# New Delay-Dependent Stability Criteria For Neural Networks With Two Additive Time-varying delay components

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*Abstract*—In this paper, the problem of stability criteria of neural networks (NNs) with two-additive time-varying delay compenents is investigated. The relationship between the time-varying delay and its lower and upper bounds is taken into account when estimating the upper bound of the derivative of Lyapunov functional. As a result, some improved delay stability criteria for NNs with two-additive time-varying delay components are proposed. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

*Keywords*—Delay-dependent stability; Time-varying delays; Lyapunov functional; Linear matrix inequality(LMI).

## I. INTRODUCTION

**N**EURAL networks(NNs) have been studied over the past few decades extensively and have found many applications in many areas, such as pattern recognition, signal processing, associative memory, static image processing, and combinatorial optimization. And time-delay also often occurs in many industrial and engineeering systems, such as manufacturing systems, telecommunication and economic systems, and is a major cause of instability and poor performance. In recent years, much efforts has been invested in the analysis of time-delay systems, such as delayed stochastic system, delayed stochastic genetic regulatory networks, delayed stochastic complex networks [1]-[4]. Up to now, stability of NNs with time delay has also received attention [5]-[14], since time delay is frequently encountered in NNs, and it is often a source of instability and oscillations in a system. Both delay-independent [8]-[17] and delay-dependent [18]-[27] stability criteria for NNs have been proposed in recent years. Since delay-independent criteria tend to be conservative, especially when the delay is small or it varies in an interval, much attention has been paid to the delay-dependent type. But note that the delay-dependent stability results mentioned above can only provide stability conditions for neural networks with one single delay in the state.

Recently, a new model for neural networks with two additive time-varying delays has been considered in [6], [7]and [26]. By constructing a new Lyapunov functional and using some

advanced techniques, a new asymptotic stability criterion for neural networks with two successive delay components is derived in [6]. By choosing a new class of Lyapunov functional, some new delay-dependent asymptotic stability criteria are derived to guaran tee the stability of the delayed neural networks in [26].

In this paper, the problem of stability criteria of neural networks (NNs) with two-additive time-varying delay compenents is investigated. The relationship between the time-varying delay and its lower and upper bounds is taken into account.When estimating the upper bound of the derivative of Lyapunov functional. As a result, some improved delay stability criteria for NNs with two-additive time-varying delay components are proposed. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

## II. PROBLEM FORMULATION AND SOME PRELIMINARES

Consider the following delayed neural networks with two-additive time-varying delays:

$$\dot{y}(t) = -Ay(t) + Bg(y(t)) + Dg(y(t-d_1(t)-d_2(t))) + u \quad (1)$$

where $y(\cdot) = [y_1(\cdot), y_2(\cdot), ..., y_n(\cdot)]^T \in \mathscr{R}^n$ is the neuron state vector, $g(y(\cdot)) = [g_1(y(t)), g_2(y(t)), ..., g_n(y(t))]^T \in \mathscr{R}^n$ denotes the neuron activation function, and $u = [u_1, u_2, ..., u_n]^T \in \mathscr{R}^n$ is a constant input vector. $B, D \in \mathscr{R}^{n \times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. $A = diag(a_1, a_2, ..., a_n)$ with $a_i > 0$, $i = 1, 2, .., n$. $d_1(t)$ and $d_2(t)$ are two time-varying satisfying:

$$0 \leq d_{11} \leq d_1(t) \leq d_{12}, \quad 0 \leq d_{21} \leq d_2(t) \leq d_{22};$$

$$\dot{d}_1(t) \leq \mu_1, \quad \dot{d}_2(t) \leq \mu_2 \quad (2)$$

where $d_{12} \geq d_{11}$, $d_{22} \geq d_{21}$ and $\mu_1$, $\mu_2$ are constants. Note that $d_{11}$, $d_{21}$ may not be equal to 0. We denote

$$d(t) = d_1(t) + d_2(t), \quad d_1 = d_{11} + d_{21}, \quad d_2 = d_{12} + d_{22};$$

$$\mu = \mu_1 + \mu_2, \quad h_1 = d_{12} - d_{11}, \quad h_2 = d_{22} - d_{21} \quad (3)$$

In addition, it is assumed that each neuron activation function in system (1), $g_i(\cdot)(i = 1, 2, ..., n)$ is bounded and satisfies the following condition:

$$0 \leq \frac{g_i(x) - g_i(y)}{x - y} \leq k_i \quad (4)$$

where $k_i$ $(i = 1, 2, ..., n)$ are positive constants, $x, y \in \mathscr{R}$.

