} &\left(c_{44}\frac{\partial^2}{\partial x \partial z} + c_{13}\left(\frac{\partial^2}{\partial x \partial z} + \beta\frac{\partial}{\partial x}\right)\right)u(x,z) + \left(c_{44}\frac{\partial^2}{\partial x^2} + c_{33}\left(\frac{\partial^2}{\partial z^2} + \beta\frac{\partial}{\partial z}\right)\right)v(x,z) \\ &+\left(e_{15}\frac{\partial^2}{\partial x^2} + e_{33}\left(\frac{\partial^2}{\partial z^2} + \beta\frac{\partial}{\partial z}\right)\right)\phi(x,z) = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &\left(e_{15}\frac{\partial^2}{\partial x \partial z} + e_{31}\left(\frac{\partial^2}{\partial x \partial z} + \beta\frac{\partial}{\partial x}\right)\right)u(x,z) + \left(e_{15}\frac{\partial^2}{\partial x^2} + e_{33}\left(\frac{\partial^2}{\partial z^2} + \beta\frac{\partial}{\partial z}\right)\right)v(x,z) \\ &+\left(-\varepsilon_{11}\frac{\partial^2}{\partial x^2} - \varepsilon_{33}\left(\frac{\partial^2}{\partial z^2} + \beta\frac{\partial}{\partial z}\right)\right)\phi(x,z) = 0. \end{aligned} \tag{10}$$

For the mixed-Mode crack (Mode-I and II), the electro-elasticity equations (8)-(10) in the graded material are subject to the following boundary conditions:

$$\left[\tau_{33}, \tau_{13}, D_3\right]\left(x, 0^{\pm}\right) = \left[\tau_{30}, \tau_{10}, D_{30}\right](x), \qquad |x| \le l, \tag{11}$$

$$\left[\tau_{33}, \tau_{13}, D_3\right]\left(x, 0^{+}\right) = \left[\tau_{33}, \tau_{13}, D_3\right]\left(x, 0^{-}\right), \qquad \forall x, \tag{12}$$

$$\left[u, v, \phi\right]\left(x, 0^{+}\right) = \left[u, v, \phi\right]\left(x, 0^{-}\right), \qquad |x| \ge l, \tag{13}$$

$$\left[u, v, \phi\right](x, z) = 0, \qquad \forall x, z \to \pm\infty, \tag{14}$$

The applied electro-mechanical loadings on the crack faces are given in (11). The continuity condition for stresses and electric displacement along the crack plane is indicated in (12). Equations (13) describes the continuity of the displacement field and the electric potential along the crack plane outside the crack. Also, regularity conditions given by (14) require that the displacement and electric field remain bounded and therefore must vanish as x, z goes to $\pm\infty$.

III. Solution Procedure of the Embedded Crack Problem

The electro-elasticity equation (8)-(10) is a system of partial differential equations (PDEs) that need to be solved to determine the unknown variables, namely, the displacement field (u,v) and the electric scalar potential (ϕ). The standard Fourier transform is applied with respect to the x-coordinate as

$$\left[u, v, \phi\right](x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\bar{u}, \bar{v}, \bar{\phi}\right](s, z)\, e^{isx}\, ds. \tag{15}$$

Here, s is the Fourier transform variable and $\bar{u}$, $\bar{v}$ and $\bar{\phi}$ represent the transformed unknown variables. This transformation reduces the PDEs given by (8)-(10) to a system of sixth-order ordinary differential equations (ODEs) with $\bar{u}$, $\bar{v}$ and $\bar{\phi}$ as the dependent and z as the independent variables. The resulting ODEs can be solved in a straightforward manner using well-known mathematical techniques [15]. The sixth-order characteristic polynomial associated with the ODE is given as

$$X^3 + a_1 X + a_0 = 0, \text{ with } X = m^2 + \beta m + a_2, \tag{16}$$

where the coefficients a_0, a_1 and a_2 depend only on the electro-elastic constants.

