International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:9, No:4, 2015

New Insight into Fluid Mechanics of Lorenz Equations

Yu-Kai Ting, Jia-Ying Tu, Chung-Chun Hsiao

Abstract**—**New physical insights into the nonlinear Lorenz equations related to flow resistance is discussed in this work. The chaotic dynamics related to Lorenz equations has been studied in many papers, which is due to the sensitivity of Lorenz equations to initial conditions and parameter uncertainties. However, the physical implication arising from Lorenz equations about convectional motion attracts little attention in the relevant literature. Therefore, as a first step to understand the related fluid mechanics of convectional motion, this paper derives the Lorenz equations again with different forced conditions in the model. Simulation work of the modified Lorenz equations without the viscosity or buoyancy force is discussed. The time-domain simulation results may imply that the states of the Lorenz equations are related to certain flow speed and flow resistance. The flow speed of the underlying fluid system increases as the flow resistance reduces. This observation would be helpful to analyze the coupling effects of different fluid parameters in a convectional model in future work.

Keywords**—**Galerkin method, Lorenz equations, Navier-Stokes equations.

I. Introduction

NAVIER-Stokes equations [1] are widely considered for describing the dynamic equilibrium of many fluid systems. The equations consider a control volume of a fluid system, which obeys the laws of conservation of energy, conservation of mass, and conservation of momentum and the continuity equation. The typical Navier-stokes equations are expressed by

$$\rho\left(\frac{\partial}{\partial t}\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla\mathbf{p}+\nabla\cdot\mathbf{T}+\mathbf{f}, \tag{1}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2}$$

where $\mathbf{v}(u, v, w)$ is the flow velocity field, ρ is the flow density, $\mathbf{p}$ is the pressure field, $\mathbf{T}$ is the component of stress tensors, and $\mathbf{f}$ represents the other external body forces. Equation (1) shows dynamic equilibrium of the forces, where the right-hand side of (1) includes the internal and external forces acting on the flow control volume, and (2) is the continuity equation. In the fluid mechanics literature, Navier-stokes equations are general to include many different forced conditions for establishing mathematical models of atmosphere system [2], ocean currents, flow in a pipe [3], and many other fluid systems. Often the Navier-stokes equations are expanded to a set of partial differentiation equations (PDEs) for further analysis work.

For example, E. N. Lorenz had made an attempt to derive the mathematical model for atmosphere convection and weather prediction [4], [5] based on the Navier-Stokes equations. In his model, it is expected that small-scale disturbances, such as breeze, would not influence the large-scale weather phenomena. However, Lorenz detected that the prediction or solutions of the atmosphere models considerably vary with small disturbances and slight changes in the initial conditions. This kind of small disturbances incurring considerable solution variations could not be omitted or simply be attributed to numerical computation problems.

In 1963, Lorenz used Barry Saltzman's models and methods [6] to explore the small perturbations issues. The fluid model was based on the Navier-Stokes and thermal diffusion equations to discuss about the natural convective motion [7]. In their work, the complex PDEs are transformed into three nonlinear ordinary differential equations (ODEs) via the Fourier and Galerkin methods as

$$\begin{aligned}\dot{x}&=-\sigma x+\sigma y,\\ \dot{y}&=-xz-y+\gamma x,\\ \dot{z}&=xy-\beta z.\end{aligned} \tag{3}$$

The three nonlinear ODEs are known as the Lorenz equations. In (3), x, y, and z are the state variables, and σ, β, and γ are the associated dynamic parameters. Time-domain simulation of the Lorenz equations exhibit irregular/non-periodic trajectories in a three dimensional state space. In addition, as the simulation time increased, two spirals appeared in the space. Thus, Lorenz showed that deterministic ODEs with small changes in the initial conditions eventually resulted in difficult-to-predict and nonperiodic solutions.

Because the ODE expression provides a concise and systematic foundation for dynamics analysis, in the last decade, many studies focused on stability, chaotic behavior, and synchronization control of Lorenz equations. However, not many studies trace back to the original fluid model in PDE form, to discuss the physical implication for the ODE parameters and state variables.