Note that by using the Brouwers fixed-point theorem, it can be proven that there at least exists one equilibrium

point for system (1). In the following, the equilibrium point $y^* = [y_1^*, y_2^*, ..., y_n^*]^T$ of system (1) is shifted to the origin by the transformation $z(\cdot) = y(\cdot) - y^*$, which converts the system to the system:

$$\dot{z}(t) = -Az(t) + Bf(z(t)) + Df(z(t - d_1(t) - d_2(t))) \quad (5)$$

where $z(\cdot) = [z_1(\cdot), z_2(\cdot), ..., z_n(\cdot)]^T$ is the state vector of the transformed system $f(z(\cdot)) = [f_1(z(\cdot)), f_2(z(\cdot)), ..., f_n(z(\cdot))]^T$ and $f_i(z_i(\cdot)) = g_i(z_i(\cdot) + y_i^*) - g_i(y_i^*), (i = 1, 2, .., n)$. note that the function $f_i(\cdot)$ $(i = 1, 2, .., n)$ satisfy the following condition:

$$0 \leq \frac{f_i(z_i)}{z_i} \leq k_i, \quad i = 1, 2, ..., n \quad (6)$$

which is equivalent to

$$f_i(z_i)[f_i(z_i) - k_i z_i] \leq 0, \quad i = 1, 2, ..., n \quad (7)$$

In this paper, we will present our practically stability criteria for DNN in (6). Before giving our main result, we present the Lemmas which are employed for future derivations.

**Lemma 1.** ([26]) For any constant matrix $R \in \mathscr{R}^{n \times n}, R = R^T > 0$, scalar $d_2 > d_1 > 0$, such that the following integrations are well defined, then

$(d_2 - d_1) \int_{t-d_2}^{t-d_1} y^T(s)Ry(s)ds \geq \int_{t-d_2}^{t-d_1} y^T(s)dsR \int_{t-d_2}^{t-d_1} y(s)ds;$

$\frac{d_2^2 - d_1^2}{2} \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} y^T(s)Ry(s)ds \geq \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} y^T(s)dsR \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} y(s)ds$

**Lemma 2.** ([27]) For any constant matrix $R \in \mathscr{R}^{n \times n}$, $R = R^T > 0$, scalar $d > 0$ and a vector-valued function $y : [t - d, t] \to \mathscr{R}^n$, the following integrations is well defined:

$$-d \int_{t-d}^{t} \dot{y}^T(s)R\dot{y}(s)ds \leq \begin{bmatrix} y(t) \\ y(t-d) \end{bmatrix}^T \begin{bmatrix} -R & R \\ * & -R \end{bmatrix} \begin{bmatrix} y(t) \\ y(t-d) \end{bmatrix}$$

## III. DELAY-DEPENDENT STABILITY CRITERIA

In this section, the following Lyapunov-Krasovskii functional is constructed:

$$V(z_t) = \sum_{i=1}^{7} V_i(z_t)$$
where

$$V_1(z_t) = z^T(t)Pz(t) + 2\sum_{i=1}^{n} \lambda_i \int_0^{z_i(t)} f_i(s)ds$$

$$V_2(z_t) = \begin{bmatrix} \int_{t-d_2}^{t} z(s)ds + \int_{t-d_2}^{t-d_1} z(s)ds \\ \int_{-d_2}^{0} \int_{t+\theta}^{t} \dot{z}(s)dsd\theta + \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} \dot{z}(s)dsd\theta \end{bmatrix}^T$$
$$\times \begin{bmatrix} M_{11} & M_{12} \\ * & M_{22} \end{bmatrix}$$
$$\times \begin{bmatrix} \int_{t-d_2}^{t} z(s)ds + \int_{t-d_2}^{t-d_1} z(s)ds \\ \int_{-d_2}^{0} \int_{t+\theta}^{t} \dot{z}(s)dsd\theta + \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} \dot{z}(s)dsd\theta \end{bmatrix}$$

$$V_3(z_t) = \int_{t-d(t)}^{t} [z^T(s)Q_1 z(s) + f^T(z(s))Q_2 f(z(s))]ds$$
$$+ \int_{t-d_1(t)}^{t} z^T(s)Q_3 z(s)ds + \int_{t-d_2(t)}^{t} z^T(s)Q_4 z(s)ds$$
$$+ \int_{t-d_1}^{t} z^T(s)Q_5 z(s)ds + \int_{t-d_2}^{t} z^T(s)Q_6 z(s)ds$$
$$+ \int_{t-d_{11}}^{t} z^T(s)Q_7 z(s)ds + \int_{t-d_{12}}^{t} z^T(s)Q_8 z(s)ds$$