Employing the Fourier transform, the mixed-boundary value problem is converted into the following three integral equations, with the unknown variables being the jumps of the physical fields across the crack surfaces:

$$\begin{aligned} f_1(x) &= u^{(1)}(x,0) - u^{(2)}(x,0), \\ f_2(x) &= v^{(1)}(x,0) - v^{(2)}(x,0), \\ f_3(x) &= \phi^{(1)}(x,0) - \phi^{(2)}(x,0), \end{aligned} \tag{17}$$

where, the superscript *j=1, 2* is used to differentiate the solutions between that corresponding to the half-plane $z \geq 0$ and $z \leq 0$, respectively. Therefore, using (6), the non-local quantities can be expressed as

$$\begin{aligned} \left[\tau_{33},\tau_{13},D_3\right](x,z) &= \int_0^{\infty}\left[\int_{-\infty}^{\infty}\alpha(|X'-X|)\left[\sigma_{33}^{(1)},\sigma_{13}^{(1)},D_3^{C(1)}\right](x',z')dx'\right]dz' \\ &+ \int_{-\infty}^{0}\left[\int_{-\infty}^{\infty}\alpha(|X'-X|)\left[\sigma_{33}^{(2)},\sigma_{13}^{(2)},D_3^{C(2)}\right](x',z')dx'\right]dz'. \end{aligned} \tag{18}$$

Here, the non-local quantities τ_{33}, τ_{13} and D_3 are expressed in terms of the unknown, $\bar{f}_j(s)$ $(j = 1..3)$, the Fourier transform of $f_i(\mathrm{x})$ $(\mathrm{i} = 1..3)$. After simplifications, the boundary conditions (11) can be applied to obtain the following:

$$\Lambda(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K_i(s) e^{isx} ds = \eta_0, \qquad |x| \leq l, \tag{19}$$

in which,

$$\Lambda = \begin{bmatrix} \tau_{33} \\ \tau_{13} \\ D_3 \end{bmatrix},\ K_i = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix},\ \eta_0 = \begin{bmatrix} \tau_{30} \\ \tau_{10} \\ D_{30} \end{bmatrix},$$

where the expressions of the kernels, K_1, K_2 and K_3 in the integral equations (19) are

$$K_i(s) = \bar{f}^T(s) D^*(s), \tag{20}$$

where,

$$\bar{f} = \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{bmatrix} \text{ and } D^* = \begin{bmatrix} d_1^* & e_1^* & f_1^* \\ d_2^* & e_2^* & f_2^* \\ d_3^* & e_3^* & f_3^* \end{bmatrix}.$$

Here, $d_i^*(s)$, $e_i^*(s)$ and $f_i^*(s)$ $(i = 1..3)$ are known functions which depend on the material properties.

Now, the mixed-Mode crack problem has been reduced to a system of three integral equations given (19) that can be solved to determine the unknown functions $\bar{f}_1(s)$, $\bar{f}_2(s)$, and $\bar{f}_3(s)$.

As the lattice parameter "*a*" tends to zero, the nonlocal theory approaches the classical (or local) theory. From a mathematical standpoint, as "*a*" tends to zero, the kernels of the three integral equations (19) $d_i^*(s)$, $e_i^*(s)$, and $f_i^*(s)$ reduce to a non-zero constant resulting in "singular" integral equations. These "singular" integral equations are the same as those obtained from the classical electro-elasticity theory. The important aspect is that when "*a*" is not zero, the integral equations are not singular as in the nonlocal case, but when "*a*" is equal to zero, the integral equations are singular as in the classical case. Due to this fundamental difference between the two cases, different methodologies are generally required to solve the singular and nonsingular integral equations. In classical electro-elasticity theory, the singularities must be extracted from the kernels of the singular integral equations and a solution in terms of orthogonal polynomials is sought [16]. In this case, the resulting stress field is singular and the results are in general given in terms of stress intensity factors that can be used as a fracture criterion based on linear elastic fracture mechanics. However, for the case of nonsingular integral equations, there are no singularities in the kernels and the stress field obtained is nonsingular. Unlike classical theory, the calculated stresses can potentially be used to establish a fracture criterion without the need for stress intensity factors.

