Multiple soliton solutions of (2+1)-dimensional potential Kadomtsev-Petviashvili equation

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Abstract—We employ the idea of Hirota’s bilinear method, to obtain some new exact soliton solutions for high nonlinear form of (2+1)-dimensional potential Kadomtsev-Petviashvili equation. Multiple singular soliton solutions were obtained by this method. Moreover, multiple singular soliton solutions were also derived.

Keywords—Hirota bilinear method, potential Kadomtsev-Petviashvili equation, Multiple soliton solutions, Multiple singular soliton solutions.

I. INTRODUCTION

Many important phenomena and dynamic processes in physics, mechanics, chemistry and biology can be represented by nonlinear partial differential equations. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models.

In recent years, many kinds of powerful methods have been proposed to find solutions of nonlinear partial differential equations, numerically and/or analytically, e.g., the tanh function method [1], the homogeneous balance method [2], the tanhcoth method [3], the Exp-function method [4], the Wronskian form, we will obtain some exact and new solutions

$$u(x, t) = e^{kx + my - ct}$$

$$u(x, t) = R \left\{ \ln f(x, y, t) \right\}_{xt},$$

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$$f(x, y, t) = 1 + C_1 f_1(x, y, t) = 1 + C_1 e^{\theta_1}.$$
where \( u = u(x, y, z, t) : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_t \rightarrow \mathbb{R} \).

To determine multiple-soliton solutions for Eq. (8), we follow the steps presented above. We first consider \( C_1 = C_2 = C_3 = 1 \). Substituting

\[
\theta_i = k_ix + m_ity - \frac{k_i^4 + 3m_i^2}{4k_i}t,
\]

into the linear terms of Eq. (8) to find the relation

\[
w_i = \frac{k_i^4 + 3m_i^2}{4k_i}, \quad i = 1, 2, \ldots, N
\]

and consequently, \( \theta_i \) becomes

\[
\theta_i = k_ix + m_ity - \frac{k_i^4 + 3m_i^2}{4k_i}t.
\]

To determine \( R \), we substitute

\[
u(x, y, t) = R(\ln f(x, y, t))_x.
\]

where

\[
f(x, y, t) = 1 + f_1(x, y, t) + e^{k_ix + m_ity - \frac{k_i^4 + 3m_i^2}{4k_i}t}.
\]

into Eq. (8) and solve to find that \( R = 2 \).

This means that the single singular soliton solution is given by

\[
u(x, y, t) = 2\ln(1 + f(x, y, t)),
\]

For the two-soliton solutions, we substitute

\[
u(x, y, t) = 2(\ln f(x, y, t))_x.
\]

where

\[
f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2},
\]

into Eq. (8), where \( \theta_1 \) and \( \theta_2 \) are given in Eq. (11) to obtain

\[
a_{12} = \frac{-4(k_1 - k_2)(w_1 - w_2) + (k_1 - k_2)^4 + 3(m_1 - m_2)^2}{4(k_1 + k_2)(w_1 + w_2) - (k_1 + k_2)^4 - 3(m_1 + m_2)^2},
\]

and

\[
w_s = \frac{k_s^4 + 3m_s^2}{4k_s}, \quad s = 1, 2,
\]

for \( |k_i| \neq |k_j| \) and \( |m_i| \neq |m_j| \).

It is interesting to point out that for \( m_s = k_s, s = 1, 2, 3 \), the phase shift reduces to

\[
a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2},
\]

for \( |k_1| \neq |k_2| \), hence

\[
a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},
\]

for \( |k_i| \neq |k_j| \). This in turn gives

\[
f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}e^{\theta_1+\theta_2},
\]

where

\[
\theta_i = k_ix + k_iy - \frac{1}{4}k_i(k_i^2 + 3)t, i = 1, 2,
\]

which is a two soliton solution(Fig. 1).

Similarly, to determine the three soliton solutions, we set

\[
f(x, y, t) = 1 + e^{\theta_1} + e^{\theta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}e^{\theta_1+\theta_2} + a_{123}e^{\theta_1+\theta_2+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{123}e^{\theta_1+\theta_2+\theta_3},
\]

To determine the three soliton solutions explicitly, we substitute the last result for \( f(x, y, t) \) into Eqs. (28), (See Fig. 2).

The higher level soliton solutions, for \( n \geq 4 \) can be obtained in a parallel manner. The obtained results confirm that the \((2+1)\)-dimensional potential Kadomtsev-Petviashvili equation is completely integrable and possesses multiple soliton solutions of any order.
IV. MULTIPLE SINGULAR SOLITON SOLUTIONS OF THE
POTENTIAL KADOMTSEV-PETVIASHVILI EQUATION:

We first consider $C_1 = C_2 = C_3 = -1$. Substituting

$$u(x, y, t) = e^{\phi_i}, \quad \theta_i = k_i x + m_i y - w_i t$$

into the linear terms of Eq.(8) to find the relation

$$w_i = \frac{k_i^4 + 3 m_i^2}{4 k_i}, \quad i = 1, 2, \ldots, N$$

and consequently, $\theta_i$ becomes

$$\theta_i = k_i x + m_i y - \frac{k_i^4 + 3 m_i^2}{4 k_i} t.$$ \(\text{(25)}\)

To determine $R$, we substitute

$$u(x, y, t) = R \left(\ln f(x, y, t)\right)_x,$$

where $f(x, y, t) = 1 - f_2(x, y, t) = 1 + e^{k_1 x + m_1 y + k_1^4 t}$ into Eq.(8) and solve to find that $R = 2$.

