Mutually Independent Hamiltonian Cycles of $C_n \times C_n$

Kai-Siou Wu and Justie Su-Tzu Juan*

Abstract—In a graph G, a cycle is Hamiltonian cycle if it contains all vertices of G. Two Hamiltonian cycles $C_1 = \langle u_0, u_1, u_2, ..., u_{n-1}, u_0 \rangle$ and $C_2 = \langle v_0, v_1, v_2, ..., v_{n-1}, v_0 \rangle$ in G are independent if $u_0 = v_0, u_i \neq v_i$ for all $1 \leq i \leq n-1$. In G, a set of Hamiltonian cycles $C = \{C_1, C_2, ..., C_k\}$ is mutually independent if any two Hamiltonian cycles of C are independent. The mutually independent Hamiltonicity $\text{IHC}(G) = k$ means there exists a maximum integer k such that there exists k-mutually independent Hamiltonian cycles start from any vertex of G. In this paper, we prove that $\text{IHC}(C_n \times C_n) = 4$, for $n \geq 3$.

Keywords—Hamiltonian, independent, cycle, Cartesian product, mutually independent Hamiltonicity.

I. INTRODUCTION

CYCLES the fundamental class of network topologies for parallel and distributed computing, is suitable for designing simple algorithms with low communication costs [1]–[3]. For solving various algebraic problems, graph problems, and some parallel applications, there are many efficient parallel algorithms designed on cycles, such as those in image and signal processing, can be found in [1], [4]. If the network can be embed cycles, then we can use these algorithms as control/data flow structures for distributed computing in arbitrary networks so that the algorithms designed on cycles can be executed on the embedded cycles.

When designing a structure in interconnection network, we often transform this problem to graph for discussing and studying. Formally, A processor will be transformed to be a vertex and the connection between two processors will be transformed to be an edge. By this transformation, it can transform an interconnection network to a graph.

For definitions and notations, we follow [5]. A *graph* $G = (V, E)$, V, E are finite set, where V is *vertex* set and E is *edge* set define as $E \subseteq \{(u, v) : (u, v) \text{ is an unordered pair of } V\}$. Two vertices u, v are *adjacent* if $(u, v) \in E(G), u, v \in V(G)$. A *path* $P[v_0, v_k] = \langle v_0, v_1, ..., v_k \rangle$ in G, define as a sequence of adjacent vertices and for all $0 \leq i < j \leq k, v_i \neq v_j$. Let $P(i) = v_{i-1}$ be the ith vertex of path P. A *cycle* $C[v_0, v_k] = \langle v_0, v_1, ..., v_k \rangle$ in G is a sequence of adjacent vertices where $v_0 = v_k, k \geq 3$ and for all $0 \leq i < j \leq k, v_i \neq v_j$. In G, a cycle is *Hamiltonian cycle* if it contains all vertices of G. Two Hamiltonian cycles $C_1 = \langle u_0, u_1, u_2, ..., u_n, u_0 \rangle$ and $C_2 = \langle v_0, v_1, v_2, ..., v_n, v_0 \rangle$ in G are *independent* if $u_0 = v_0, u_i \neq v_i$ for all $1 \leq i \leq n$. A set of Hamiltonian cycles $C = \{C_1, C_2, ...C_n\}$ start from the same vertex in G is *mutually independent* if any two Hamiltonian cycles of

C are independent. The *mutually independent Hamiltonicity*, $\text{IHC}(G) = k$, of a graph G means there exists a maximum integer k such that there exists k-mutually independent Hamiltonian cycles start from any vertex of G. In this paper we define C_n is a cycle with $|V(C_n)| = n$.

Broadcasting is an information dissemination process that involves one node in a network sending pieces of information to all other nodes in the network. Constructing a set of k-mutually independent Hamiltonian cycles enables us to efficiently broadcast a message formed of k pieces on a graph G. The common starting vertex acts as the source to send the k pieces of the given message along the k parallel Hamiltonian cycles. This broadcasting can be done in $O(n)$ time using an all-port model.

Given two graph $G = (V_G, E_G), H = (V_H, E_H)$, the Cartesian product of G and $H, G \times H$, is a graph which $V = V_G \times V_H$, $E = \{((p_1, q_1), (p_2, q_2)) : (p_1, q_1), (p_2, q_2) \in V, \text{ and } (p_1 = p_2, (q_1, q_2) \in E_H, \text{ or } q_1 = q_2, (p_1, p_2) \in E_G)\}$. For a vertex $u \in V(G)$, the *neighbor* of u, $N_G(u)$, is the set $\{v : (u, v) \in E(G)\}$. And $\deg_G(u) = |N_G(u)|$ is called the *degree* of u. In a graph G, we use $\Delta(G)$ and $\delta(G)$ to denote the *maximum degree* and *minimum degree* of G such that $\Delta(G) = $ maximum degree, $\delta(G) = $ minimum degree. When we consider $\text{IHC}(G)$, these mutually independent Hamiltonian cycles are start from any vertex of G, then we can get a property as follows:

Remark 1. For any graph G, $\text{IHC}(G) \leq \delta(G)$.