The system given by (19) is solved by treating it as a single integral equation of the first kind with discontinuous kernel [13]. Integral equations of the first kind are generally ill-posed in the sense of Hadamard; i.e., small perturbations of the data can yield arbitrarily large changes in the solution [13] and hence a numerical solution is often difficult. Zhou et al. [13] showed that this difficulty can be overcome by using the Schmidt method ([17], [18]). Therefore, the Schmidt method is employed here to solve the system of three integral equations (19). The jumps of mechanical displacements and electric potential are expanded by the following series:

$$[f_1, f_2, f_3](x) = \begin{cases} \sum_{n=0}^{\infty}[a_n, b_n, c_n] P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\frac{x}{l}\right)\left(1-\frac{x^2}{l^2}\right)^{\frac{1}{2}}, & |x| \leq l, \\ 0, & |x| > l, \end{cases} \tag{21}$$

where a_n, b_n, and c_n are unknown coefficients, and $P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)$ is a Jacobi polynomial [19].

For the domain $|x| \leq l$, where $f_i(x)$ $(i = 1..3)$ is nonzero, the Fourier transforms of (21) can be written as [20]

$$\left[\bar{f}_1, \bar{f}_2, \bar{f}_3\right](s) = \sum_{n=0}^{\infty}[a_n, b_n, c_n] Q_n \frac{1}{s} J_{n+1}(sl), \tag{22}$$

where $Q_n = 2\sqrt{\pi}(-1)^n i^n \Gamma(n+1+1/2)/n!$. Here, $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions of order *n*, respectively.

By substituting (22) in (20), we rewrite (19) to obtain the following system of equations which can now be solved for the coefficients a_n, b_n, and c_n by the Schmidt method:

$$\sum_{n=0}^{\infty}\Gamma_n^T(x) \mathrm{A}_n = \Re_0(x), \qquad -l \leq x \leq l, \tag{23}$$

in which,

$$\Gamma_n = \begin{bmatrix} E_n^{(1)} & E_n^{(2)} & E_n^{(3)} \\ F_n^{(1)} & F_n^{(2)} & F_n^{(3)} \\ G_n^{(1)} & G_n^{(2)} & G_n^{(3)} \end{bmatrix},\ \Re_0 = \begin{bmatrix} U_0 \\ V_0 \\ W_0 \end{bmatrix},\ \mathrm{A}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix},$$

where $E_n^{(j)}, F_n^{(j)}, G_n^{(j)}, U_0, V_0,$ and W_0 $(j=1..3)$ are known functions.

IV. Results and Discussions

The mechanical stresses (τ_{13}, τ_{33}) and electric displacement (D_3) in the medium are evaluated by determining the unknown coefficients $a_n, b_n,$ and c_n from (23) as explained in section III. Of particular interest is the calculation of the mechanical stresses and electric displacements in the vicinity of the crack tips, and determining their variation as a function of the functionally graded parameter (β), lattice parameter (a) and crack length (l). PZT-5H is used as a model material for the calculations presented here, properties of which are provided in Table I based on the data in [13].

TABLE I
Material Properties of PZT-5H Based on [13]

$c_{11} = 12.6E10Nm^{-2}$, $c_{13} = 5.3E10Nm^{-2}$, $c_{33} = 11.7E10Nm^{-2}$,
$c_{44} = 3.53E10Nm^{-2}$, $e_{15} = 17Cm^{-2}$, $e_{31} = -6.5Cm^{-2}$, $e_{33} = 23.3Cm^{-2}$,
$\varepsilon_{11} = 15.1E-9C^2N^{-1}m^{-2}$, $\varepsilon_{33} = 13E-9C^2N^{-1}m^{-2}$,