Therefore, as a first step to understand the physical meaning behind the Lorenz equations, this paper modifies the forced terms in the Saltzman's PDE model to examine the changes in the ODE model. Section II of this paper will introduce the derivation of the Lorenz equations in detail. Then, in Section III, the PDE based on the Saltzman's convection model are modified according to different forced conditions; thus, the

This work was supported by the Taiwan Ministry of Science and Technology, under grant MOST 102-2221-E-007-109-MY2 "Evaluation of Optimal Dynamic Substructuring Tests: Development and Application of Substructurability Theory".

Y. K. Ting and C. C. Hsiao are with the Department of Power Mechanical Engineering, National Tsing Hua University, Hsinchu 300, Taiwan (e-mail: yukeaily@hotmail.com, en54612002@hotmail.com)

J. Y. Tu is with the Department of Power Mechanical Engineering, National sing Hua University, Hsinchu 300, Taiwan (Tel.: +886-3-5742497; e-mail: jytu@pme.nthu.edu.tw)

ODE state variables can be analyzed individually. Simulation results of the modified Lorenz model are discussed in Section IV, showing the new physical insights as to flow resistance. Finally, the conclusion of work is provided in Section V.

II. Review of Derivation of Lorenz Equations

The derivation of Lorenz equations with reference to [4]-[6] is presented in this section; the parameters and variables involved in the derivation and in this paper are summarized in Table I. First, the convection model underlying the Lorenz equations is introduced. As shown in Fig. 1, the distance between the two parallel plates is H, and the origin of the coordinate system is defined at the center of the lower plate. Accordingly, d is the width between the z axis and the lateral boundary, and T_H and T_0 denote the boundary temperatures at $z = H$ and $z = 0$, respectively. In addition, the upper and lower plates are assumed to have free boundary conditions without stress; the lateral boundaries are dispensed, which means that the liquid is free-flow in the x-axis direction. On the basis of Navier-Stokes equations, the convection model considered by Saltzman and Lorenz further assumed that: (a) the y-axis dynamics is irrelevant and only the motion in the x-z plane is considered; (b) the fluid is incompressible despite of temperature variations; (c) the conductive state temperature changes linearly from $z = 0$ to $z = H$; (d) the dynamic equilibrium is independent of pressure.

TABLE I
NOTATION FOR DEVIATION OF LORENZ EQUATIONS

Symbol	Description
$\mathbf{v}$	flow velocity field
T	temperature
θ	temperature departure
ψ	stream function
g	acceleration of gravity
v	kinematic viscosity rate
α	thermal expansion rate
κ	thermal diffusion rate
k	wave number in the x direction
n	wave number in the z direction
H	distance between the plates
d	width
σ	Prandtl number
γ	Rayleigh number
R_ψ	residual function of stream function
R_θ	residual function of temperature departure function

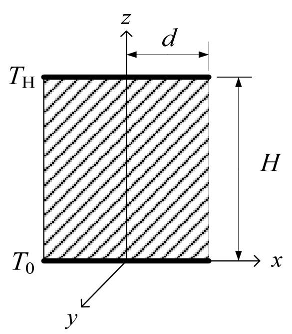

Fig. 1 Convection model in the x-z plane

According to the four assumptions and Fig. 1, the specified Laplace operator defined in this model is written as

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{4}$$

The y-direction terms are omitted according to the (a) assumption. Then, the convection models described by Saltzman and Lorenz are shown as

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla_1\mathbf{v} = v\nabla_1^2\mathbf{v} + \alpha g\left(T - T_0\right), \tag{5}$$

$$\nabla_1 \cdot \mathbf{v} = 0, \tag{6}$$

$$\frac{\partial}{\partial t}T + \mathbf{v}\cdot\nabla_1 T = \kappa\nabla_1^2 T. \tag{7}$$