The mutually independent Hamiltonicity of a graph can be represented by a Latin square. A *Latin square* of order n is an $n \times n$ matrix which each row and each column is an integer from 1 to n, and in the same row or column, each number occur exactly once. If we remove any k rows of Latin square for $1 \leq k < n$, we can get a *Latin rectangle*.

Mutually independent Hamiltonian cycle problem is an important property for graph theory, there are many studies for this problem on different graph [6]–[12]. In 2005, Sun et al. study on the mutually independent Hamiltonicity of hypercube [13]. In 2007, Hsieh et al. study on fault-free mutually independent Hamiltonian cycles in hypercubes with faulty edges [14]. Lin et al. study on the pancake graphs and the star graphs in 2009 [15]. Chang et al. study on (n, k)-star graphs in 2009 [10]. In this paper, we study on mutually independent Hamiltonicity of $C_n \times C_n$. And we prove that for $n \geq 3$, $\text{IHC}(C_n \times C_n) = 4$.

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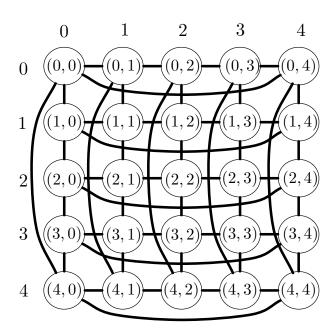
Fig. 1.   Symbol definition of $C_5 \times C_5$.

II. SYMBOL DEFINITION

Let $Z_n = \{0, 1, 2, ... n - 1\}$, $n \in N$ and let $V(C_n) = Z_n$. We refer to *toroidal mesh graph* [16] and define graph $C_n \times C_n$, $n \geq 3$ as follows: The vertex set $V(C_n \times C_n) = \{(x, y) : x, y \in Z_n\}$; The edge set $E(C_n \times C_n) = \{((u_1, u_2), (v_1, v_2)) : (u_1, u_2), (v_1, v_2) \in V(C_n \times C_n)$ and $u_1 = v_1$, $|u_2 - v_2| \in \{1, n - 1\}$ or $u_2 = v_2$, $|u_1 - v_1| \in \{1, n - 1\}\}$. We use this symbol definition to construct $C_5 \times C_5$ as Fig. 1. A graph G is a *k-regular* graph if for all $x \in V(G)$, $\deg_G(x) = k$. And a graph is *vertex transitive* if every vertex can be mapped to any other vertex by some automorphism [2]. By definition, $C_n \times C_n$ is a 4-regular graph with n^2 vertices and it is vertex transitive.

In $C_n \times C_n$, we use C^i to denote the *ith subgraph* of $C_n \times C_n$ for $i \in Z_n$, where the vertex set $V(C^i) = \{(i, v) : v \in Z_n\}$; The edge set $E(C^i) = \{((i, v_1), (i, v_2)) : |v_1 - v_2| \in \{1, n - 1\}$ and $0 \leq v_1 < v_2 \leq n - 1$, $v_1, v_2 \in V(C^i)\}$. Then $C_n \times C_n$ can be decomposed into n subgraph C^i for $i \in Z_n$, and each C^i is isomorphic to C_n. Let (i, x) and $(i, y) \in V(C^i)$, then we define $P_{i,x,y,+} = \langle (i, x), (i, x+1), (i, x+2), ..., (i, y) \rangle$ is a path of C^i which is from (i, x) to (i, y) and the index of vertices are increased. And $P_{i,x,y,-} = \langle (i, x), (i, x-1), (i, x-2), ..., (i, y) \rangle$ is a path of C^i which is from (i, x) to (i, y) and the index of vertices are decreased. Note that the operation $+$ or $-$ should be calculate on Z_n. In a cycle C, we use $C(t) = x$ for $1 \leq t \leq |C|$ to denote the tth vertex of C from the start vertex.

III. MAIN RESULT

We study on mutually independent Hamiltonicity of $C_n \times C_n$ in this section. First, we prove some lemmas and use them to show that IHC($C_n \times C_n$) = 4. We show results as follows.

0	1	2	$\cdots$	$n-1$
$n-1$	0	1	$\cdots$	$n-2$
$n-2$	$n-1$	0	$\cdots$	$n-3$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
1	2	3	$\cdots$	0

Fig. 2.   A Latin square order n, which subtract 1 from each element.

	$t_1 - n + 1$	$t_1 - ...$	$t_1 - 2$	$t_1 - 1$	t_1	$t_1 + 1$	$t_1 + 2$	$t_1 + ...$	$t_1 + n - 1$
P_1					0	$n-1$	$\cdots$	2	1
$b \neq 0$					0	$n-1$	$\cdots$	2	1
$b = 1$				0	$n-1$	$\cdots$	2	1	
$b = 2$			0	$n-1$	$\cdots$	2	1		
$b = ...$		0	$n-1$	$\cdots$	2	1			
$b = n-1$	0	$n-1$	$\cdots$	2	1				

Fig. 3.   A matrix of $u = a = v = 0, b \in Z_n, b \neq 0$.