$$V_4(z_t) = \int_{-d_{12}}^{0} \int_{t+\theta}^{t} \dot{z}^T(s)Z_1 \dot{z}(s)dsd\theta + \int_{-d_{12}}^{-d_{11}} \int_{t+\theta}^{t} \dot{z}^T(s)$$
$$Z_2 \dot{z}(s)dsd\theta + \int_{-d_{22}}^{0} \int_{t+\theta}^{t} \dot{z}^T(s)Z_3 \dot{z}(s)dsd\theta$$
$$+ \int_{-d_{22}}^{-d_{21}} \int_{t+\theta}^{t} \dot{z}^T(s)Z_4 \dot{z}(s)dsd\theta$$

$$V_5(z_t) = d_2 \int_{-d_2}^{0} \int_{t+\theta}^{t} \dot{z}^T(s)Z_5 \dot{z}(s)dsd\theta$$
$$+ (d_2 - d_1) \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} \dot{z}^T(s)Z_6 \dot{z}(s)dsd\theta$$

$$V_6(z_t) = d_2 \int_{-d_2}^{0} \int_{t+\theta}^{t} z^T(s)Z_7 z(s)dsd\theta$$
$$+ (d_2 - d_1) \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} z^T(s)Z_8 z(s)dsd\theta$$

$$V_7(z_t) = \frac{d_2^2}{2} \int_{-d_2}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{z}^T(s)Z_9 \dot{z}(s)dsd\lambda d\theta$$
$$+ \frac{d_2^2 - d_1^2}{2} \int_{-d_2}^{-d_1} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{z}^T(s)Z_{10} \dot{z}(s)dsd\lambda d\theta$$

$$(8)$$

Where $P = P^T > 0$, $Q_l = Q_l^T > 0$ $(l = 1, 2, ..., 10)$, $Z_i = Z_i^T > 0$ $(i = 1, 2, ..., 10)$, $\begin{bmatrix} M_{11} & M_{12} \\ * & M_{22} \end{bmatrix} > 0$ and $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n) \geq 0$ are to be determined.

**Theorem 1**: For given scalar $d_{ij}$ $(i = 1, 2; j = 1, 2)$, and $\mu_i$ $(i = 1, 2)$, the system described by (2), (3), (5) and (6) is global asymptotically stable if there exist symmetric positive matrices $P, Q_i, Z_i$ $(i = 1, 2, ..., 10)$ $\begin{bmatrix} M_{11} & M_{12} \\ * & M_{22} \end{bmatrix}$, positive diagonal matrices, $T_i = diag(t_{1i}, t_{2i}, ..., t_{ni})$, $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_n)(i = 1, 2)$ and any matrices $\mathcal{N}, \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}, P_1, P_2$ with appropriate dimensions, such that the following LMIs hold:

$$\hat{\Pi}_{11} = \begin{bmatrix} \Pi & -d_{11}\mathcal{N} & -h_1\mathcal{L} & -d_{21}\mathcal{S} & -h_2\mathcal{T} \\ * & -d_{11}Z_1 & 0 & 0 & 0 \\ * & * & -h_1 Z_{12} & 0 & 0 \\ * & * & * & -d_{21}Z_3 & 0 \\ * & * & * & * & -h_2 Z_{34} \end{bmatrix} < 0$$

$$(9)$$

$$\hat{\Pi}_{12} = \begin{bmatrix} \Pi & -d_{11}\mathcal{N} & -h_1\mathcal{L} & -d_{22}\mathcal{S} & -h_2\mathcal{U} \\ * & -d_{11}Z_1 & 0 & 0 & 0 \\ * & * & -h_1 Z_{12} & 0 & 0 \\ * & * & * & -d_{22}Z_3 & 0 \\ * & * & * & * & -h_2 Z_4 \end{bmatrix} < 0$$

$$(10)$$

$$\hat{\Pi}_{21} = \begin{bmatrix} \Pi & -d_{12}\mathcal{N} & -h_1\mathcal{R} & -d_{21}\mathcal{S} & -h_2\mathcal{T} \\ * & -d_{12}Z_1 & 0 & 0 & 0 \\ * & * & -h_1 Z_2 & 0 & 0 \\ * & * & * & -d_{21}Z_3 & 0 \\ * & * & * & * & -h_2 Z_{34} \end{bmatrix} < 0 \quad (11)$$

