

N-Sun Decomposition of Complete Graphs and Complete Bipartite Graphs

R. Anitha, and R. S. Lekshmi

Abstract—Graph decompositions are vital in the study of combinatorial design theory. Given two graphs G and H , an H -decomposition of G is a partition of the edge set of G into disjoint isomorphic copies of H . An n -sun is a cycle C_n with an edge terminating in a vertex of degree one attached to each vertex. In this paper we have proved that the complete graph of order $2n$, K_{2n} can be decomposed into $n-2$ n -suns, a Hamilton cycle and a perfect matching, when n is even and for odd case, the decomposition is $n-1$ n -suns and a perfect matching. For an odd order complete graph K_{2n+1} , delete the star subgraph $K_{1, 2n}$ and the resultant graph K_{2n} is decomposed as in the case of even order. The method of building n -suns uses Walecki's construction for the Hamilton decomposition of complete graphs. A spanning tree decomposition of even order complete graphs is also discussed using the labeling scheme of n -sun decomposition. A complete bipartite graph $K_{n, n}$ can be decomposed into $n/2$ n -suns when $n/2$ is even. When $n/2$ is odd, $K_{n, n}$ can be decomposed into $(n-2)/2$ n -suns and a Hamilton cycle.

Keywords—Hamilton cycle, n -sun decomposition, perfect matching, spanning tree.

I. INTRODUCTION

BY a graph $G = (V, E)$ we mean a finite undirected graph without loop unless otherwise stated. A cycle of length n is denoted by C_n . An n -sun is a cycle C_n with an edge terminating from each vertex of C_n [1]. Thus every n -sun graph contains exactly one cycle of length n and n pendant vertices. A decomposition of a graph is a collection of edge-disjoint subgraphs G_1, G_2, \dots, G_n of G such that every edge of G belongs to exactly one G_i . Graph decompositions, known for its applications in combinatorial design theory, have been studied since the mid nineteenth century. Several decades after its introduction, Walecki has the credit of constructing Hamilton cycle decomposition of complete graphs [2]-[4]. Complete graphs are decomposed in many ways. In this paper we have decomposed complete graphs of even order, K_{2n} into n -suns. The decomposition is based on Walecki's construction of complete graphs into Hamilton cycles. The aim is to provide a systematic approach to the decomposition with a

labeling scheme. By an orderly removal of edges from the cycles in n -suns, spanning tree decomposition is also possible in K_{2n} . Each of these spanning trees has the specialty of containing a perfect matching of K_{2n} . Complete bipartite graphs $K_{n, n}$ are equally significant. Their n -sun decompositions are also given.

II. PRELIMINARIES

Let G be a graph with n vertices and m edges. A graph in which any two distinct points are adjacent is called a *complete graph*. A spanning cycle in a graph is called a *Hamilton cycle* of the graph. A *perfect matching* or *1-factor*, denoted as I , of a graph G of even order is a set of mutually non-adjacent edges, which covers all vertices of G [5]. A Hamilton cycle of a graph of even order is the union of two perfect matchings of G .

A *Hamilton decomposition* is a partitioning of the edge set of G into Hamilton cycles if G is $2d$ -regular or into Hamilton cycles and a perfect matching if G is $(2d+1)$ -regular [4]. The complete graph K_n has Hamilton decomposition for all $n > 2$. Any complete graph K_{2n} can be decomposed into $(n-1)/2$ Hamilton cycles if n is odd and $(n-2)/2$ Hamilton cycles if n is even [6].

In the decomposition of K_{2n} into n -suns we choose C_n to be the Hamilton cycles of of its subgraph, K_n .

III. MAIN DEFINITIONS AND RESULTS

An *n -sun decomposition* of the complete graph K_{2n} is partitioning the edge set into n -suns and a perfect matching if n is odd and n -suns, a perfect matching and a Hamilton cycle if n is even.

A graph G is said to have *total n -sun decomposition* if every edge belongs to exactly one n -sun of the decomposition. An example graph is shown in Fig. 1. Note that regularity of graphs does not play a role in total n -sun decomposition.

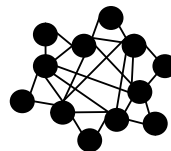


Fig. 1 A graph with total n -sun decomposition

A graph with $2n$ vertices may have an n -sun as its subgraph but need not have n -sun decomposition and not all regular

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graphs have an n-sun decomposition which can be observed from Fig. 2.