Equation (5) is modified from (1) with the addition of buoyancy due to temperature, and the constant pressure field is omitted. The continuity of fluid is addressed in (6), and (7) shows the thermal diffusion equation to predict the thermal convection. In addition, κ, T, v, α, and g denote the thermal diffusion rate, temperature, kinematic viscosity rate, thermal expansion rate, and gravitational acceleration, respectively; see the notation summary in Table I. Furthermore, because of the (c) assumption, the temperature departure function, $\theta(x, z, t)$, is defined as

$$\theta = \left(T - T_0\right) - \frac{z}{H}\left(T_H - T_0\right). \tag{8}$$

Then, the associated variables are normalized though the factors [8]

$$x^* = \frac{x}{d},\ \mathbf{v}^* = \frac{d}{\kappa}\mathbf{v},\ t^* = \frac{\kappa}{d^2}t, \tag{9}$$

$$\theta^* = \frac{T - T_0}{T_H - T_0} - z^*. \tag{10}$$

Thus, x^*, $\mathbf{v}^*$, t^*, θ^*, and z^* are the dimensionless variables associated with x, $\mathbf{v}$, t, θ, and z, respectively. In the rest of this paper, the equations and variables are all dimensionless, and the asterisk symbol is omitted for brevity. With the substitution of (9) and (10) into (5)-(7), the normalized model are expanded and arranged to

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} = \dfrac{v}{\kappa}\nabla_1^2 u \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} = \dfrac{v}{\kappa}\nabla_1^2 w + \dfrac{\alpha g\left(T_0 - T_H\right)d^3}{\kappa^2}\theta \end{cases}, \tag{11}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{12}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = w + \nabla_1^2\theta, \tag{13}$$

where w is the dimensionless velocity along the z axis. From (12), a stream function is defined as $\psi(x, z, t)$, which satisfies

$$u = -\frac{\partial \psi}{\partial z}, \text{ and } w = \frac{\partial \psi}{\partial x}. \tag{14}$$

Substitution of (14) into (11)-(13) reduces the three equations to

$$\begin{aligned} \frac{\partial}{\partial t}\nabla_1^2\psi - \frac{\partial \psi}{\partial z}\frac{\partial}{\partial x}\nabla_1^2\psi + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\nabla_1^2\psi = \frac{\nu}{\kappa}\nabla_1^4\psi \\ + \frac{\alpha g\left(T_0 - T_H\right)d^3}{\kappa^2}\frac{\partial \theta}{\partial x}, \end{aligned} \tag{15}$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} = \nabla_1^2\theta + \frac{\partial \psi}{\partial x}. \tag{16}$$

To transform the PDEs of (15) and (16) into ODEs, the Fourier series expansion method is used, which assumes that the stream and temperature functions, $\psi(x, z, t)$ and $\theta(x, z, t)$, are given by

$$\psi\left(x,z,t\right) = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\psi_0 \exp\left[\pi Hi\left(\frac{k}{d}x + \frac{n}{H}z\right)\right], \tag{17}$$

$$\theta\left(x,z,t\right) = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\theta_0 \exp\left[\pi Hi\left(\frac{k}{d}x + \frac{n}{H}z\right)\right]. \tag{18}$$

In (17) and (18), k and n are the wave numbers in the x direction and z direction, respectively. In addition, $\theta_0 = \theta(x, z, 0)$ and $\psi_0 = \psi(x, z, 0)$ are the initial conditions. Expand (17) and (18) and preserve the first-order terms, (17) and (18) become

$$\psi\left(x,z,t\right) = a\left(t\right)\sin\left(k\pi x\right)\sin\left(\pi z\right), \tag{19}$$

$$\theta\left(x,z,t\right) = b\left(t\right)\cos\left(k\pi x\right)\sin\left(\pi z\right) - c\left(t\right)\sin\left(2\pi z\right), \tag{20}$$

where $a(t)$, $b(t)$, and $c(t)$ are the time-dependent functions. Then, the Galerkin method [9] is used to solve $a(t)$, $b(t)$, and $c(t)$ in Appendix, to yield the Lorenz equations in (3). In [5], Lorenz stated that the new states variables, x, y, z in (3) are proportional to the intensity of the convection motion, the temperature difference of ascending and descending flows at the two ends along the x axis, and the distortion of the vertical temperature, respectively. However, the detailed explanations and verification are not provided.

