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New exact solutions for the (3+1)-dimensional breaking soliton equation

Mohammad Taghi Darvishi, Maliheh Najafi and Mohammad Najafi

Abstract—In this work, we obtain some analytic solutions for the (3+1)-dimensional breaking soliton after obtaining its Hirota's bilinear form. Our calculations show that, three-wave method is very easy and straightforward to solve nonlinear partial differential equations.

Keywords—(3+1)-dimensional breaking soliton equation, Hirota's bilinear form.

I. Introduction

In recent years, many kinds of powerful methods have been proposed to find solutions of nonlinear partial differential equations, numerically and/or analytically, e.g., the variational iteration method [1], [2], [3], the homotopy perturbation method [4], [5], [6], [7], [8], parameter expansion method [9], [10], [11], spectral collocation method [12], [13], [14], [15], [16], homotopy analysis method [17], [18], [19], [20], [21], [22], and the Exp-function method [23], [24], [25], [26], [27], [28].

The (2+1)-dimensional nonlinear breaking soliton equation has the following form

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} = 0, \tag{1}$$

this equation describes the (2+1)-dimensional interaction of the Riemann wave propagated along the y-axis with a long wave propagated along the x-axis [29]. Wazwaz [30] introduced an extension to equation (1) by adding the last three terms with y replaced by z. His work, enables us to establish the following (3+1)-dimensional breaking soliton equation

$$\begin{gathered} u_{xt} - 4\,u_x\,(u_{xy}+u_{xz}) - 2\,u_{xx}\,(u_y+u_z) - \\ (u_{xxxy}+u_{xxxz}) = 0, \end{gathered} \tag{2}$$

where $u = u(x,y,z,t) : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z \times \mathbb{R}_t \to \mathbb{R}$.
Recently, Dai et al. [31], suggested the three-wave method for nonlinear evolution equations. The basic idea of this method applies the Painlevé analysis to make a transformation as

$$u = T(f) \tag{3}$$

for some new and unknown function f. Then we use this transformation in a high dimensional nonlinear equation of the general form

$$F(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, \cdots) = 0, \tag{4}$$

M.T. Darvishi, Maliheh Najafi and **Mohammad Najafi**: Department of Mathematics, Faculty of Science, Razi University, Kermanshah 67149, Iran. **e-mail**: darvishimt@yahoo.com, malihe_math87@yahoo.com and m_najafi82@yahoo.com.

where $u = u(x,y,z,t)$ and F is a polynomial of u and its derivatives. By substituting (3) in (4), the first one converts into the Hirota's bilinear form, which it will solve by taking a special form for f and assuming that the obtained Hirota's bilinear form has three-wave solutions, then we can specify the unknown function f. For more details see [31], [32]. In this paper we solve equation (1) by the three-wave method and obtain some exact and new solutions for it.

II. The (3+1)-dimensional breaking soliton equation

In this section, we investigate explicit formula of solutions of the following (3+1)-dimensional breaking soliton equation

$$\begin{gathered} u_{xt} - 4\,u_x\,(u_{xy}+u_{xz}) - 2\,u_{xx}\,(u_y+u_z) - \\ (u_{xxxy}+u_{xxxz}) = 0. \end{gathered} \tag{5}$$

To solve this equation we suppose that

$$\theta = y + k\,z \tag{6}$$

then equation (5) reduces to

$$u_{xt} - 4\,(k+1)u_x u_{x\theta} - 2\,(k+1)u_{xx}u_\theta - \;(k+1)u_{xxx\theta} = 0. \tag{7}$$

To solve equation (7), we introduce a new dependent variable u by

$$u = 2(\ln f)_x \tag{8}$$

where f is an unknown real function which will be determined. Substituting equation (8) into equation (7), we have

$$\begin{gathered} [2(\ln f)_x]_{xt} - 4\,(k+1)\,[2(\ln f)_x]_x\,[2(\ln f)_x]_{x\theta} - \\ 2\,(k+1)\,[2(\ln f)_x]_{xx}\,[2(\ln f)_x]_\theta \\ -(k+1)\,[2(\ln f)_x]_{xxx\theta} = 0, \end{gathered} \tag{9}$$

which can be integrated once with respect to x to give

$$\begin{gathered} [2(\ln f)]_{xt} - 3\,(k+1)\,[2(\ln f)]_{xx}\,[2(\ln f)]_{x\theta} - \\ (k+1)[2(\ln f)]_{xxx\theta} + 2\partial_x^{-1}((\ln f)_{xxx}\,(\ln f)_{x\theta} - \\ (\ln f)_{xx}\,(\ln f)_{xx\theta}) = C, \end{gathered} \tag{10}$$