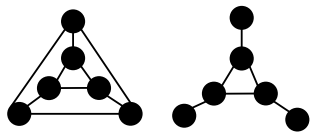


Fig. 2 A 3-regular graph and its 3-sun

Lemma 3.1: A necessary condition for the n-sun decomposition of K_{2n} into t isomorphic copies is that $|E|$ is divisible by $(1+2t)n$ when n is odd and $(3+2t)n$ when n is even.

Proof: When n is odd, the total of $n(2n-1)$ edges in K_{2n} are divided as follows. Since there are 2n edges in an n-sun, 2nt edges are in t isomorphic copies of n-suns and n edges in the perfect matching. Also $n(2n-1) = 2nt + n + 2n$ implies $t = n - 1$. The same line of reasoning is applicable for odd n in which $t = n - 2$.

By the above lemma, K_{2n} can be decomposed into n-1 n-suns and a perfect matching when n is odd. For n even, K_{2n} can be decomposed into n-2 n-suns, a Hamilton cycle and a perfect matching. We use a labeling scheme to decompose K_{2n} ($n > 2$) into n-suns. A proper labeling of G with k vertices is a bijection $f: V(G) \rightarrow \{1, 2, \dots, k\}$.

Theorem 3.2: The complete graph K_{2n} has n-sun decomposition for all odd $n > 2$.

Proof: Consider the complete graph K_{2n} where n is odd. Split the vertex set $V = \{v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, \dots, v_{2n}\}$ of K_{2n} into two such that $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V_2 = \{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$. Let U and W be the induced subgraph of K_{2n} with vertex subset V_1 and V_2 respectively. Then U and W are complete graphs of odd order and have $n(n-1)/2$ edges each. The remaining n^2 edges of K_{2n} form an edge cut (whose removal disconnects K_{2n}). To construct n-suns, we find Hamilton cycles of U and W using Walecki's construction and to each Hamilton cycle, adjoin n edges from the edge cut. A similar functional notation of [2] is adopted in finding the Hamilton cycles of U and W.

Since n is odd, U can be decomposed into $(n-1)/2$ Hamilton cycles and a perfect matching and hence the maximum number of edge disjoint n-suns possible in U is $(n-1)/2$.

In U, let C be the Hamilton cycle $v_1v_2v_3v_nv_4v_{n-1}v_5v_{n-2} \dots v_{\frac{n-1}{2}+4}v_{\frac{n-1}{2}+1}v_{\frac{n-1}{2}+3}v_{\frac{n-1}{2}+2}v_1$ and α be the permutation $(v_1)(v_2v_3 \dots v_n)$. Then $C, \alpha(C), \alpha^2(C), \dots, \alpha^{\frac{n-3}{2}}(C)$ is a Hamilton cycle decomposition of U. For simplicity, let Φ_k denote the k^{th} Hamilton cycle and $\Phi_k(v_i)$ denote vertex v_i in that Hamilton cycle, where $k = 1, 2, \dots, \frac{n-1}{2}$. Then $\Phi_k = \alpha^{k-1}(C)$.

Append the edges to the Hamilton cycles as

$$\Phi_k(v_i) = \begin{cases} n+k+i-1 & \text{if } k+i-1 \leq n \\ [(k+i-1) \bmod n] + n & \text{if } k+i-1 > n \end{cases}, i = 1, 2, \dots, n.$$

Similarly in W let C' be the Hamilton cycle $v_{n+1}v_{n+2}v_{n+3}v_{2n}v_{n+4}v_{2n-1}v_{n+5}v_{2n-2} \dots v_{\frac{n-1}{2}+n+4}v_{\frac{n-1}{2}+n+1}v_{\frac{n-1}{2}+n+3}v_{\frac{n-1}{2}+n+2}v_{n+1}$ and β be the permutation $(v_{n+1})(v_{n+2}v_{n+3} \dots v_{2n})$. Then

$C', \beta(C'), \beta^2(C'), \dots, \beta^{\frac{n-3}{2}}(C')$ is a Hamilton cycle decomposition of W. Let $\Phi'_k(v_i)$ denote vertex v_i in the k^{th}

Hamilton cycle Φ'_k in W, where $k = 1, 2, \dots, \frac{n-1}{2}$. Then

$$\Phi'_k = \beta^{k-1}(C'). \quad \text{Append the edges as}$$

$$\Phi'_k(v_{n+i}) = \begin{cases} k+i & \text{if } k+i \leq n \\ (k+i) \bmod n & \text{if } k+i > n \end{cases} \text{ where } i = 1, 2, \dots, n.$$