$$\hat{\Pi}_{22} = \begin{bmatrix} \Pi & -d_{12}\mathcal{N} & -h_1\mathcal{R} & -d_{22}\mathcal{S} & -h_2\mathcal{U} \\ * & -d_{12}Z_1 & 0 & 0 & 0 \\ * & * & -h_1 Z_2 & 0 & 0 \\ * & * & * & -d_{22}Z_3 & 0 \\ * & * & * & * & -h_2 Z_4 \end{bmatrix} < 0 \quad (12)$$

where

$$\Pi = \begin{bmatrix} \Xi_{11} & O & \Xi_{13} & \Xi_{14} \\ * & \Xi_{22} & O & O \\ * & * & \Xi_{33} & \Xi_{34} \\ * & * & * & \Xi_{44} \end{bmatrix} + \begin{bmatrix} \Upsilon_1 + \Upsilon_1^T & \Upsilon_2 + \Upsilon_2^T \end{bmatrix}$$

$$\Xi_{11} = \begin{bmatrix} \Pi_{11} & 0 & 0 & 0 & Z_5 \\ * & \Phi_{22} & 0 & 0 & 0 \\ * & * & \Phi_{33} & 0 & Z_6 \\ * & * & * & \Phi_{44} & 0 \\ ** & * & * & * & \Phi_{55} \end{bmatrix}$$

$$\Xi_{22} = \begin{bmatrix} \Pi_{66} & 0 & 0 & 0 & 0 \\ * & -Q_7 & 0 & 0 & 0 \\ * & * & -Q_8 & 0 & 0 \\ * & * & * & -Q_9 & 0 \\ ** & * & * & * & -Q_{10} \end{bmatrix}$$

$$\Xi_{33} = \begin{bmatrix} \Pi_{11,11} & \Lambda + P_2 B & P_2 D \\ * & \Omega_{22} & 0 \\ * & * & \Omega_{33} \end{bmatrix}$$

$$\Xi_{44} = \begin{bmatrix} -Z_7 & 0 & 0 & 0 \\ * & -Z_7 & 0 & 0 \\ * & * & -Z_7 & 0 \\ * & * & * & -Z_7 \end{bmatrix}$$

$$\Xi_{13} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & -P_1 D \\ 0 & 0 & T_2 \Sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Xi_{14} = \begin{bmatrix} \Psi_{14} & \Psi_{15} & \Psi_{16} & \Psi_{17} \\ 0 & 0 & 0 & 0 \\ \Psi_{34} & \Psi_{35} & \Psi_{36} & \Psi_{37} \\ 0 & 0 & 0 & 0 \\ \Psi_{54} & \Psi_{55} & \Psi_{156} & \Psi_{57} \end{bmatrix}$$

$$\Upsilon_1 = \begin{bmatrix} \Phi_1 & 0 & 0 & \Phi_2 & 0 & \Phi_3 & \mathcal{R} & -\mathcal{L} & \mathcal{U} \end{bmatrix}$$
$$\Upsilon_2 = \begin{bmatrix} -\mathcal{T} & 0 & 0 & 0 & -\mathcal{V} & -\mathcal{W} & -\mathcal{V} & -\mathcal{W} \end{bmatrix}$$

$\Pi_{11} = Q_1 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10} + d_2^2 Z_7 + h_{21}^2 Z_8 - Z_5 - P_1 A - A^T P_1$;

$\Pi_{11,11} = d_{12}Z_1 + h_1 Z_2 + d_{22}Z_3 + h_2 Z_4 + d_2^2 Z_5 + (d_2 - d_1)^2 Z_6 + \frac{d_4^4}{4} Z_9 + \frac{(d_2^2 - d_1^2)^2}{4} Z_{10} - P_2 - P_2^T$;

$\Phi_{22} = -(1-\mu)Q_1; \Phi_{33} = -Q_5 - Z_6; \Phi_{44} = -(1-\mu_1)Q_3$;

$\Phi_{55} = -Z_5 - Z_6 - Q_6; \Phi_{66} = -(1-\mu_2)Q_4$;

$\Omega_{22} = Q_2 - T_1 - T_1^T; \Omega_{33} = -(1-\mu)Q_2 - T_2 - T_2^T$;