Lemma 1. For any odd positive integer $n \geq 3$, IHC(C_n) = 2.
Proof. Let $V(C_n) = Z_n$, it is easy to see that there exits two Hamiltonian cycles H_1, H_2 as follow:
$H_1 : \langle 0, 1, 2, 3, ..., n-1, 0 \rangle$; $H_2 : \langle 0, n-1, n-2, n-3, ..., 1, 0 \rangle$. So, for all $1 \leq i \leq n, H_1(i) = i - 1$; $H_2(1) = 0$ and for all $2 \leq j \leq n, H_2(j) = n - j + 1$. Thus, for all $2 \leq k \leq n, H_1(k) + H_2(k) = n$. Since n is odd, $H_1(k) \neq H_2(k)$. Then there exists two independent Hamiltonian cycles start from vertex 0. Since cycle is vertex transitive, there exists two independent Hamiltonian cycles start from any vertex of C_n. Hence, IHC(C_n) ≥ 2. By Remark 1, we can know that IHC(C_n) = 2. $\square$

It is not too difficult to see that the following lemma is true.

Lemma 2. For any graph G, if there exists two P_1, P_2 and a subgraph $C_n \subseteq G$ with $V(C_n) = \{x_0, x_1, ..., x_{n-1}\}$. For all $1 \leq i \leq 2, 0 \leq j \leq n - 1, P_i(t_i + j) \in V(C_n)$. Let $P_1(t_1 + j) = x_{u-j}, P_2(t_2 + j) = x_{v-j}$ for $0 \leq j \leq n - 1$. If $u - a = a \bmod n, t_1 - t_2 = b \bmod n$ for some, $a, b \in Z_n$ and $a \neq b$, then these 2 paths are not at the same vertex of C_n in the same time.

Fig. 3 is an illustration of Lemma 2.

Theorem 1. For $n \geq 3$ and n is odd, IHC($C_n \times C_n$) = 4.
Proof. Since $C_n \times C_n$ is 4-regular graph, IHC($C_n \times C_n$) ≤ 4 by Remark 1. Next, we will show that IHC($C_n \times C_n$) ≥ 4 by constructing 4-mutually independent Hamiltonian cycles from any vertex of $C_n \times C_n$. Without loss of generality, we may assume that start vertex $e = (0, 0)$ because $C_n \times C_n$ is vertex transitive. We construct 4-mutually independent Hamiltonian cycles H_1, H_2, H_3 and H_4 start from e as follows:
$H_1 = \langle (0, 0), P_{0,0,n-1,+}, (0, n - 1), (1, n - 1), P_{1,n-1,n-2,+}, (1, n - 2), (2, n - 2), P_{2,n-2,n-3,+}, (2, n - 3), ..., (n - 1, 1), P_{n-1,1,0,+}, (n - 1, 0), (0, 0) \rangle$.
$H_2 = \langle (0, 0), (1, 0), P_{1,0,1,-}, (1, 1), (2, 1), P_{2,1,2,-}, (2, 2), (3, 2), P_{3,2,3,-}, (3, 3), ..., (n - 1, n - 2), P_{n-1,n-2,n-1,-},$
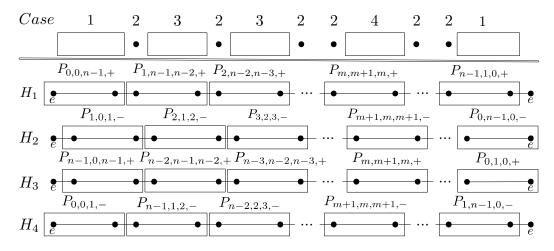
Fig. 4. The construction of 4 Hamiltonian cycles of $C_n \times C_n$ start at e

$(n-1)$, $(n-1)$, $(0, n-1)$, $P_{0,n-1,0,-}$, $(0,0)\rangle$.

$H_3 = \langle (0,0), (n-1,0), P_{n-1,0,n-1,+}, (n-1,n-1),$
$(n-2,n-1), P_{n-2,n-1,n-2,+}, (n-2,n-2), (n-3,n-2),$
$P_{n-3,n-2,n-3,+}, (n-3,n-3), ..., (0,1), P_{0,1,0,+}, (0,0)\rangle$.

$H_4 = \langle (0,0), P_{0,0,1,-}, (0,1), (n-1,1), P_{n-1,1,2,-},$
$(n-1,2), (n-2,2), P_{n-2,2,3,-}, (n-2,3), ..., (1,n-1),$
$P_{1,n-1,0,-}, (1,0), (0,0)\rangle$.