Finally, the perfect matchings of U and W form a perfect matching of K_{2n} . The matching is given as (v_i, v_j)

$$\text{where } i = 1, 2, \dots, n \text{ and } j = \begin{cases} \frac{3n-1}{2} + i, & \text{if } i \leq \frac{n+1}{2} \\ \left[\left(\frac{3n-1}{2} + i \right) \bmod n \right] + n, & \text{if } i > \frac{n+1}{2} \end{cases}. \text{ This}$$

perfect matching along with the n-suns decomposes K_{2n} .

An illustration of the n-sun decomposition of K_{14} into six 7-suns and a perfect matching is shown in Fig. 4 of Appendix.

Corollary 3.3: $K_{2n} + C_n$ can be decomposed into n n-suns when n is odd.

From the previous theorem there are n-1 n-suns of K_{2n} . Add n multi edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), \dots, (v_{n-1}, v_n), (v_n, v_1)$ to form an n-cycle of K_{2n} . The perfect matching of K_{2n} in the previous theorem with the multi edges forms another n-sun.

Spanning trees are well known in the literature as minimally connected subgraphs of a graph. They find immense applications in networks whenever there is a necessity of unique paths between vertices. K_{2n} can be decomposed into n spanning trees [2], [4]. The next corollary gives a labeling scheme for the spanning tree decomposition of K_{2n} . By properly removing one edge from each cycle of n-sun and adding them with the perfect matching, we get the required decomposition.

Corollary 3.4: K_{2n} can be decomposed into n spanning trees each containing a perfect matching.

In Theorem 3.4, delete edges (v_{k+1}, v_{k+2}) from $\alpha^{k-1}(C)$ and edges (v_{n+k+1}, v_{n+k+2}) from $\beta^{k-1}(C)$, $k = 1, 2, \dots, \frac{n-1}{2}$. These

edges with the perfect matching give a spanning tree. The edge deleted n-suns form another n-1 spanning trees.

Theorem 3.5: The complete graph K_{2n} has n-sun decomposition for all even $n > 2$.

Proof: The maximum number of n-sun decomposition possible when n is even is n-2. The procedure for the decomposition is the same as that for odd n except for a slight

change in the labels. Let the vertex set of K_{2n} be split as $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V_2 = \{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$. Let X and Y be the induced subgraph of K_{2n} with vertex subset V_1 and V_2 respectively. Here let C be the Hamilton cycle $V_1 V_2 V_3 V_n V_4 V_{n-1} V_5 V_{n-2} \dots V_{\frac{n}{2}} V_{\frac{n}{2}+3} V_{\frac{n}{2}+1} V_{\frac{n}{2}+2} V_1$ and α be the permutation $(v_1)(v_2 v_3 \dots v_n)$ as in the odd case. Then $C, \alpha(C), \alpha^2(C), \dots, \alpha^{\frac{n-4}{2}}(C)$ is a Hamilton cycle decomposition of X .

Let $\Phi_k(v_i)$ denote vertex v_i in the k^{th} Hamilton cycle of X , where $k = 1, 2, \dots, \frac{n-2}{2}$. Then $\Phi_k = \alpha^{k-1}(C)$. Append the edges to the Hamilton cycles of X as $\Phi_k(v_i) = \begin{cases} n+k+i-1 & \text{if } k+i-1 \leq n \\ [(k+i-1) \bmod n] + n & \text{if } k+i-1 > n \end{cases}, k = 1, 2, \dots, \frac{n-2}{2}$ and $i = 1, 2, \dots, n$.