III. Modification of Lorenz Equation

This section changes the forced conditions in (5) and (7) to obtain different form of modified Lorenz equations; the force terms are removed step by step. It is expected that the state evolution in relation to the different forced condition may show some helpful hints on the physical implication about the state variables. First, the buoyancy term in the right-hand side of Navier-stokes equation is removed; thus (5) becomes

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla_1\mathbf{v} = \nu\nabla_1^2\mathbf{v}. \tag{21}$$

Second, when the viscosity force is omitted, the Navier-stokes equations is reduced to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \alpha g\left(T - T_0\right). \tag{22}$$

Equations (21) and (22) describe different fluid models induced by viscosity and buoyancy, respectively, and (6) and (7) are unchanged. With a similar approach to that introduced in Section II, the two modified Lorenz models related to viscosity and buoyancy are derived as:
Viscosity + diffusion:

$$\begin{aligned} \dot{x} &= -\sigma x \\ \dot{y} &= -xz - y + \gamma x. \\ \dot{z} &= xy - \beta z \end{aligned} \tag{23}$$

Buoyancy + diffusion:

$$\begin{aligned} \dot{x} &= \sigma y \\ \dot{y} &= -xz - y + \gamma x. \\ \dot{z} &= xy - \beta z, \end{aligned} \tag{24}$$

Equations (23) and (24) show that only the terms related to the $\dot{x}$ dynamics are changed, in comparison with that in (3); the remaining equations related to $\dot{y}$ and $\dot{z}$ are not affected.

Furthermore, modification of the thermal diffusion equation in (7) is discussed. With the removal of the diffusion term, (7) becomes

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = 0, \tag{25}$$

and the Navier-stokes and continuity equations are unchanged. Thus, (25) leads to the following modified Lorenz equations:
Viscosity + buoyancy

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz. \\ \dot{z} &= xy \end{aligned} \tag{26}$$

In this model, the equations related to $\dot{y}$ and $\dot{z}$ are different from those in (3), and the $\dot{x}$ equation is unchanged. In this section, three different form of Lorenz equations are proposed; simulation studies will be performed in the next section.

IV. Simulation Studies

Although three modified Lorenz equations with respect to (23), (24), and (26) are presented, only the (23) and (24) cases are discussed in this section to focus on the new observation. The chosen parameters for the simulation studies were $\sigma = 10$, $r = 28$, and $\beta = 8 / 3$, with reference to [5], [6]. The initial conditions of $(x, y, z) = (30, 0, 0)$ were considered, and the simulation work was conducted in the Matlab and Simulink environments, with the simulation time set to three seconds. At this stage, we assume that the physical meaning of x, y and z are uncertain. However, the time responses of the states in relation to the forced conditions may give some hints about the physical meaning of x, y and z.

ISSN: 2517-9950
Vol:9, No:4, 2015

First, the simulation results of (23), the viscosity + diffusion case, are discussed. The numerical simulations show that the time responses of $x(t)$, $y(t)$ and $z(t)$ finally settled down to zero. This means that with the removal of the buoyancy force, the chaotic dynamics do not happen. Furthermore, the simulation results of the buoyancy + diffusion case in (24) are drawn in Fig. 3. When the viscosity force was removed from the convectional model, only $y(t)$ and $z(t)$ gradually settled down to zero or a constant value, whereas $x(t)$ is diverged and could be unstable and lead to chaotic responses. In addition, the time responses of $x(t)$, $y(t)$, and $z(t)$ are arranged into a three-dimensional plot in Fig. 2, which interesting shows a vortex pattern.

From the comparison of (23) and (24), it is envisaged that the chaotic behavior is incurred by the buoyancy force, and the viscosity force dissipates the flow energy. In addition, Fig. 3 shows that as $x(t)$ increases, $y(t)$ and $z(t)$ decrease. Therefore, it is envisaged that $y(t)$ or $z(t)$ is related to the flow resistance, and $x(t)$ is related to the flow velocity. In other words, because the viscosity force is approximately zero, the flow resistance reduces and the flow velocity increases with time. Further analysis about the physical meaning and long-time simulation of the states is the authors' ongoing work.