We represent H_1, H_2, H_3 and H_4 as graph in Fig. 4. According to vertices $H_i(t)$, for $2 \leq t \leq n^2, i \in \{1,2,3,4\}$ belong to different subgraph C^j, for $j \in Z_n$ or not, we have following cases:

Case 1 : $2 \leq t \leq n$ and $n(n-1) + 2 \leq t \leq n^2$.

For any $2 \leq t_1 \leq n$ and $n(n-1)+2 \leq t_2 \leq n^2$. By the construction of H_1, H_2, H_3 and H_4, we can know that $H_1(t_1), H_4(t_1) \in V(C^0)$; $H_2(t_1) \in V(C^1)$; $H_3(t_1) \in V(C^{n-1})$ and $H_1(t_2) \in V(C^1)$; $H_2(t_2), H_3(t_3) \in V(C^0)$; $H_4(t_2) \in V(C^{n-1})$. By Lemma 1, $H_1(t_1) \neq H_4(t_1)$ and $H_2(t_2) \neq H_3(t_2)$. In the same time, $H_2(t_1), H_3(t_1); H_1(t_2), H_4(t_2)$ are in different subgraph, so $H_i(t) \neq H_j(t)$, for $i,j \in \{1,2,3,4\}, i \neq j$, for $2 \leq t \leq n$ or $n(n-1)+2 \leq t \leq n^2$. That is, H_1, H_2, H_3 and H_4 are mutually independent for this case.

Case 2 : $t = in+1$, for all $1 \leq i \leq n-1$.

From the construction of 4 Hamiltonian cycles, we can know that for all $1 \leq i \leq n-1, t = in+1$, $H_1(t) = (i, n-i); H_2(t) = (i,i); H_3(t) = (n-i, n-i); H_4(t) = (n-i, i)$. Since n is odd, $n-i \neq i$. It means that when $t = in+1$, for $1 \leq i \leq n-1$ $H_1(t), H_2(t), H_3(t), H_4(t)$ are distinct.

Case 3 : $in+2 \leq t \leq in+n$, for all $1 \leq i \leq n-2$ and $i \neq (n-1)/2$.

For all $1 \leq i \leq n-2$ and $i \neq (n-1)/2$ and $in+2 \leq t \leq in+n$, $H_1(t) \in V(C^i), H_2(t) \in V(C^{i+1}), H_3(t) \in V(C^{n-i-1})$, and $H_4(t) \in V(C^{n-i})$. It is trivial that $i \neq i + 1, n-i+1 \neq n-i$. Since $i \neq (n-1)/2$, then $i \neq n-i-1; i+1 \neq n-i$. Besides, $i \neq n-i$ and $i+1 \neq n-i-1$, for n is odd. Hence, H_1, H_2, H_3 and H_4 are mutually independent in this range.

Case 4 : $in+2 \leq t \leq in+n$, when $i = (n-1)/2$.

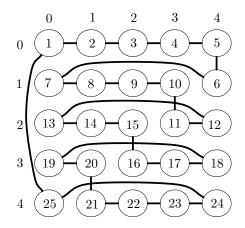When $t = in+2$, $H_1(t-1) = H_3(t); H_4(t-1) = H_2(t)$,

Fig. 5. Hamiltonian cycles H_1 in $C_5 \times C_5$.

and for all $in+2 \leq t \leq in+n, i = (n-1)/2, H_1(t), H_3(t) \in V(C^i); H_2(t), H_4(t) \in V(C^{i+1})$. In this interval, $\langle H_1(in+1), H_1(in+2), ..., H_1(in+n) \rangle = \langle H_3(in+2), H_3(in+3), ..., H_3(in+n+1) \rangle = P_{i,i+1,i,+}$ and $\langle H_2(in+2), H_2(in+3), ..., H_2(in+n+1) \rangle = \langle H_4(in+1), H_4(in+2), ..., H_4(in+n) \rangle = P_{i+1,i,i+1,-}$.

From the above discussion H_1, H_3 and H_2, H_4 are independent. Besides, H_1, H_3 and H_2, H_4 are in different subgraphs. Thus, H_1, H_2, H_3 and H_4 are mutually independent in this range.

From the above 4 cases, we can know that H_1, H_2, H_3 and H_4 start from e are mutually independent. Hence, $\text{IHC}(C_n \times C_n) \geq 4$ and it implies $\text{IHC}(C_n \times C_n) = 4$ for odd integer $n \geq 3$. □

We construct 4-mutually independent Hamiltonian cycles of $C_5 \times C_5$ in Fig. 5, 6, 7, 8 as an example of Theorem 1. Next, we discuss the number of $\text{IHC}(C_n \times C_n)$ for n is an even positive integer with $n \geq 4$.