Similarly in Y , $C', \beta(C'), \beta^2(C'), \dots, \beta^{\frac{n-4}{2}}(C')$ is a Hamilton cycle decomposition where C' is the cycle $V_{n+1} V_{n+2} V_{n+3} V_{2n} V_{n+4} V_{2n-1} V_{n+5} \dots V_{\frac{3n+3}{2}} V_{\frac{3n+1}{2}} V_{\frac{3n+2}{2}} V_{n+1}$ and β is the permutation $(v_{n+1})(v_{n+2} v_{n+3} \dots v_{2n})$. Let $\Phi'_k(v_i)$ denote vertex v_i in the k^{th} Hamilton cycle $\Phi'_k = \beta^{k-1}(C')$ of Y where $k = 1, 2, \dots, \frac{n-2}{2}$. Append the pendant edges using the labeling scheme $\Phi'_k(v_{n+i}) = \begin{cases} k+i & \text{if } k+i \leq n \\ (k+i) \bmod n & \text{if } k+i > n \end{cases}, i = 1, 2, \dots, n$. The perfect matching of $X: \{(v_i, v_j)\}$ and that of $Y: \{(v_{n+i}, v_{n+j})\}$ forms a perfect matching of K_{2n} where $i = 1, 2, \dots, \frac{n}{2}$ and $j = \begin{cases} \frac{n}{2} + 1, & \text{if } i = 1 \\ n - i + 2, & \text{if } i > 1 \end{cases}$. The remaining edges form a Hamilton

cycle whose labeled structure is shown in Fig. 3.

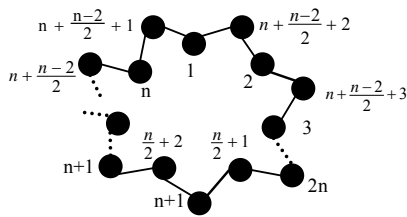


Fig. 3 The Hamilton cycle structure in the n -sun decomposition of K_{2n} , n even

A decomposition of K_{12} into four 6-suns, a perfect matching and a Hamilton cycle is shown in Fig. 5 of Appendix.

Corollary 3.6: Let $K_{2n} - I$ denote the subgraph of K_{2n} with a perfect matching removed. Then $(K_{2n} - I) + 2C_n$ can be decomposed into n n -suns.

Add two sets of n multi edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_n,$

$v_{n-1}), (v_n, v_1)$ to K_{2n} . These multi edges form two n -cycles. Since a Hamilton cycle is the union of two perfect matchings, append one matching each to the two n -cycles to obtain the required decomposition.

Corollary 3.7: An odd order complete graph can be decomposed into n -suns, a Hamilton cycle, a perfect matching and a star graph when n is even. For odd n , n -suns, a perfect matching and a star partition the edges of K_{2n+1} .

Proof: In the case of odd order complete graphs K_{2n+1} , n -sun decomposition is not possible, since the maximum matching is $2n$. By the removal of a star graph $K_{1, 2n}$, the resultant is a complete graph of even order which has n -sun decomposition.

In the next section we discuss about the n -sun decomposition of complete bipartite graphs $K_{m, m}$, $m = 2n$. We split $K_{m, m}$ into two complete bipartite subgraphs $K_{n, n}$ and the remaining edges form an edge cut of $K_{m, m}$. We find Hamilton cycles in each subgraph $K_{n, n}$ and append edges from the edge cut for the pendants of the n -sun. Since the minimum cycle length in bipartite graphs is four and to append pendants, $m \geq 4$.

In [7] decomposition of r -partite graphs into edge-disjoint Hamilton cycles is discussed. We brief now the procedure to find edge-disjoint Hamilton cycles for bipartite graphs which uses consecutive perfect matchings (1-factors).

Let the vertices of $K_{n, n}$ be partitioned as $U = \{u_1, u_2, \dots, u_n\}$ and $V = \{v_1, v_2, \dots, v_n\}$. Let the set of perfect matching be where $\{F_j = \bigcup_{i=1}^n u_i v_{i+j}, j = 1, 2, \dots, n\}$ and the suffices of v are taken modulo n . Then $H_k = F_{2k-1} \cup F_{2k}, k = 1, 2, \dots, n/2$ is a set of edge disjoint Hamilton cycles of $K_{n, n}$. When n is even, $K_{n, n}$ can be decomposed into $n/2$ edge-disjoint Hamilton cycles. For odd n , the decomposition is $(n-1)/2$ Hamilton cycles plus a perfect matching. When n is odd the edges of the perfect matching is $\{u_i v_i\}$ where $i = 1, 2, \dots, n$.

An n -sun decomposition of the complete bipartite graph $K_{n, n}$ is partitioning the edge set into n -suns and n copies of C_4 if $n/2$ is odd and n -suns if $n/2$ is even.

Theorem 3.8: The complete bipartite graph $K_{n, n}$ has n -sun decomposition for all $n/2$ even.