$\Omega_{14} = \Omega_{15} = (d_2 + h_{21})M_{12}; \Omega_{16} = \Omega_{17} = (d_2 + h_{21})M_{22}$;

$\Phi_1 = \mathcal{N} + \mathcal{S} + d_2\mathcal{V} + h_{21}\mathcal{W}; \Phi_2 = \mathcal{L} - \mathcal{N} - \mathcal{R}$;

$\Phi_3 = \mathcal{T} - \mathcal{S} - \mathcal{U}; \sum = diag(k_1, k_2, ..., k_n)$;

$\Psi_{11} = P - P_1 - A^T P_2; \Psi_{12} = P_1 B + T_1 \Sigma$;

$\Psi_{14} = \Psi_{15} = \Psi_{34} = \Psi_{35} = M_{11} - M_{12}$;
$\Psi_{16} = \Psi_{17} = \Psi_{36} = \Psi_{37} = M_{12} - M_{22}$;
$\Psi_{54} = \Psi_{55} = -M_{11} - M_{11}^T + M_{12} + M_{12}^T$;
$\Psi_{56} = \Psi_{57} = -M_{12} - M_{12}^T + M_{22} + M_{22}^T$;

**Proof**: Calculating the derivatives of $V_i(z_t)$ $(i = 1, 2, .., 7)$, along the trajectories of system (5) yields.

$$\dot{V}_1(z_t) = 2z^T(t)P\dot{z}(t) + 2f^T(z(t))\Lambda\dot{z}(t) \quad (13)$$

$$\dot{V}_2(z_t) = 2 \begin{bmatrix} \int_{t-d_2}^t z(s)ds + \int_{t-d_2}^{t-d_1} z(s)ds \\ \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}(s)dsd\theta + \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}(s)dsd\theta \end{bmatrix}^T$$
$$\times \begin{bmatrix} M_{11} & M_{12} \\ * & M_{22} \end{bmatrix}$$
$$\times \begin{bmatrix} z(t) + z(t-d_1) - 2z(t-d_2) \\ (d_2 + h_{21})\dot{z}(t) - (z(t) + z(t-d_1) - 2z(t-d_2)) \end{bmatrix} \quad (14)$$

$$\dot{V}_3(z_t) \leq z^T(t)[Q_1 + Q_3 + Q_4 + Q_5 + Q_6 + Q_7 + Q_8 + Q_9 + Q_{10}]z(t) - (1-\mu)z^T(t-d(t))Q_1 z(t-d(t))$$
$$- (1-\mu_1)z^T(t-d_1(t))Q_3 z(t-d_1(t))$$
$$- (1-\mu_2)z^T(t-d_2(t))Q_4 z(t-d_2(t))$$
$$- z^T(t-d_1)Q_5 z(t-d_1) - z^T(t-d_2)Q_6 z(t-d_2)$$
$$- z^T(t-d_{11})Q_7 z(t-d_{11}) - z^T(t-d_{12})Q_8 z(t-d_{12})$$
$$- z^T(t-d_{21})Q_9 z(t-d_{21}) - z^T(t-d_{22})Q_{10} z(t-d_{22})$$
$$- (1-\mu)f^T(z(t-d(t)))Q_2 f(z(t-d(t)))$$
$$+ f^T(z(t))Q_2 f(z(t)) \quad (15)$$

$$\dot{V}_4(z_t) = \dot{z}^T(t)[d_{12}Z_1 + h_1 Z_2 + d_{22}Z_3 + h_2 Z_4]\dot{z}(t)$$
$$- \int_{t-d_{12}}^{t-d_1(t)} \dot{z}^T(s)(Z_{12})\dot{z}(s)ds - \int_{t-d_1(t)}^t \dot{z}^T(s)Z_1\dot{z}(s)ds$$
$$- \int_{t-d_1(t)}^{t-d_{11}} \dot{z}^T(s)Z_2\dot{z}(s)ds - \int_{t-d_{22}}^{t-d_2(t)} \dot{z}^T(s)(Z_{34})\dot{z}(s)ds$$
$$- \int_{t-d_2(t)}^t \dot{z}^T(s)Z_3\dot{z}(s)ds - \int_{t-d_2(t)}^{t-d_{21}} \dot{z}^T(s)Z_4\dot{z}(s)ds$$

Let

$M_1 = \frac{1}{d_1(t)} \int_{t-d_1(t)}^t \dot{z}(s)ds, M_2 = \frac{1}{d_{12}-d_1(t)} \int_{t-d_{12}}^{t-d_1(t)} \dot{z}(s)ds,$