Theorem 2. For $n \geq 4$ and n is even, $\text{IHC}(C_n \times C_n) = 4$.
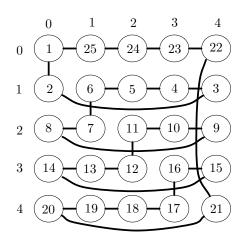
Fig. 6.   Hamiltonian cycles H_2 in $C_5 \times C_5$.
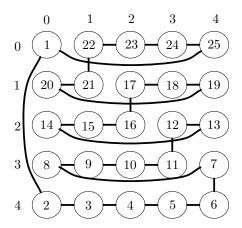

Fig. 7.   Hamiltonian cycles H_3 in $C_5 \times C_5$.
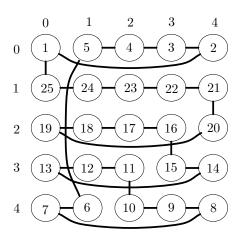

Fig. 8.   Hamiltonian cycles H_4 in $C_5 \times C_5$.

Proof. Since $C_n \times C_n$ is 4-regular graph, $IHC(C_n \times C_n) \leq 4$ by Remark 1. Without loss of generality, we may assume that start vertex $e = (0,0)$ because $C_n \times C_n$ is vertex transitive. For $n = 4$, we construct 4-mutually independent Hamiltonian

t	1	2	3	4	5	6	7	8	9
H_1	(0,0)	(0,1)	(1,1)	(1,0)	(1,3)	(1,2)	(2,2)	(2,1)	(2,0)
H_2	(0,0)	(1,0)	(1,3)	(1,2)	(0,2)	(0,3)	(3,3)	(3,0)	(3,1)
H_3	(0,0)	(3,0)	(2,0)	(2,3)	(2,2)	(2,1)	(3,1)	(3,2)	(3,3)
H_4	(0,0)	(0,3)	(3,3)	(3,2)	(3,1)	(0,1)	(0,2)	(1,2)	(1,3)

t	10	11	12	13	14	15	16	17
H_1	(2,3)	(3,3)	(3,0)	(3,1)	(3,2)	(0,2)	(0,3)	(0,0)
H_2	(3,2)	(2,2)	(2,3)	(2,0)	(2,1)	(1,1)	(0,1)	(0,0)
H_3	(0,3)	(0,2)	(0,1)	(1,1)	(1,2)	(1,3)	(1,0)	(0,0)
H_4	(1,0)	(1,1)	(2,1)	(2,2)	(2,3)	(2,0)	(3,0)	(0,0)

Fig. 9.   In $C_4 \times C_4$, 4-mutually independent Hamiltonian cycles start at e.

cycles H_1, H_2, H_3 and H_4 start from e as Fig. 9. For $n \geq 6$, We construct 4-mutually independent Hamiltonian cycles H_1, H_2, H_3 and H_4 start from e as follows:

$H_1 = \langle(0,0), (0,1), (1,1), P_{1,1,2,-}, (1,2), (2,2), P_{2,2,3,-}, (2,3), (3,3), P_{3,3,4,-}, (3,4), ..., (n-2, n-2), P_{n-2,n-2,n-1,-}, (n-2, n-1), (n-1, n-1), (0, n-1), (0, n-2), (n-1, n-2), (n-1, n-3), (0, n-3), (0, n-4), (n-1, n-4), (n-1, n-5), ..., (n-1, 1), (n-1, 0), (0, 0)\rangle$.

$H_2 = \langle(0,0), (1,0), P_{1,0,1,-}, (1,1), (2,1), P_{2,1,2,-}, (2,2), (3,2), P_{3,2,3,-}, (3,3), ..., (0, n-1), P_{0,n-1,0,-}, (0,0)\rangle$.

$H_3 = \langle(0,0), (n-1, 0), P_{n-1,0,n-2,+}, (n-1, n-2), (0, n-2), P_{0,n-2,1,-}, (0,1), (1,1), P_{1,1,2,-}, (1,2), (2,2), P_{2,2,3,-}, (2,3), (3,3), P_{3,3,4,-}, (3,4), ..., (n-2, n-2), P_{n-2,n-2,n-1,-}, (n-2, n-1), (n-1, n-1), (0, n-1), (0,0)\rangle$.

$H_4 = \langle(0,0), P_{0,0,1,-}, (0,1), (n-1, 1), P_{n-1,1,2,-}, (n-1, 2), (n-2, 2), P_{n-2,2,1,+}, (n-2, 1), (n-3, 1), P_{n-3,1,0,+}, (n-3, 0), (n-4, 0), P_{n-4,0,1,-}, (n-4, 1), (n-5, 1), P_{n-5,1,0,+}, (n-5, 0), ..., (3, 1), P_{3,1,0,+}, (3, 0), (2, 0), P_{2,0,n-1,+}, (2, n-1), (1, n-1), P_{1,n-1,0,-}, (1, 0), (0,0)\rangle$.

We represent H_1, H_2, H_3 and H_4 as graph in Fig. 10. According to $H_i(t), 2 \leq t \leq n^2, 1 \in \{1, 2, 3, 4\}$ belong to the same subgraph C^j for some $j \in Z_n$ or not, we have following cases:

Case 1 : $t = 2, n^2 - 1$, or n^2.