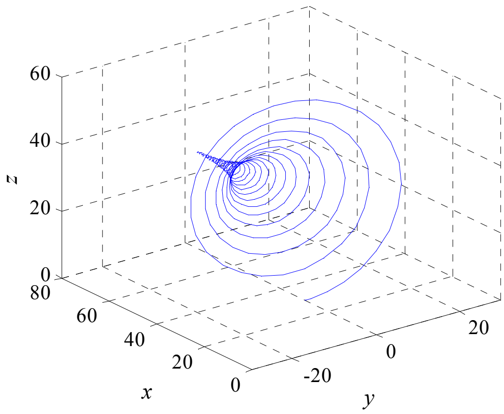

Fig. 2 Three-dimensional plot of simulation result of the modified Lorenz equations without the viscosity term

V. Conclusions

The physical motion of Lorenz equations is discussed in this paper. First of all, the derivation of Lorenz equations from the main literature is introduced. Then, the forced conditions of the fluid convectional model are modified in order to compare and examine the state response in relation to different forced conditions. Here, two cases related to the viscosity and buoyancy forces are considered, and simulation studies of the modified models are presented. The results imply that the chaotic dynamics is incurred by the buoyancy force, and the viscosity force tends to dissipate flow energy. In addition, it is envisaged that the x state is associated with the flow velocity and the y or z states is related to the flow resistance dynamics. Further work regarding the exact physical meaning of the state variables comprises the authors' ongoing work.

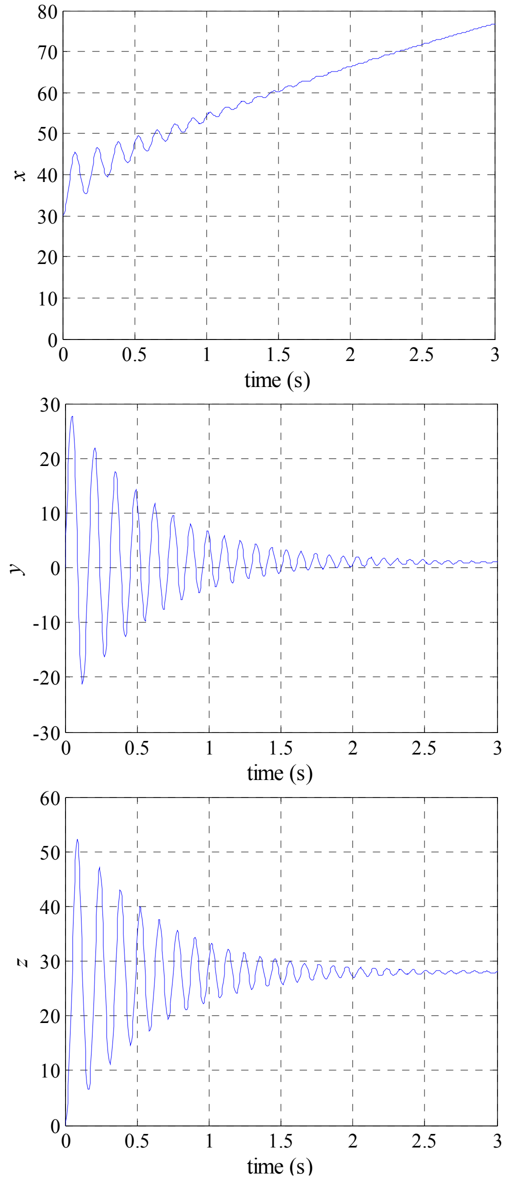

Fig. 3 Simulation results of the modified Lorenz equations without the viscosity term

Appendix

The Galerkin method in [5], [9] is introduced. First, (15) and (16) are re-arranged to the residual functions, R_θ and R_ψ, as

$$\begin{aligned}R_\psi(x,z,t) = {} & \frac{\partial}{\partial t}\nabla_1^2\psi - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\nabla_1^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\nabla_1^2\psi \\ & -\frac{\nu}{\kappa}\nabla_1^4\psi - \frac{\alpha g\left(T_0 - T_H\right)d^3}{\kappa^2}\frac{\partial\theta}{\partial x},\end{aligned} \tag{27}$$

$$R_\theta\left(x,z,t\right) = \frac{\partial\theta}{\partial t} - \frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z} - \nabla_1^2\theta - \frac{\partial\psi}{\partial x}. \tag{28}$$