$M_3 = \frac{1}{d_1(t)-d_{11}} \int_{t-d_1(t)}^{t-d_{11}} \dot{z}(s)ds, M_4 = \frac{1}{d_2(t)} \int_{t-d_2(t)}^t \dot{z}(s)ds,$

$M_5 = \frac{1}{d_{22}-d_2(t)} \int_{t-d_{22}}^{t-d_2(t)} \dot{z}(s)ds, M_6 = \frac{1}{d_2(t)-d_{21}} \int_{t-d_2(t)}^{t-d_{21}} \dot{z}(s)ds$

Then

$$\dot{V}_4(z_t) \leq \dot{z}^T(t)[d_{12}Z_1 + h_1 Z_2 + d_{22}Z_3 + h_2 Z_4]\dot{z}(t)$$
$$- (d_{12} - d_1(t))M_2^T Z_{12} M_2 - d_1(t)M_1^T Z_1 M_1$$
$$- (d_1(t) - d_{11})M_3^T Z_2 M_3 - (d_{22} - d_2(t))M_5^T Z_{34} M_5$$
$$- d_2(t)M_4^T Z_3 M_4 - (d_2(t) - d_{21})M_6^T Z_4 M_6 \quad (16)$$

By the Lamma 2, we can obtain

$$\dot{V}_5(z_t) \leq d_2^2 \dot{z}^T(t) Z_5 \dot{z}(t) + (d_2 - d_1)^2 \dot{z}^T(t) Z_6 \dot{z}(t)$$
$$+ \begin{bmatrix} z(t) \\ z(t-d_2) \end{bmatrix}^T \begin{bmatrix} -Z_5 & Z_5 \\ * & -Z_5 \end{bmatrix} \begin{bmatrix} z(t) \\ z(t-d_2) \end{bmatrix}$$
$$+ \begin{bmatrix} z(t-d_1) \\ z(t-d_2) \end{bmatrix}^T \begin{bmatrix} -Z_6 & Z_6 \\ * & -Z_6 \end{bmatrix} \begin{bmatrix} z(t-d_1) \\ z(t-d_2) \end{bmatrix} \tag{17}$$

By the Lemma 1, we can obtain

$$\dot{V}_6(z_t) \leq d_2^2 z^T(t) Z_7 z(t) + (d_2 - d_1)^2 z^T(t) Z_8 z(t)$$
$$- \int_{t-d_2}^t z^T(s) ds Z_7 \int_{t-d_2}^t z(s) ds \tag{18}$$
$$- \int_{t-d_2}^{t-d_1} z^T(s) ds Z_8 \int_{t-d_2}^{t-d_1} z(s) ds$$

$$\dot{V}_7(z_t) \leq \frac{d_2^4}{4} \dot{z}^T(t) Z_9 \dot{z}(t) + \frac{(d_2^2 - d_1^2)^2}{4} \dot{z}^T(t) Z_{10} \dot{z}(t)$$
$$- \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}^T(s) ds d\theta Z_9 \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}(s) ds d\theta$$
$$- \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}^T(s) ds d\theta Z_{10} \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}(s) ds d\theta \tag{19}$$

In addition, using the Leibniz-Newton formula, for any appropriately dimensional matrices $\mathcal{N}, \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{W}$, the following equations are true:

$$2\zeta^T(t)\mathcal{N}[z(t) - z(t-d_1(t)) - d_1(t)M_1] = 0 \tag{20}$$

$$2\zeta^T(t)\mathcal{L}[z(t-d_1(t)) - z(t-d_{12}) - (d_{12}-d_1(t))M_2] = 0 \tag{21}$$

$$2\zeta^T(t)\mathcal{R}[z(t-d_{11}) - z(t-d_1(t)) - (d_1(t)-d_{11})M_3] = 0 \tag{22}$$

$$2\zeta^T(t)\mathcal{S}[z(t) - z(t-d_2(t)) - d_2(t)M_4] = 0 \tag{23}$$

$$2\zeta^T(t)\mathcal{T}[z(t-d_2(t)) - z(t-d_{22}) - (d_{22}-d_2(t))M_5] = 0 \tag{24}$$

$$2\zeta^T(t)\mathcal{U}[z(t-d_{21}) - z(t-d_2(t)) - (d_2(t)-d_{21})M_6] = 0 \tag{25}$$