For $t = 2$, $H_1(2) = (0,1), H_2(2) = (1,0), H_3(2) = (n-1, 0), H_4(2) = (0, n-1)$. And for $t = n^2 - 1$, $H_1(n^2 - 1) = (n-1, 1), H_2(n^2 - 1) = (0, 2), H_3(n^2 - 1) = (n-1, n-1), H_4(n^2 - 1) = (1, 1)$. For $t = n^2$, $H_1(n^2) = (n-1, 0), H_2(n^2) = (0, 1), H_3(n^2) = (0, n-1), H_4(n^2 - 1) = (1, 0)$. Hence, H_1, H_2, H_3 and H_4 are mutually independent in this case.

Case 2 : $(i-1)n + 3 \leq t \leq in + 2$, for all $1 \leq i \leq n - 2$.

For any $1 \leq i \leq n - 2$, for all $(i-1)n + 3 \leq t \leq in + 2$, $H_1(t), H_2(t) \in V(C^i)$. Two paths H_1 and H_2 go the same way in C^i. Let $H_1((i-1)n+3) = (i, u), H_2((i-1)n+3) = (i, v)$. By the construction of H_1 and H_2, we can know that $v \equiv (u - 2) \bmod n$. By Lemma 2 and $H_1(in + 2) \in V(C^i)$, $H_2(in + 2) \in V(C^{i+1})$ for any $1 \leq i \leq n - 2$, H_1 and H_2 are independent for $(i-1)n + 3 \leq t \leq in + 2, 1 \leq i \leq n - 2$. Then, we have the following subcases:

Case 2.1 : $3 \leq t \leq n$.

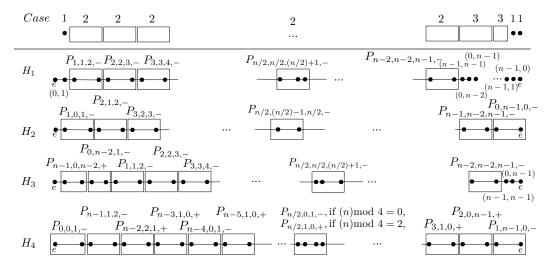$H_1(t), H_2(t) \in C^i, H_3(t) \in C^{n-1}$ and $H_4(t) \in C^0$, for

Fig. 10. The construction of 4 Hamiltonian cycles start at e

$3 \leq t \leq n$. Since H_1 and H_2 are independent and H_3, H_4 are in different subgraph with H_1 and H_2, then H_1, H_2, H_3 and H_4 are mutually independent in this subcase.

Case 2.2 : $n + 1 \leq t \leq 2n - 2$.

$H_1(t) \in V(C^1)$ for $t \in \{n+1, n+2\}$ and $H_1(t) \in V(C^2)$ for $n + 3 \leq t \leq 2n - 2$. $H_2(t) \in V(C^1)$ for $t = n + 1$, and $H_2(t) \in V(C^2)$ for $n + 2 \leq t \leq 2n - 2$. $H_3(t) \in V(C^0)$ and $H_4(t) \in V(C^{n-1})$ for $n + 1 \leq t \leq 2n - 3$. Since H_1 and H_2 are independent and H_3, H_4 are in different subgraph with H_1 and H_2, then H_1, H_2, H_3 and H_4 are mutually independent for $n + 1 \leq t \leq 2n - 2$.

Case 2.3 : $2n - 1 \leq t \leq 2n + 2$.

For $2n - 2 \leq t \leq 2n + 1$, $H_1(t)$ and $H_2(t) \in V(C^2)$, $H_3(t) \in V(C^1)$. $H_1(2n+2) \in V(C^2)$, $H_2(2n+2) \in V(C^3)$, and $H_2(2n + 2) \in V(C^1)$. $H_4(t) \in V(C^{n-1})$ for $t \in \{2n - 1, 2n\}$ and $H_4(t) \in V(C^{n-2})$ for $t \in \{2n+1, 2n+2\}$. Since H_1 and H_2 are independent and H_3, H_4 are in different subgraph with H_1 and H_2, then H_1, H_2, H_3 and H_4 are mutually independent in this subcase.

Case 2.4 : $(i-1)n+3 \leq t \leq in+2$, for all $3 \leq i \leq n-2$.

Case 2.4.1 : $(i-1)n+3 \leq t \leq in-2$, for all $3 \leq i \leq n-2$.

When t in this interval, $H_1(t), H_2(t) \in V(C^i)$, $H_3(t) \in V(C^{i-2})$, $H_4(t) \in V(C^{n-i+1})$. Since $i \neq i - 2, i \neq n - i + 1$ and $i - 2 \neq n - i + 1$ for n is even, and H_1 and H_2 are independent, then H_1, H_2, H_3 and H_4 are mutually independent in this subcase.