A. Results Validation

In order to validate the solution methodology adopted, a particular case of a homogeneous medium (i.e. with a small value for non-homogeneity parameter, β) and vanishing piezoelectric coefficient is considered first. Under these conditions, the present problem reduces to the one studied by Zhou et al [21]. Referring to Fig. 4 in [21], the normalized stress value near the crack tip, for $a/\delta l = 0.002$ is approximately 14.25 for β tending to zero. For the same case, from the present formulation a value of 14.199 is obtained for the normalized stress (τ_{33}/τ_0) which is in excellent agreement with [21]. As a second case of validation, by considering vanishing piezoelectric coefficient in the present work, and assuming the second and third boundary conditions in (11) go to zero, a comparison can be made with results in [22]. Fig. 2 shows the comparison of the normalized normal stress, $\tau_{33}(x,0)/\tau_0$ for a crack embedded in an infinite functionally graded medium subjected to uniform normal surface traction τ_0 with $a/\delta l = 0.001$, $l = 1.0$, and $\beta = 1.0$. The results are in excellent agreement with [22].

B. Parametric Study

(i) Figs. 3 (a)-(c) show the variation of normalized normal stress, tangential stress, and electric displacement with x, and it is clear that all quantities have finite values at $x = \pm l$ (crack tips). Also, all quantities of interest are asymmetric across $x = 0$ which is a result of the application of tangential traction on the crack as expected in a mixed-Mode crack scenario.

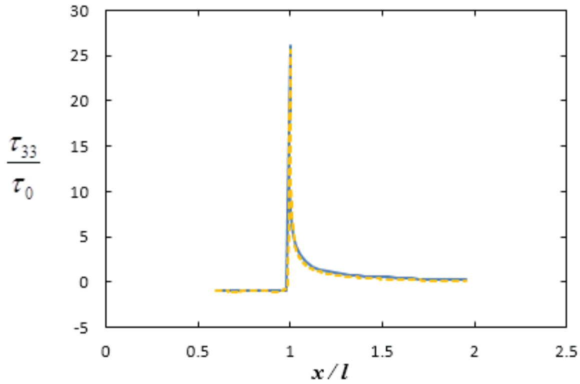

Fig. 2 (a) Comparison of the normalized normal stress, $\tau_{33}(x,0)/\tau_0$ obtained from the present work (solid line) with that of Liang [22] (dashed line). Results are shown for a case of a crack embedded in an infinite functionally graded medium subjected to uniform normal crack surface traction τ_0 with $a/\delta l = 0.001$, $l = 1.0$ and $\beta = 1.0$

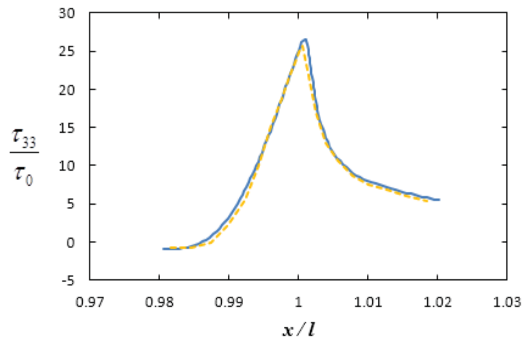

Fig. 2 (b) Locally enlarged graph of Fig. 2 (a)

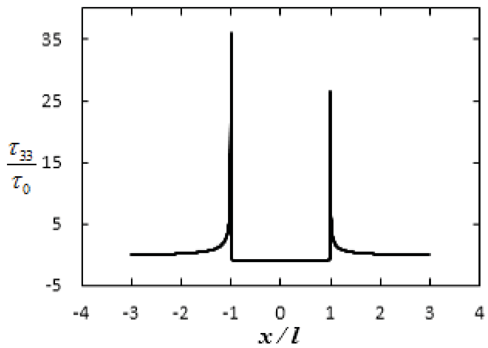

Fig. 3 (a) Variation of the normal stress $\tau_{33}(x,0)/\tau_0$, with x for $a/\delta l = 0.001$, $l = 1.0$, and $\beta = 1.0$ under uniform combined mechanical and electric loading