$$2\zeta^T(t)\mathcal{V}[d_2 z(t) - \int_{t-d_2}^t z(s) ds - \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}(s) ds d\theta] = 0 \tag{26}$$

$$2\zeta(t)^T \mathcal{W}[h_{21} z(t) - \int_{t-d_2}^{t-d_1} z(s) ds - \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}(s) ds ds] = 0 \tag{27}$$

$$2[z(t)^T P_1 + \dot{z}^T(t) P_2][- \dot{z}(t) - Az(t) + Bf(z(t)) + Df(z(t-d(t)))] = 0 \tag{28}$$

where $\zeta^T(t) = [z^T(t) \ z^T(t-d(t)) \ z^T(t-d_1) \ z^T(t-d_1(t))$ $z^T(t-d_2) \ z^T(t-d_2(t)) \ z^T(t-d_{11}) \ z^T(t-d_{12}) \ z^T(t-d_{21})$ $z^T(t-d_{22}) \ \dot{z}^T(t) \ f^T(z(t)) \ f^T(z(t-d(t))) \int_{t-d_2}^t z^T(s) ds$ $\int_{t-d_2}^{t-d_1} z^T(s) ds \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}^T(s) ds d\theta \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}^T(s) ds d\theta]$

Furthermore, there exsits positive diagonal matrices $T_1, T_2$, such that the following inequalities hold based on (6)

$$0 \leq -2f^T(z(t)) T_1 f(z(t)) + 2z^T(t) T_1 \Sigma f(z(t)) \tag{29}$$

$$0 \leq -2f^T(z(t-d(t))) T_2 f(z(t-d(t))) + 2z^T(t-d(t)) T_2 \Sigma f(z(t-d(t))) \tag{30}$$

Hence, according to (8) and (13)-(30), we can obtain

$$\dot{V}(z_t) \leq \xi^T(t) \tilde{\Pi} \xi(t) \tag{31}$$

where $\xi^T(t) = [\zeta^T(t) \ M_1^T \ M_2^T \ M_3^T \ M_4^T \ M_5^T \ M_6^T]$
$\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_1 & \tilde{\Pi}_2 \\ * & \tilde{\Pi}_3 \end{bmatrix}$

$$\tilde{\Pi}_1 = \begin{bmatrix} \Pi & -d_1(t)\mathcal{N} & -(d_{12}-d_1(t))\mathcal{L} & -(d_1(t)-d_{11})\mathcal{R} \\ * & -d_1(t)Z_1 & 0 & 0 \\ * & * & -(d_{12}-d_1(t))Z_{12} & 0 \\ * & * & * & -(d_1(t)-d_{11})Z_2 \end{bmatrix}$$

$$\tilde{\Pi}_2 = \begin{bmatrix} -d_2(t)\mathcal{S} & -(d_{22}-d_2(t))\mathcal{T} & -(d_2(t)-d_{21})\mathcal{U} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{\Pi}_3 = \begin{bmatrix} -d_2(t)Z_3 & 0 & 0 \\ * & -(d_{22}-d_2(t))Z_{34} & 0 \\ * & * & -(d_2(t)-d_{21})Z_4 \end{bmatrix}$$

where $Z_{12} = Z_1 + Z_2$, and $Z_{34} = Z_3 + Z_4$. If $\tilde{\Pi} < 0$, then there exists a scalar $\varepsilon > 0$, such that

$$\dot{V}(z_t) \leq -\varepsilon \xi^T(t) \xi(t) \leq -\varepsilon z^T(t) z(t) \tag{32}$$

According to the paper [26], we can know that when $d_1(t) \to d_{11}, d_1(t) \to d_{12}, d_2(t) \to d_{21}$ and $d_2(t) \to d_{22}$, the LMI $\tilde{\Pi}$ are equal to (9)-(12) which are define in Theorem 1, so we can conclude that the system described by (2), (3), (5) and (6) is asymptotically stable if the LMIs (9)-(12) hold.

**Remark 1**: It is seen that $d_1(t), d_2(t), d_{12}-d_1(t)$, $d_{22}-d_2(t)$ and $d_1(t)-d_{11}, d_2(t)-d_{21}$ are not simple enlarged as $d_{12}, d_{22}, d_{12}-d_{11}, d_{22}-d_{21}$, respectively. Instead, the relationship that $d_1(t) + (d_{12}-d_1(t)) = d_{12}, d_2(t) + (d_{22}-d_2(t)) = d_{22}$ and $(d_1(t)-d_{11}) + (d_{12}-d_1(t)) = d_{12}-d_{11}$, $(d_2(t)-d_{21}) + (d_{22}-d_2(t)) = d_{22}-d_{21}$ are considered.