The residual functions are given by substituting (19) and (20) into (27) and (28). The stream function ψ is considered zero because the vertical and horizontal velocities vanish at the free boundaries; in addition, the temperature departure θ is zero at the upper and lower plates. Then, apply the integration, define the boundary conditions, the three simultaneous equations related to $a(t)$, $b(t)$, and $c(t)$ yield

$$\begin{aligned}\frac{\partial}{\partial t}a(t)=&-\frac{\nu}{\kappa}\pi^2\left(k^2+1\right)a(t)\\&-\frac{\alpha g\left(T_0-T_H\right)d^3}{\kappa^2}\frac{k\pi}{\pi^2\left(k^2+1\right)}b(t),\end{aligned}\tag{29}$$

$$\frac{\partial}{\partial t}b(t)=-k\pi^2a(t)c(t)-\pi^2\left(k^2+1\right)b(t)-k\pi a(t),\tag{30}$$

$$\frac{\partial}{\partial t}c(t)=\frac{1}{2}k\pi^2a(t)b(t)-4\pi^2c(t).\tag{31}$$

With separation of variable for (29)-(31), the new variables are defined as

$$\begin{aligned}t&=\frac{\tau}{\pi^2\left(k^2+1\right)}\\a(t)&=\frac{\sqrt{2}\left(k^2+1\right)}{k}x(\tau)\\b(t)&=-\frac{\sqrt{2}\pi^3\left(k^2+1\right)^3\nu\kappa}{k^2\alpha g\left(T_0-T_H\right)d^3}y(\tau)\\c(t)&=-\frac{\pi^3\left(k^2+1\right)^3\nu\kappa}{k^2\alpha g\left(T_0-T_H\right)d^3}z(\tau),\end{aligned}\tag{32}$$

The variables, $a(t)$, $b(t)$, and $c(t)$, in (29)-(31) are replaced by (32); thus, (29)-(31) are transformed into the Lorenz equations in (3), where the parameters σ, γ, and β are given by

$$\sigma=\nu\cdot\frac{1}{\kappa},\tag{33}$$

$$\gamma=\frac{k^2}{\pi^4\left(k^2+1\right)^3}\frac{\alpha g\left(T_0-T_H\right)d^3}{\nu\kappa}=\frac{k^2}{\pi^4\left(k^2+1\right)^3}Ra,\tag{34}$$

$$\beta=\frac{4}{k^2+1}.\tag{35}$$

This completes the derivation of Lorenz equations.

ACKNOWLEDGMENT

The authors gratefully acknowledge the support of the Taiwan Ministry of Science and Technology, under grant MOST 102-2221-E-007-109-MY2 “Evaluation of Optimal Dynamic Substructuring Tests: Development and Application of Substructurability Theory”, for their support in the pursuance of this work.

REFERENCES

[1] B. R. Munson, *Fluid Mechanics*. Singapore: Wiley Pte. Ltd., 2013.

[2] A. Shapiro, "The use of an exact solution of the Navier-Stokes equations in a validation test of a three-dimensional nonhydrostatic numerical model," *Monthly Weather Review,* vol. 121, pp. 2420-5, 08/ 1993.

[3] L. Jirkovsky and A. Muriel, "Pipe flow and wall Turbulence Using a Modified Navier-Stokes Equation," *Communications in Theoretical Physics,* vol. 57, pp. 477-81, 03/ 2012.

[4] E. N. Lorenz, "Maximum simplification of the dynamic equations," *Tellus,* vol. 12, pp. 243-254, 1960.

[5] E. N. Lorenz, "Deterministic nonperiodic flow," *Journal of the Atmospheric Sciences,* vol. 20, pp. 130-141, 1963.

[6] B. Saltzman, "Finite amplitude free convection as an initial value problem—I," *Journal of the Atmospheric Sciences,* vol. 19, pp. 329-341, 1962.

[7] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. New York: Dover Publications, 1981.

[8] R. C. Hilborm, *Chaos and Nonlivear Dynamics: An Introduction for Scientists and Engineers*. New York: Oxford University Press, 1994.

[9] A. Ern, *Theory and Practice of Finite Elements*. New York: Springer, 2004.