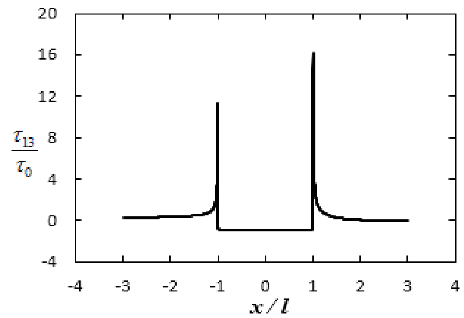

Fig. 3 (b) Variation of the tangential stress $\tau_{13}(x,0)/\tau_0$, with x for $a/\delta l = 0.001$, $l = 1.0$, and $\beta = 1.0$ under uniform combined mechanical and electric loading

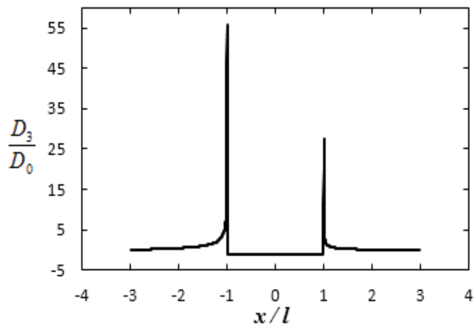

Fig. 3 (c) Variation of the electric displacement $D_3(x,0)/D_0$ with x for $a/\delta l = 0.001$, $l = 1.0$, and $\beta = 1.0$ under uniform combined mechanical and electric loading

(ii) The normal stress and the electric displacement fields near the inner (left) crack tips are larger than the ones near the outer (right) crack tips as shown in Figs. 3 (a) and (c). On the other hand, the tangential stress field is smaller at the inner crack tip as shown in Fig. 3 (b). This asymmetry is due to the presence of horizontal mechanical loadings on the crack surfaces.

(iii) A particularly interesting observation is that the maximum stress and electric displacement do not occur exactly at the crack tip, but in its immediate vicinity as shown in Figs. 3 (a) and (b). Similar observations were drawn by [23]. It is also noted that the distance between the crack tip and the point at which the maximum occurs is small and depends on the crack length and the lattice parameter. Another observation is that the values of stresses and electric displacement reduce quite rapidly away from the crack tips and approach zero as $x \to \infty$. Away from the crack tips, this behavior is qualitatively similar to results obtained via classical methods.

(iv) The normalized stresses and electric displacement fields at the crack tips tend to decrease with increase in the lattice parameter as shown in Figs. 5 (a)-(c). Therefore, it can be concluded that FGPMs characterized by higher lattice parameters are more resilient to fracture.