where $\partial_x^{-1}\partial_x = 1$. Taking $C = 0$, therefore, equation (10) can be written as

$$(D_x D_t + D_x^3\,D_\theta) f\cdot f + 2\,f^2\,\partial_x^{-1}(D_x(\ln f)_{xx}\cdot(\ln f)_{x\theta}) = 0, \tag{11}$$

where the D-operator is defined by

$$D_x^m D_y^k D_z^p D_t^n f(x,y,z,t)\cdot g(x,y,z,t) = (\tfrac{\partial}{\partial x_1}-\tfrac{\partial}{\partial x_2})^m(\tfrac{\partial}{\partial y_1}-\tfrac{\partial}{\partial y_2})^k(\tfrac{\partial}{\partial z_1}-\tfrac{\partial}{\partial z_2})^p(\tfrac{\partial}{\partial t_1}-\tfrac{\partial}{\partial t_2})^n [f(x_1,y_1,z_1,t_1)g(x_2,y_2,z_2,t_2)],$$

and the right hand side is computed in

$$x_1=x_2=x,\ y_1=y_2=y,\ z_1=z_2=z,\ t_1=t_2=t.$$

We suppose that

$$\partial_x^{-1}(D_x(\ln f)_{xx}\cdot(\ln f)_{x\theta})=0, \tag{12}$$

note that to have a correct solution for equation (5) we must consider (12) in our algebraic systems of equations, which that will be our modification from the three-wave method. Therefore, by our assumption, equation (11) reduces to

$$(D_xD_t+D_x^3D_\theta)f\cdot f=0. \tag{13}$$

Now we suppose that the solution of equation (11) as

$$f(x,\xi,t)=\mathrm{e}^{-\xi_1}+\delta_1\cos(\xi_2)+\delta_2\,\mathrm{e}^{\xi_1} \tag{14}$$

where

$$\xi_i=a_ix+b_i\theta+c_it,\quad i=1,2 \tag{15}$$

and a_i, b_i, c_i, δ_i are some constants to be determined later. Substituting equation (14) into equation (13) and equating all coefficients of $\sin(\xi_2)$, $\cos(\xi_2)$, $\exp(\xi_1)$ and $\exp(-\xi_1)$ to zero, we get the following set of algebraic equation for a_i,b_i,c_i, δ_i, $(i=1,2)$

$$\begin{aligned}&3\,a_2{}^2a_1b_1+3\,a_1{}^2b_2a_2-kb_2a_2{}^3+3\,ka_1{}^2b_2a_2+c_1a_1\\&-ka_1{}^3b_1-a_1{}^3b_1-b_2a_2{}^3-a_2c_2+3\,ka_2{}^2a_1b_1=0,\\&ka_2{}^3b_1+3\,b_2a_2{}^2a_1-3\,a_1{}^2b_1a_2-ka_1{}^3b_2+c_2a_1\\&-3\,ka_1{}^2b_1a_2+3\,kb_2a_2{}^2a_1-a_1{}^3b_2+a_2{}^3b_1+c_1a_2=0,,\\&-4\,k\delta_1{}^2a_2{}^3b_2-4\,\delta_1{}^2a_2{}^3b_2-\delta_1{}^2a_2c_2-16\,ka_1{}^3\delta_3b_1\\&-16\,a_1{}^3\delta_3b_1+4\,c_1a_1\delta_3=0,\end{aligned} \tag{16}$$

and from our assumption, that is, from equation (12) we have

$$\begin{aligned}&a_2{}^4b_1-a_1{}^3a_2b_2-a_2{}^3a_1b_2+a_2{}^2b_1a_1{}^2=0,\\&-4\,a_1{}^2a_2{}^2b_2+4\,a_1{}^3a_2b_1+4\,a_2{}^3b_1a_1-4\,a_1{}^4b_2=0.\end{aligned} \tag{17}$$

Solving the system of equations (16) and (17) with the aid of Maple, yields the following cases:

A. Case 1:

$$\begin{aligned}&b_1=\tfrac{b_2a_1}{a_2},c_1=\tfrac{b_2a_1\left(a_1{}^2-3\,a_2{}^2\right)(k+1)}{a_2},\\&c_2=b_2\left(a_1{}^2-ka_2{}^2+3\,ka_1{}^2-a_2{}^2\right),\delta_2=-\tfrac{\delta_1{}^2a_2{}^2}{4\,a_1{}^2}\end{aligned} \tag{18}$$

for some arbitrary real constants a_1, a_2, b_2, k and δ_1. Substitute equations (18) into equation (8) with equation (14), we obtain the solution as

$$f(x,y,z,t)=\mathrm{e}^{-\xi_1}+\delta_1\cos(\xi_2)+\delta_2\mathrm{e}^{\xi_1}$$

and

$$u(x,y,z,t)=2\,\frac{-a_1\mathrm{e}^{-\xi_1}-\delta_1\sin(\xi_2)\,a_2+\delta_2a_1\mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1}+\delta_1\cos(\xi_2)+\delta_2\mathrm{e}^{\xi_1}} \tag{19}$$

for

$$\xi_1=a_1x+\tfrac{b_2a_1(y+kz)}{a_2}+\tfrac{b_2a_1\left(-3\,a_2{}^2+ka_1{}^2-3\,ka_2{}^2+a_1{}^2\right)t}{a_2},$$

and

$$\xi_2=a_2x+b_2\,(y+kz)+\left(3\,a_1{}^2b_2-kb_2a_2{}^2+3\,ka_1{}^2b_2-b_2a_2{}^2\right)t,$$

and

$$\delta_2=-\tfrac{\delta_1{}^2a_2{}^2}{4\,a_1{}^2}.$$

If $\delta_2>0$, then we obtain the exact breather cross-kink solution

$$u(x,y,z,t)=2\,\frac{-2\,a_1\sqrt{\delta_2}\sinh(\xi_1-\beta)-\delta_1\sin(\xi_2)\,a_2}{2\sqrt{\delta_2}\cosh(\xi_1-\beta)+\delta_1\cos(\xi_2)}$$

for

$$\beta=\frac{1}{2}\,\ln(\delta_2).$$

If $\delta_2<0$, then we obtain the exact breather cross-kink solution

$$u(x,y,z,t)=2\,\frac{-2\,a_1\sqrt{-\delta_2}\cosh(\xi_1-\beta)-\delta_1\sin(\xi_2)\,a_2}{2\sqrt{-\delta_2}\sinh(\xi_1-\beta)+\delta_1\cos(\xi_2)}$$

for

$$\beta=\frac{1}{2}\,\ln(-\delta_2).$$

B. Case 2:

$$a_1=ia_2,c_2=0,b_2=0,c_1=-4\,a_2{}^2b_1\,(k+1) \tag{20}$$

for some arbitrary real constants a_2,b_1,k,δ_1 and δ_2. Substitute equation (20) into equation (8) with equation (14), we obtain the solution as

$$f(x,y,z,t)=\mathrm{e}^{-\xi_1}+\delta_1\cos(\xi_2)+\delta_2\mathrm{e}^{\xi_1}$$

and

$$u(x,y,z,t)=2\,\frac{-ia_2\mathrm{e}^{-\xi_1}-\delta_1\sin(\xi_2)\,a_2+i\delta_2a_2\mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1}+\delta_1\cos(\xi_2)+\delta_2\mathrm{e}^{\xi_1}} \tag{21}$$

for

$$\xi_1=ia_2x+b_1\,(y+kz)-4\,a_2{}^2b_1\,(k+1)\,t$$

$$\xi_2=a_2x.$$

We make the dependent variable transformation in equation (21) as follows

$$a_2 = -i\,A_2, \tag{22}$$

where A_2 is real. We obtain new form for equation (21) as follows

$$u\,(x,y,z,t) = 2\,\frac{-A_2\mathrm{e}^{-\xi_1^*} + i\delta_1 \sin\left(\xi_2^*\right) A_2 + \delta_2 A_2 \mathrm{e}^{\xi_1^*}}{\mathrm{e}^{-\xi_1} + \delta_1 \cos\left(\xi_2\right) + \delta_2 \mathrm{e}^{\xi_1^*}} \tag{23}$$

for

$$\xi_1^* = A_2\,x + b_1\,(y+kz) + 4\,A_2{}^2 b_1\,(k+1)\,t$$

$$\xi_2^* = -i\,A_2\,x.$$

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,z,t) = 2\,\frac{-2\,A_2\sqrt{\delta_2}\sinh(\xi_1^* - \beta) + i\delta_1 \sin\left(\xi_2^*\right) A_2}{2\,\sqrt{\delta_2}\cosh(\xi_1^* - \beta) + \delta_1 \cos\left(\xi_2^*\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(\delta_2).$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,z,t) = 2\,\frac{-2\,A_2\sqrt{-\delta_2}\cosh(\xi_1^* - \beta) + i\delta_1 \sin\left(\xi_2^*\right) A_2}{2\,\sqrt{-\delta_2}\sinh(\xi_1^* - \beta) + \delta_1 \cos\left(\xi_2^*\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_2).$$