This means that the single singular soliton solution is given by

$$u(x, y, t) = -2 \frac{k_1 e^{k_1 x + m_1 y} - k_1^4 + 3 m_1^2}{4 k_1} \frac{k_1 e^{k_1 x + m_1 y} - k_1^4 + 3 m_1^2}{4 k_1} + 1.$$ \(\text{(27)}\)

For the two-soliton solutions, we substitute

$$u(x, y, t) = 2 \left(\ln f(x, y, t)\right)_x$$

where

$$f(x, y, t) = 1 - e^{\phi_1} - e^{\phi_2} + a_{12} e^{\phi_1 + \phi_2},$$ \(\text{(28)}\)

into Eq.(8), where $\theta_1$ and $\theta_2$ are given in Eq.(25) to obtain

$$a_{12} = -4 \frac{(k_1 - k_2) (w_1 - w_2) + (k_1 - k_2)^4 + 3 (m_1 - m_2)^2}{4 (k_1 + k_2) (w_1 + w_2) - (k_1 + k_2)^4 - 3 (m_1 + m_2)^2},$$ \(\text{(29)}\)

which is a two soliton solution(Fig. 3).

And

$$w_s = \frac{k_s^4 + 3 m_s^2}{4 k_s}, \quad s = 1, 2,$$

for $|k_i| \neq |k_j|$ and $|m_i| \neq |m_j|$, hence

$$a_{ij} = \frac{(k_i - k_j) (w_i - w_j) + (k_i - k_j)^4 + 3 (m_i - m_j)^2}{4 (k_i + k_j) (w_i + w_j) - (k_i + k_j)^4 - 3 (m_i + m_j)^2},$$ \(\text{(30)}\)

and

$$w_s = \frac{k_s^4 + 3 m_s^2}{4 k_s}, \quad s = 1, 2, 3,$$

for $|k_i| \neq |k_s|$ and $|m_i| \neq |m_s|$.

It is interesting to point out that for $m_s = k_s, s = 1, 2, 3$, the phase shift reduces to

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2},$$ \(\text{(31)}\)

for $|k_1| \neq |k_2|$, hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},$$ \(\text{(32)}\)

for $|k_i| \neq |k_j|$. This in turn gives

$$f(x, y, t) = 1 - e^{\phi_1} - e^{\phi_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{\theta_1 + \theta_2},$$ \(\text{(33)}\)

where

$$\theta_i = k_i x + k_i y - \frac{1}{4} k_i (k_i^2 + 3), \quad i = 1, 2,$$

Fig. 2. The three soliton solution with $k_1 = -1, k_2 = 1.2$ and $k_3 = 1.6$.

Fig. 3. The two soliton solution with $k_1 = 1$ and $k_2 = -1.2$. 

It is interesting to point out that for $m_s = k_s, s = 1, 2, 3$, the phase shift reduces to

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2},$$ \(\text{(31)}\)

for $|k_1| \neq |k_2|$, hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2},$$ \(\text{(32)}\)

for $|k_i| \neq |k_j|$. This in turn gives

$$f(x, y, t) = 1 - e^{\phi_1} - e^{\phi_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} e^{\theta_1 + \theta_2},$$ \(\text{(33)}\)

where

$$\theta_i = k_i x + k_i y - \frac{1}{4} k_i (k_i^2 + 3), \quad i = 1, 2,$$ 

which is a two soliton solution(Fig. 3).
Similarly, to determine the three soliton solutions, we set
\[ f(x, y, t) = 1 - e^{\theta_1} - e^{\theta_2} - e^{\theta_3} + a_{12} e^{\theta_1 + \theta_2} + a_{13} e^{\theta_1 + \theta_3} + a_{23} e^{\theta_2 + \theta_3} - a_{12} a_{23} a_{13} e^{\theta_1 + \theta_2 + \theta_3}. \]

To determine the three soliton solutions explicitly, we substitute the last result for \( f(x, y, t) \) into Eqs. (28), (See Fig. 4).

The higher level singular soliton solutions, for \( n \geq 4 \) can be obtained in a parallel manner.

**V. Conclusion**

In this paper, by using the Hirota bilinear method, we obtained some explicit formulas of solutions for the (2+1)-dimensional potential Kadomtsev-Petviashvili equation. Multiple soliton solutions were formally derived. Moreover, multiple singular soliton solutions of any order was derived as well. The results of other works are special cases of our results.

**References**