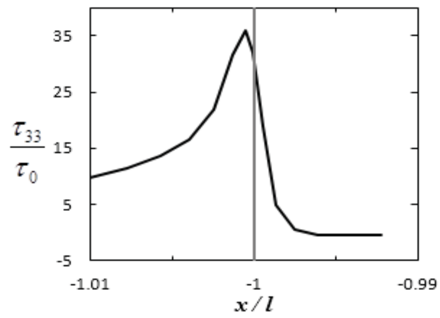

Fig. 4 (a) Locally enlarged graph of Fig. 3 (a) near the crack left tip

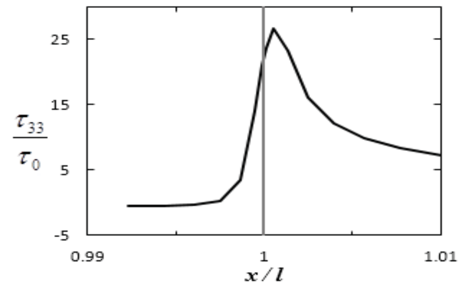

Fig. 4 (b) Locally enlarged graph of Fig. 3 (a) near the crack right tip

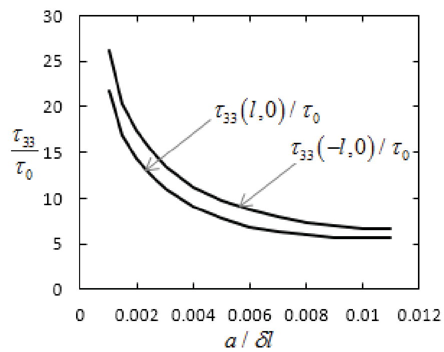

Fig. 5 (a) Effect of the lattice parameter a on the normal stress $\tau_{33}(\pm l,0)/\tau_0$ along the crack line for $l = 1.0$ and $\beta = 0.4$ under uniform combined mechanical and electric loading

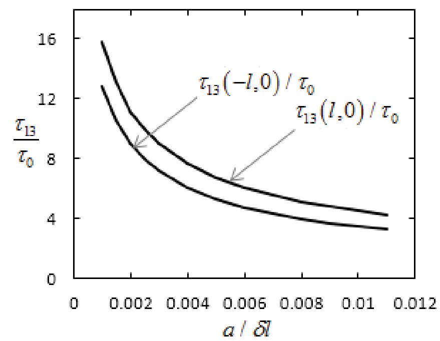

Fig. 5 (b) Effect of the lattice parameter a on the tangential stress $\tau_{13}(\pm l,0)/\tau_0$ along the crack line for $l = 1.0$ and $\beta = 0.4$ under uniform combined mechanical and electric loading

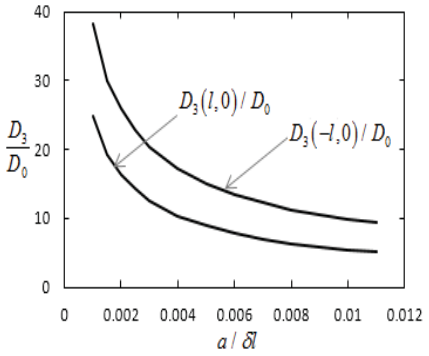

Fig. 5 (c) Effect of the lattice parameter *a* on the electric displacement $D_3(\pm l,0)/D_0$ along the crack line for $l=1.0$ and $\beta=0.4$ under uniform combined mechanical and electric loading

(v) For the case of $a \neq 0$, at the crack tips, finite values of field quantities are obtained as seen from Fig. 5. Also, it is clear that as "*a*" tends to zero, the values of the field quantities increase rapidly with decreasing values of "*a*". In the classical limit when $a=0$, it is known that the stresses and electric displacements are singular at the crack tips and the observations in Fig. 5 provide a good qualitative agreement with the classical theory.

(vi) With a/δ held constant, the values of the normalized stresses and electric displacement at the crack tip increase with increasing crack length (*l*) as shown in Fig. 6.

(vii) It is evident from the discussion in (iv) and (vi) that the maximum stress depends on the lattice parameter (*a*) (which is an internal length scale related to the microstructure of the material) and the crack length (*l*) (which is an external length scale related at the macroscopic level). Hence, it can be concluded that the maximum stresses calculated through Eringen's nonlocal formalism enable the unification of macroscopic and microscopic scales.

(viii) From Figs. 7 (a) and (c), it is clear that for $\beta > 0$, the normalized stresses and electric displacement at the crack tips tend to increase with increasing gradient parameter. Also, for $\beta < 0$, the values increase with reduction of the gradient parameter. The minimum values are attained close to $\beta = 0$. Unlike normal stress and electric displacement, the tangential stress fields vary almost linearly with the gradient parameter (Fig. 7 (b)). Therefore, the gradient parameter (β) can be used to control the stress and electric displacement fields at the crack tips.