C. Case 3:

$$\begin{aligned} &a_1 = ia_2, b_1 = ib_2, \\ &c_1 = -i\left(8\,kb_2a_2{}^2 + 8\,b_2a_2{}^2 + c_2\right), \delta_2 = \tfrac{\delta_1{}^2}{4} \end{aligned} \tag{24}$$

for some arbitrary real constants a_2, b_2, c_2, k and δ_1. Substitute equation (24) into equation (8) with equation (14), we obtain the solution as follows

$$f\,(x,y,z,t) = \mathrm{e}^{-\xi_1} + \delta_1 \cos\left(\xi_2\right) + \delta_2 \mathrm{e}^{\xi_1}$$

and

$$u\,(x,y,z,t) = 2\,\frac{-ia_2\mathrm{e}^{-\xi_1} - \delta_1 \sin\left(\xi_2\right) a_2 + i\delta_2 a_2 \mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_1 \cos\left(\xi_2\right) + \delta_2 \mathrm{e}^{\xi_1}} \tag{25}$$

for

$$\xi_1 = ia_2x + ib_2\,(y+kz) - i\left(8\,kb_2a_2{}^2 + 8\,b_2a_2{}^2 + c_2\right) t$$

$$\xi_2 = a_2x + b_2\,(y+kz) + c_2t$$

and

$$\delta_2 = \frac{\delta_1{}^2}{4}. \tag{26}$$

We make the dependent variable transformation in equation (25) as follows

$$\begin{aligned} a_2 &= i\,A_2, \\ b_2 &= i\,B_2, \\ c_2 &= i\,C_2, \end{aligned} \tag{27}$$

where A_2, B_2 and C_2 are real. We obtain new form for equation (25) as

$$u\,(x,y,z,t) = -2\,\frac{A_2\left(\mathrm{e}^{-\xi_1^*} - \delta_1 \sinh\left(\xi_2^*\right) - \delta_2 \mathrm{e}^{\xi_1^*}\right)}{\mathrm{e}^{-\xi_1^*} + \delta_1 \cosh\left(\xi_2^*\right) + \delta_2 \mathrm{e}^{\xi_1^*}} \tag{28}$$

for

$$\xi_1^* = A_2x + B_2y + B_2kz + \left(8\,B_2kA_2{}^2 + 8\,B_2A_2{}^2 - C_2\right) t$$

$$\xi_2^* = A_2x + B_2y + B_2kz + C_2t.$$

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,z,t) = -2\,\frac{A_2\left(2\,\sqrt{\delta_2}\sinh(\xi_1^* - \beta) - \delta_1 \sinh\left(\xi_2^*\right)\right)}{2\,\sqrt{\delta_2}\cosh(\xi_1^* - \beta) + \delta_1 \cosh\left(\xi_2^*\right)}$$

for

$$\beta = \frac{1}{2}\,\ln(\delta_2) \quad , \quad \delta_2 = \frac{\delta_1{}^2}{4}.$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,z,t) = -2\,\frac{A_2\left(2\,\sqrt{-\delta_2}\cosh(\xi_1^* - \beta) - \delta_1 \sinh\left(\xi_2^*\right)\right)}{2\,\sqrt{-\delta_2}\sinh(\xi_1^* - \beta) + \delta_1 \cosh\left(\xi_2^*\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_2) \quad , \quad \delta_2 = \frac{\delta_1{}^2}{4}.$$

III. Conclusions

In this paper, we introduced a modification of three-wave method, and we obtained some analytic solutions for the (3+1)-dimensional breaking soliton equation in its bilinear form. We can apply this modification when a PDE does not have a bilinear closed form. By comparison of three-wave method and another analytic methods, like HAM, HTA and EHTA methods, we can see that the new idea is very easy and straightforward which can be applied on another nonlinear partial differential equations.

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