Case 2.4.2 : $in-1 \leq t \leq in$, for all $3 \leq i \leq n-2$.

When t in this interval, $H_1(t), H_2(t) \in V(C^i)$, $H_3(t) \in C^{i-1}, H_4(t) \in C^{n-i+1}$. So $i \neq i - 1, i \neq n - i + 1$ since n is even. Furthermore, $H_3(in - 1) = (i - 1, i - 1)$, $H_4(in) = (i-1, i)$, $H_4(in-1) = (n-i+1, 2)$ or $(n-i+1, n-1)$, and $H_4(in) = (n-i+1, 1)$ or $(n-i+1, 0)$ for any $3 \leq i \leq n-2$. When $i = n/2+1$, $i - 1 \neq 2$ or $n - 1$ and $i \neq 1$ or 0 because $n \geq 6$. Thus, $H_3(t) \neq H_4(t)$ for $in - 1 \leq t \leq in$, for all $3 \leq i \leq n - 2$. Again, because H_1 and H_2 are independent and above analysis, H_1 and H_2, then H_1, H_2, H_3 and H_4 are mutually independent in this subcase.

Case 2.4.3 : $in+1 \leq t \leq in+2$, for all $3 \leq i \leq n-2$.

For all $3 \leq i \leq n-2, i \neq n/2, t \in \{in+1, in+2\}$, $H_1(t) \in V(C^i)$, $H_3(t) \in V(C^{i-1})$, $H_4(t) \in V(C^{n-i})$. $H_2(in + 1) \in V(C^i)$, $H_2(in+2) \in V(C^{i+1})$. Since n is even and $i \neq n/2$, then $i \neq i - 1$, $i \neq n - i$, $i - 1 \neq n - i$, $i + 1 \neq i - 1$, and $i+1 \neq n-i$. Since H_1 and H_2 are independent and H_3, H_4 are in different subgraph with H_1 and H_2, then H_1, H_2, H_3 and H_4 are mutually independent in this interval. When $i = n/2$, $i - 1 \neq i$, $i - 1 \neq n - i$, and $i - 1 \neq i + 1$ still hold. Now consider $H_1(in+1)$, $H_2(in+1)$, $H_4(in+1)$, $H_1(in+2)$, and $H_4(in + 2)$ as follows. Note that $H_1(in+1) = (n/2, (n/2) + 2)$, $H_1(in+2) = (n/2, (n/2) + 1)$, $H_2(in+1) = (n/2, n/2)$, $H_4(in + 1) = (n/2, 0)$, or $(n/2, 2)$. So $H_1(t), H_2(t), H_3(t), H_4(t)$ are mutually independent when $i = n/2$ and $in + 1 \leq t \leq in + 2$.

Case 3 : $(n - 2)n + 3 \leq t \leq n^2 - 2$.

For all $(n - 2)n + 3 \leq t \leq (n - 1)n + 1$, $H_2(t) \in C^{n-1}$, and for all $(n - 1)n + 2 \leq t \leq n^2 - 2$, $H_2(t) \in C^0$. We have following subcases:

Case 3.1 : $(n - 2)n + 3 \leq t \leq (n - 1)n + 1$.

In this case, $H_1(t) \in V(C^0) \cup V(C^{n-1})$, $H_2(t) \in V(C^{n-1})$, $H_3(t) \in V(C^{n-3}) \cup V(C^{n-2})$ and $H_4(t) \in V(C^2) \cup V(C^1)$. H_3, H_4 are in different subgraphs with H_1, H_2, we only need to consider $H_1(t)$ and $H_2(t)$ both in $V(C^{n-1})$. Let $x = (n-2)n+3$, $0 \leq j \leq \lceil n/4 \rceil - 2$. By the construction of H_1 and H_2, we get follows:

The set $\{H_1(z) \in V(C^{n-1}) : (n-2)n + 3 \leq z \leq (n-1)n + 1\} = \{H_1(x) = (n - 1, n - 1)\} \cup \{H_1(x + 3 + 4j) = (n - 1, n - 2 - 2j), H_1(x + 4 + 4j) = (n - 1, n - 3 - 2j)$: for all $0 \leq j \leq \lceil n/4 \rceil - 2\}$. The set $\{H_2(z) \in V(C^{n-1}) : (n - 2)n + 3 \leq z \leq (n - 1)n + 1\} = \{H_2(x) = (n - 1, n - 3)\} \cup H_2(x + 3 + 4j) = (n - 1, (n - 6 - 4j) \mod n), H_2(x + 4 + 4j) = (n - 1, (n - 7 - 4j) \mod n)$: for all $0 \leq j \leq \lceil n/4 \rceil - 2$. If $n - 6 - 4j > 0$, then $(n - 6 - 4j) \mod n = n - 6 - 4j(< n)$. Suppose that $n - 2 - 2j = n - 6 - 4j$, then we can get $2j = -4$. Since $j \geq 0$, a contradiction. If $n - 6 - 4j < 0$, then $(n - 6 - 4j) \mod n = n - 6 - 4j + n$. Suppose that $n - 2 - 2j - n = n - 6 - 4j$, then we can get

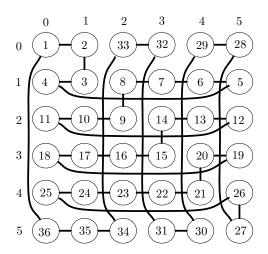Fig. 11.   Hamiltonian cycles H_1 in $C_6 \times C_6$.