(ix) As seen from Fig. 3 through 7, Eringen's nonlocal theory provides a maximum value for the stresses and electric displacement when studying mixed-Mode cracks in FGPMs. The distribution of the field variables is devoid of singularities at the crack tips which is in direct contrast to what is expected from classical (based on local assumption) and micropolar [24] continuum theories. The indeterminate nature of stresses at the crack tips is a major drawback of these theories, making it difficult to attribute a unique value for the maximum stress in the material in the presence of cracks. Therefore, both classical and micropolar continuum theories fail to establish a brittle fracture criterion based on the maximum stress hypothesis. This prompted the investigators to consider other alternatives; for instance, the popular Griffith's criterion which is based on considerations of energy balance [25]. On the other hand, as maximum stresses can be evaluated in the neighborhood of the crack tips through Eringen's nonlocal theory, a fracture criterion based on maximum stress hypothesis can be established [10]. Eringen et al. [11] and Eringen [26] investigated, respectively, the Mode-I and Mode-II crack problems in an isotropic homogenous medium. For the Mode-I problem, Eringen et al. [11] stipulated a fracture criterion based on the maximum stress hypothesis which states that the crack becomes unstable when the calculated maximum normal stress τ_{yy} exceeds the cohesive strength τ_c of the atomic bonds. Similarly, for a Mode-II problem [26], the crack becomes unstable when the calculated shear stress τ_{xy} exceeds the cohesive strength τ_c. The cohesive strength holding the atomic bonds were calculated by introducing the experimentally measured values of surface energy ([26], [11]). Therefore, through the maximum stress hypothesis, a fracture criterion was established that unified the macro and micro scales in addition to employing the natural concept of bond failure ([26], [11]). It must be noted that that Eringen and his group established this criterion for homogeneous materials treating Mode-I and Mode-II cracks separately. On the other hand, smart materials such as FGPMs can also have electrical Modes of crack opening in addition to the mechanical Modes [27]. Furthermore, for mixed-Mode fracture of FGPMs, Eringen's maximum stress fracture criterion cannot be extended in a straightforward manner and needs further consideration and can form part of future investigations.

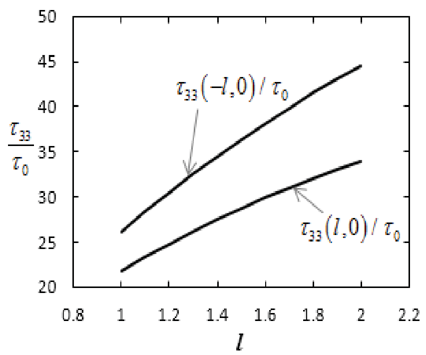

Fig. 6 (a) Effect of the length crack *l* on the normal stress $\tau_{33}(\pm l,0)/\tau_0$ along the crack line for $a/\delta l = 0.001$ and $\beta = 0.4$ under uniform combined mechanical and electric loading

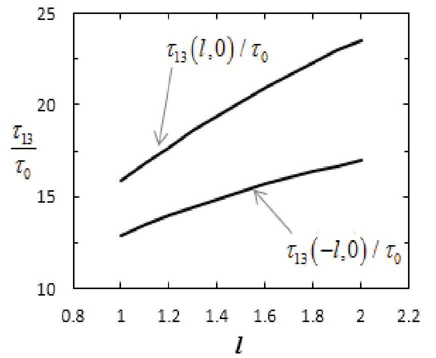

Fig. 6 (b) Effect of the length crack l on the tangential stress $\tau_{13}(\pm l,0)/\tau_0$ along the crack line for $a/\delta l=0.001$ and $\beta=0.4$ under uniform combined mechanical and electric loading

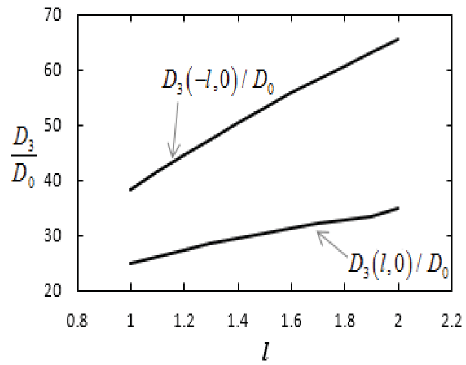

Fig. 6 (c) Effect of the length crack l on the electric displacement $D_3(\pm l,0)/D_0$ along the crack line for $a/\delta l=0.001$ and $\beta=0.4$ under uniform combined mechanical and electric loading