**Remark 2**: A novel term $V_2(z_t)$ which noted in the paper that is included in the Lyapunov functional $V(z_t)$, which plays an important role in reducing conservativeness of our results. In our paper, by taking the states $\int_{t-d_1}^t z(s) ds, \int_{t-d_2}^t z(s) ds$, $\int_{-d_2}^0 \int_{t+\theta}^t \dot{z}(s) ds d\theta$ and $\int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}(s) ds d\theta$ are augmented variables, the atability in Theorem 1 utilizes more information on state variables, which yield less conservation results.

**Remark 3**: To reduce the conservatism, the lemma 1 is used to deal with the derivative of the $\dot{V}_7(z_t)$, i.e., $-\frac{d_2^2}{2} \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}^T(s) Z_9 \dot{z}(s) ds d\theta$ and $-\frac{d_2^2-d_1^2}{2} \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}^T(s) Z_{10} \dot{z}(s) ds d\theta$ are bounded with $-(\int_{-d_2}^0 \int_{t+\theta}^t \dot{z}^T(s) ds d\theta)^T Z_9 \int_{-d_2}^0 \int_{t+\theta}^t \dot{z}^T(s) ds d\theta$ and $-(\int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}^T(s) ds d\theta)^T$ $Z_{10} \int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}^T(s) ds d\theta$ and the $-\int_{-d_2}^0 \int_{t+\theta}^t \dot{z}^T(s) ds d\theta$, $-\int_{-d_2}^{-d_1} \int_{t+\theta}^t \dot{z}^T(s) ds d\theta$ are not retained as augmented variable, not replaced by $d_2 z(t) - \int_{t-d_2}^t z(s) ds$ and $(d_2-d_1)z(t) - \int_{t-d_2}^{t-d_1} z(s) ds$, which yield less conservative results.

The case in which only two additive time-varying components appear in the state has been considered, and the idea in this paper can be easily extended to the following systems with multiple additive delay components.

## IV. EXAMPLES

In the section, a example is given to demonstrate the benefits of the proposed method. Consider the system (5) with parameters [26]:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, D = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}$$

$k_1 = 0.4, k_2 = 0.8$

$f_1(s) = 0.2(|s+1| - |s-1|), f_2(s) = 0.4(|s+1| - |s-1|)$.

when $d_{11} = d_{21} = 0$, $d_{12} \leq 0.8$, $d_{22} \leq 2.2236$, The global asymptotic stability of (5) is listed in Table 1. The corresponding upper bounds of $d_{22}$ for various $d_{12}$ derived by Theorem 1 and methods in [6], [7], [25] and [26] are listed in Table 1. It is chear that our results in this paper are significant better than those in [6], [7], [25] and [26]. On the other hand, the previous results cannot handle the case for $2.0164 \leq d_{22} \leq 2.2236$. However, it is seen that we calculated the value of $d_{22}$ for $d_{21} = d_{11} = 0.1$ in this paper.

TABLE I
ALLOWABLE UPPER BOUND OF $d_{22}$ FOR VARIOUS $d_{12}$

| $d_{11}$ | $d_{21}$ | Method | $d_{12}$ | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|
| 0 | 0 | [25] | $d_{22}$ | 0.8831 | 0.6831 | 0.4831 |
| 0 | 0 | [6] | $d_{22}$ | 0.8831 | 0.6832 | 0.4843 |
| 0 | 0 | [7] | $d_{22}$ | 1.5666 | 1.3668 | 1.1664 |
| 0 | 0 | [26] | $d_{22}$ | 2.0164 | 1.8203 | 1.6197 |
| 0 | 0 | Theorem 1 | $d_{22}$ | 2.2236 | 2.0133 | 1.8894 |
| 0.1 | 0.1 | Theorem 1 | $d_{22}$ | 2.2477 | 2.1886 | 2.0172 |

## V. CONCLUTION

This paper has investigated the delay-dependent stability problem for neural networks with two additive time-varying delay components. Some less conservative stability criteria have been obtained by considering the relationship between the time-varying delay and its lower and upper bounds when calculating the upper bound of the derivative of Lyapunov functional. A numerical example has been given to demonstrate the effectiveness of the presented criteria and their improvement over the existing results.

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