Proof: To simplify the labeling scheme, let the vertices of $K_{n, n}$ be partitioned as U and V ,

$$U = \{u_1, u_2, \dots, u_{n/2}, u'_1, u'_2, \dots, u'_{n/2}\} \text{ and}$$

$$V = \{v_1, v_2, \dots, v_{n/2}, v'_1, v'_2, \dots, v'_{n/2}\}.$$

Let us split $K_{n, n}$ into $K'_{n/2, n/2}, K''_{n/2, n/2}$ and an edge-cut with $n^2/2$ edges where $K'_{n/2, n/2}$ and $K''_{n/2, n/2}$ are the induced subgraphs formed by $\{u_1, u_2, \dots, u_{n/2}, v_1, v_2, \dots, v_{n/2}\}$ and $\{u'_1, u'_2, \dots, u'_{n/2}, v'_1, v'_2, \dots, v'_{n/2}\}$ respectively.

The maximum number of n -suns possible in the decomposition of $K_{n, n}$ is $n/2$ since the maximum number of Hamilton cycles in $K'_{n/2, n/2}$ and $K''_{n/2, n/2}$ put together is $n/2$.

APPENDIX

Let $\{F_j = \bigcup_{i=1}^{n/2} u_i v_{i+j}, j = 1, 2, \dots, n/2\}$, be the perfect matchings of $K'_{n/2, n/2}$, the suffix of v being taken modulo $n/2$. Let $H_k = F_{2k-1} \cup F_{2k}, k = 1, 2, \dots, n/4$ be a Hamilton cycle decomposition of $K'_{n/2, n/2}$. Append edges from the edge cut as follows: $H_k(u_i) = v'_p, H_k(v_i) = u'_p, p \equiv (k+i-1 \pmod{n/2}), i = 1, 2, \dots, n/2$ and $k = 1, 2, \dots, n/4$. Similarly let $\{F'_j = \bigcup_{i=1}^{n/2} u'_i v'_{i+j}, j = 1, 2, \dots, n/2\}$, be the perfect matchings of $K''_{n/2, n/2}$, the suffix of v being taken modulo $n/2$. Let $H'_k = F'_{2k-1} \cup F'_{2k}, k = 1, 2, \dots, n/4$ be a Hamilton cycle decomposition of $K''_{n/2, n/2}$. Append edges for the pendants of n -sun as $H'_k(u'_i) = v_{k+i}$ and $H'_k(v'_i) = u_{k+i}, i = 1, 2, \dots, n/2, k = 1, 2, \dots, n/4$, the suffices $k+i$ are taken modulo $n/2$. The Hamilton cycles with the appended edges form n -suns. Since every edge is in exactly one n -sun, $K_{n, n}$ has a total n -sun decomposition.

Theorem 3.9: The complete bipartite graph $K_{n, n}$ has n -sun decomposition for all $n/2$ odd.

Proof: Let the notations U, V, F_k and H_k be as in Theorem 3.8 where $n/2$ is odd, $k = 1, 2, \dots, (n-2)/4$ and the maximum number of n -suns possible is $(n-1)/2$. By the construction, the perfect matching left out in $K'_{n/2, n/2}$ and $K''_{n/2, n/2}$ after the n -suns are constructed are $F_{n/2}$ and $F'_{n/2}$ respectively. These edges along with the edge set $\{u_i v'_q\} \cup \{u'_i v_q\}, i = 1, 2, \dots, n/2$ and $q = i+(n-1)/2$ is taken modulo $n/2$, forms a Hamilton cycle. In fact $\{u_i v'_q\} \cup \{u'_i v_q\}$ is a perfect matching of $K_{n, n}$.

Examples of n -sun decompositions of $K_{4, 4}$ and $K_{6, 6}$ are shown in Fig. 6 and Fig. 7 of Appendix.

IV. CONCLUSION

The aim of this communication has been to present a new kind of decomposition of K_{2n} and $K_{n, n}$. It is hoped that this decomposition may stimulate further studies on n -sun decompositions. The n -suns and the Hamilton cycles of K_{2n} have close association since both are spanning subgraphs with exactly one cycle. When n is odd, the total number of n -suns in the decomposition of K_{2n} is $n-1$ which is exactly the same number as in the decomposition of K_{2n} into Hamilton cycles, whereas for even n , the number is $n-2$ which is one less than the number of Hamilton cycles in the decomposition. Also the deletion of any one edge of the cycle in the n -sun or Hamilton cycle results in a spanning tree where the tree contains a perfect matching. Since the maximum degree of a vertex in the n -sun is three, the spanning trees obtained from n -suns also have the same maximum degree. The important feature of these spanning trees is that the diameter is $(n/2)+1$ for all n where as in Hamilton paths the diameter is $n-1$. An n -sun decomposition of $K_{n, n}$ is also discussed. Finding a strong sufficient condition for the existence of n -sun and total n -sun decomposition of K_{2n} and $K_{n, n}$ will be well appreciated. Tree decomposition for $K_{n, n}$ using n -suns can be studied.