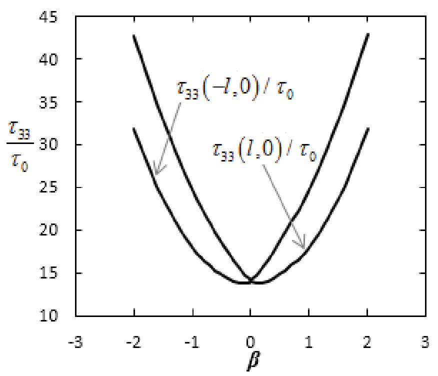

Fig. 7 (a) Effect of the functionally graded parameter β on the (a) normal stress $\tau_{33}(\pm l,0)/\tau_0$ along the crack line for $a/\delta l=0.002$ and $l=1.0$ under uniform combined mechanical and electric loading

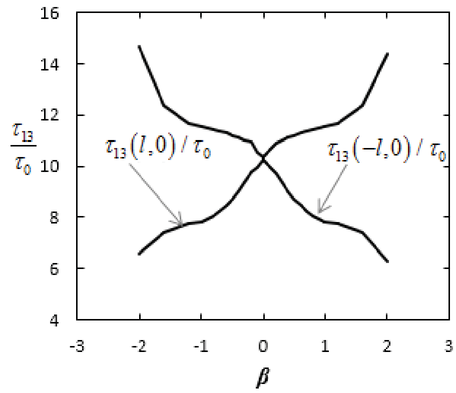

Fig. 7 (b) Effect of the functionally graded parameter β on the tangential stress $\tau_{13}(\pm l,0)/\tau_0$ along the crack line for $a/\delta l=0.002$ and $l=1.0$ under uniform combined mechanical and electric loading

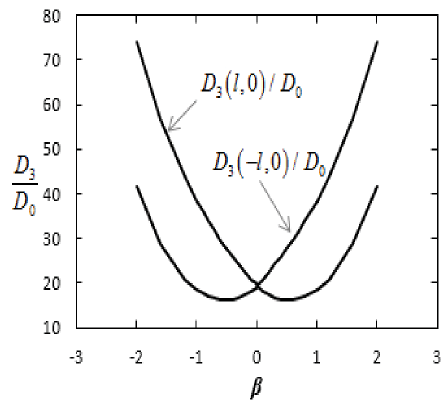

Fig. 7 (c) Effect of the functionally graded parameter β on the electric displacement $D_3(\pm l,0)/D_0$ along the crack line for $a/\delta l=0.002$ and $l=1.0$ under uniform combined mechanical and electric loading

V. Conclusion

In the present work, the mixed-Mode fracture problem of a FGPM has been solved within the framework of nonlocal continuum mechanics employing Eringen's nonlocal elasticity model. An electrically impermeable crack is embedded in an infinitely extending FGPM and is subject to electro-mechanical loadings. By using Fourier transforms, the governing equations are converted into a system of three integral equations, where the unknowns are the jumps of mechanical displacement and the electric potentials across the crack. The unknowns are expanded as a series of Jacobi polynomials to obtain a system of linear algebraic equations that are solved by using the Schmidt method. Results show that the stresses and electric displacement fields throughout the problem domain including the crack tips are devoid of any singularities unlike electro-elasticity fracture problems studied via classical (or local) elasticity theories. The present formulation using Eringen's nonlocal theory gives finite values for the maximum stress in the vicinity of crack tips – thus making it possible to employ the maximum stress hypothesis to establish a fracture criterion. It is also observed that the maximum stress and electric displacement near the crack tips depends on the functionally graded parameter, crack length and lattice parameter. Establishing a brittle fracture criterion for FGPMs based on the maximum stress hypothesis is a promising direction for research and can be pursued as part of future work.

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