Fig. 12.   Hamiltonian cycles H_2 in $C_6 \times C_6$.

$4 + 2j = n$. Since $n \geq 6$ and $j \geq \lceil n/4 \rceil - 2$, a contradiction. So, $n - 2 - 2j \neq (n - 6 - 4j) \bmod n$. In the same way, we can know that $n - 3 - 2j \neq (n - 7 - 4j) \bmod n$. Hence, H_1, H_2 are independent, that implies H_1, H_2, H_3 and H_4 are mutually independent in this subcase.

Case 3.2 : $(n-1)n + 2 \leq t \leq n^2 - 2$.

In this case, $H_1(t) \in V(C^0) \cup V(C^{n-1})$, $H_2(t) \in V(C^0)$, $H_3(t) \in V(C^{n-2})$ and $H_4(t) \in V(C^1)$. H_3, H_4 are in different subgraph with H_1, H_2, we only need to consider the vertices of H_1 and H_2 in this subcase. Let $x = (n-1)n + 2$, $y = \lfloor n/4 \rfloor - 1$, $0 \leq j \leq y$. By the construction of H_1 and H_2, we get follows:
The set $\{H_1(z) \in V(C^0) : x \leq z \leq n^2 - 2\} = \{H_1(x+4j) = (0, 2(y-j)+3), H_1(x+1+4j) = (0, 2(y-j)+2)$:for all $0 \leq j \leq y\}$. The set $\{H_2(z) \in V(C^0) : x \leq z \leq n^2 - 2\}$ $= \{H_2(x+kj) = (0, n-1-k) : 0 \leq k \leq n+3\}$. Since $2(y-j)+3 \neq n-1+4j$, $2(y-j)+2 \neq n-1-(4j+1)$, then H_1 and H_2 are independent. Thus, H_1, H_2, H_3 and H_4 are mutually independent in this subcase.

From the above 3 cases, we can know that H_1, H_2, H_3 and H_4 start from e are mutually independent and $IHC(C_n \times C_n) \geq 4$. Hence, $IHC(C_n \times C_n) = 4$ can be concluded.   □

We construct 4-mutually independent Hamiltonian cycles $C_6 \times C_6$ in Fig. 11, 12, 13, 14as an example of Theorem 2.

IV. CONCLUSION

In this paper, we discuss the mutually independent Hamiltonian cycles of $C_n \times C_n$, for $n \geq 3$ and get the optimal result on it. Note that a Hamilton cycles in any graph G is a cycle C_n with $n = |V(G)|$. So, if two graphs G_1 and G_2 are Hamiltonian with $|V(G_1)| = |V(G_2)| = n \geq 3$, then graph $C_n \times C_n$ will be a spanning subgraph of $G_1 \times G_2$. Hence, we have the following corollary.

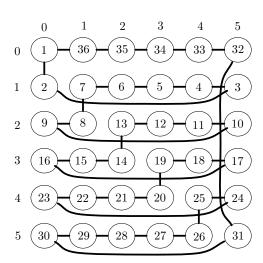**Corollary 1.** For any graphs G_1, G_2, if G_1 and G_2 are Hamiltonian and $|V(G_1)| = |V(G_2)| \geq 3$, then
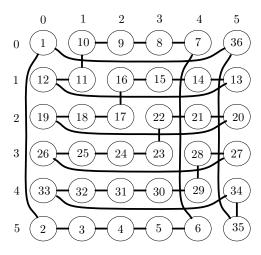

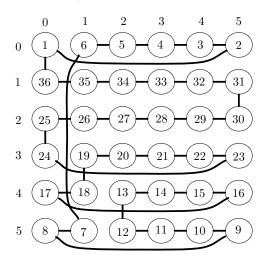Fig. 13.   Hamiltonian cycles H_3 in $C_6 \times C_6$.

Fig. 14.   Hamiltonian cycles H_4 in $C_6 \times C_6$.

$IHC(G_1 \times G_2) \geq 4.$

In the future, discuss in the mutually independent Hamiltonian cycles of $C_m \times C_n$, for $m, n \geq 3$ and $m \neq n$ is an interesting question. Furthermore, we can discuss $IHC(G)$ for $G = C_n^k = C_n \times C_n \times ... \times C_n$, or more general graph $C_{n_1} \times C_{n_2} \times ... \times C_{n_k}$, for any positive integer $n_1 \geq n_2 \geq ... \geq n_k \geq 3$.

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