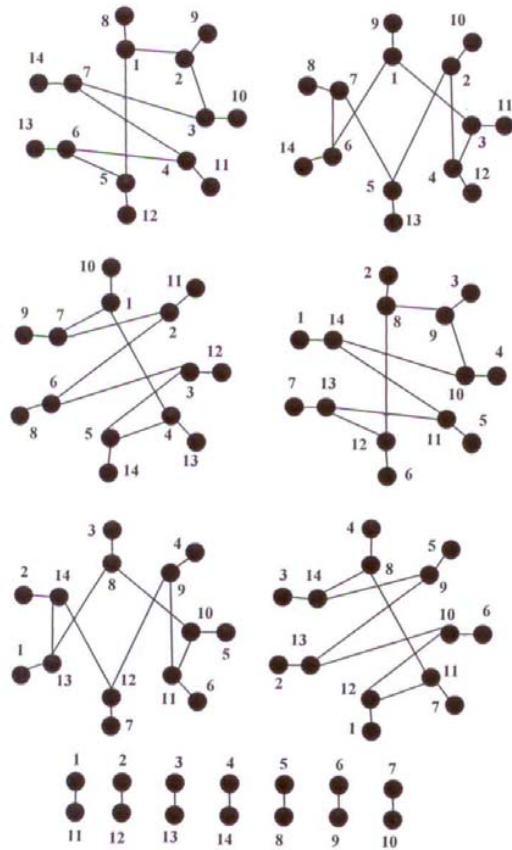


Fig. 4 A 7-sun decomposition of K_{14}

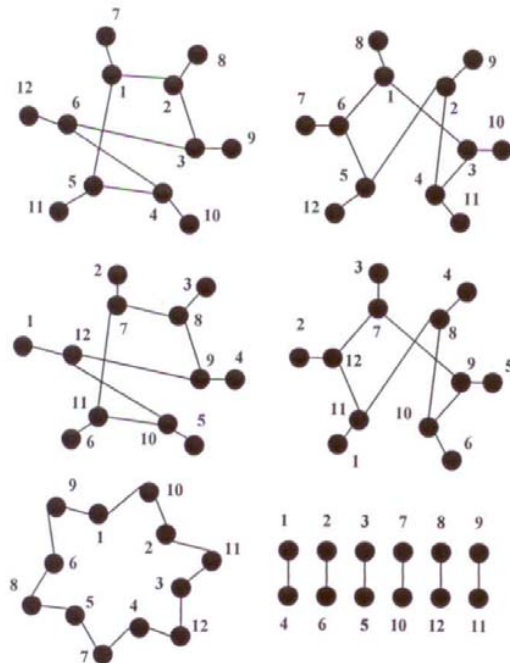


Fig. 5 A 6-sun decomposition of K_{12}

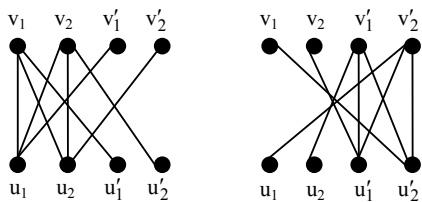


Fig. 6 A 4-sun decomposition of $K_{4,4}$

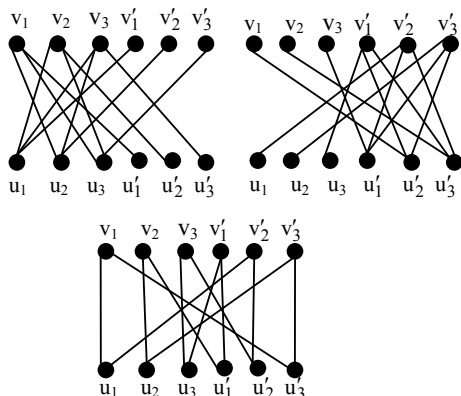


Fig. 7 A 6-sun decomposition of $K_{6,6}$

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