# Monomial Form Approach to Rectangular Surface Modeling

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*Abstract*—Geometric modeling plays an important role in the constructions and manufacturing of curve, surface and solid modeling. Their algorithms are critically important not only in the automobile, ship and aircraft manufacturing business, but are also absolutely necessary in a wide variety of modern applications, e.g., robotics, optimization, computer vision, data analytics and visualization. The calculation and display of geometric objects can be accomplished by these six techniques: Polynomial basis, Recursive, Iterative, Coefficient matrix, Polar form approach and Pyramidal algorithms. In this research, the coefficient matrix (simply called monomial form approach) will be used to model polynomial rectangular patches, i.e., Said-Ball, Wang-Ball, DP, Dejdumrong and NB1 surfaces. Some examples of the monomial forms for these surface modeling are illustrated in many aspects, e.g., construction, derivatives, model transformation, degree elevation and degress reduction.

*Keywords*—Monomial form, rectangular surfaces, CAGD curves, monomial matrix applications.

## I. Introduction

IN computer-aided design and manufacturing, curves and surfaces are widely used to produce models, prototypes, and products. Typically, curve and surface modeling schemes can been represented by polynomial bases, e.g., Bézier and B-spline. These polynomial bases are more potential than basic geometric shapes because they can represent complex models. In the recent years, when 3D models are popularly applied in a wide variety of work, the rectangular surface modeling plays an important role in many applications. However, the traditional way of the rectangular patch construction employs polynomial basis. This method consumes much computational time and it is the problem of interactive application. In this research, the monomial form approaches are proposed in surface construction and their applications.

In CAGD, the rectangular surfaces can be constructed by a roughly rectangular grid of control points and polynomial functions. The polynomial functions of surface can be derived from polynomial basis of curve using Cartesian product. The most famous polynomial basis is Bernstein polynomials[1] because it is widely used to represent the Bézier curves and surfaces. The rectangular Bézier surface are resided within a rectangular grid satisfied by convex hull property. Moreover, Bernstein polynomial can be simply implemented the surfaces. However, the degree of polynomial function is varied proportional to the number of control points. The

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high degree of rectangular Bézier surface may not practically applied.

Besides the Bézier /Bernstein polynomials, there are some polynomials of curves and surfaces that possess the convex hull property and shape preserving like Bernstein. These curves were developed to improve curve and surface construction, i.e., reduce computational time. By the restriction of Bézier curve and surface, the higher degree of curves and surfaces can not be accepted in the application due to high computational time. To apply curves and surfaces in higher degree, Said-Ball[2] developed the Ball basis functions[3][4][5]. Moreover, Hu et al.[6] presented the degree elevation and degree reduction of Said-Ball polynomials. With these properties, Said-Ball polynomial can be efficiently utilized in any degrees.

Since there have been the problem in the curve construction, the Bernstein polynomial has quadratic complexity function. Therfore, many researchers proposed the new polynomial with linear complexity function such as Wang-Ball[7], Delgado-Peña [8], NB1[9][10] and Dejdumrong [11] polynomials. However, these polynomials were complex to implement the curve and the surface.

Besides of polynomial basis, there are several techniques for surface construction, e.g., recursive algorithm, iterative form, and monomial form. The recursive algorithm is appropriate in software implementation but more static storage is needed while processing. The iterative form has a mechanism to interpolate pairs of adjacent control points for generating new points. The process is repeated until the corresponding point on a surface is acquired. However, this approach cannot be adopted to polynomial properties such as degree elevation, degree reduction, and conversion.

In this paper, the coefficient matrix or monomial form approach is demonstrated to construct the curves and surfaces. With its merits, the curves and surfaces can be modeled simply and efficiently by matrix products. Monomial matrix approach and its applications for CAGD curves were introduced in[12][13]. In this work, the coefficient matrix for the Said-Ball, Wang-Ball, DP, Dejdumrong and NB1 surfaces are formulated. Moreover, monomial matrices can be facilitated in degree elevations, degree reductions, and conversion among polynomials.

## II. Monomial Forms

In order to understand the uses of monomial matrices for surface constructions, it is necessary to review the monomial matrices for each curve[12].

**Proposition 1.** *(Bézier Monomial Form[12]) A Bézier curve of degree $n$, denoted by $B^n(t)$, with $n+1$ control points, denoted by $\{\mathbf{b}_i\}_{i=0}^n$, can be written in terms of the power basis form as follows: [14]*

$$B^n(t) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{b}_i \cdot m_{i,j} \cdot t^j, \qquad (1)$$

*where*

$$m_{i,j} = (-1)^{j-i} \binom{n}{j}\binom{j}{i}. \qquad (2)$$

**Proposition 2.** *(Said-Ball Monomial Form)[12] An $n^{th}$-degree Said-Ball curve, denoted by $S^n(t)$, given by $n+1$ control points, denoted by $\{\mathbf{V}_i\}_{i=0}^n$, can be expressed in power basis form as follows:*

$$S^n(t) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{V}_i \cdot s_{i,j} \cdot t^j, \qquad (3)$$

*where*

$$s_{i,j} = \begin{cases} (-1)^{j-i}\binom{i+\lfloor \frac{n}{2} \rfloor}{i}\binom{\lfloor \frac{n}{2}\rfloor+1}{j-i} & , \quad 0 \le i \le \lceil \frac{n}{2} \rceil - 1, \\[8pt] (-1)^{j-i}\binom{n}{i}\binom{i}{j-i} & , \quad i = \frac{n}{2}, \\[8pt] (-1)^{j-\lfloor\frac{n}{2}\rfloor-1}\binom{\lfloor\frac{n}{2}\rfloor+n-i}{n-i}\binom{n-i}{j-\lfloor\frac{n}{2}\rfloor-1} & , \quad \lfloor\frac{n}{2}\rfloor+1 \le i \le n. \end{cases} \qquad (4)$$

**Proposition 3.** *(Wang-Ball Monomial Form)[12] A Wang-Ball curve, denoted by $A^n(t)$, provided with $n+1$ control points, denoted by $\{\mathbf{p}_i\}_{i=0}^n$, can be shown as*

$$A^n(t) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{p}_i \cdot a_{i,j} \cdot t^j, \qquad (5)$$

*where*

$$a_{i,j} = \begin{cases} (-1)^{j-i}2^i\binom{i+2}{j-i} & , \quad \text{for } 0 \le i \le \lfloor \frac{n}{2} \rfloor - 1, \\[8pt] (-1)^{j-i}2^i\binom{n-i}{j-i} & , \quad \text{for } i = \lfloor \frac{n}{2} \rfloor, \\[8pt] (-1)^{j-i}2^{n-i}\binom{n-i}{j-i} & , \quad \text{for } i = \lceil \frac{n}{2} \rceil, \\[8pt] (-1)^{j-n+i}2^{n-i}\binom{n-i}{j-n+i-2} & , \quad \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n. \end{cases} \qquad (6)$$

**Proposition 4.** *(DP Monomial Form)[12] An $n^{th}$-degree DP curve, denoted by $C^n(t)$, given by a set of $n+1$ control points, denoted by $\{\mathbf{q}_i\}_{i=0}^n$, can be formulated in power basis form by*

$$C^n(t) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{q}_i \cdot c_{i,j} \cdot t^j, \qquad (7)$$

*where*

$$c_{i,j} = \begin{cases} (-1)^j\binom{n}{j}, & \text{for } i = 0, \\[6pt] (-1)^{j-1}\binom{n-i}{j-1}, & \text{for } 0 < i \le \lfloor \frac{n}{2}\rfloor - 1, \\[6pt] (-1)^{j-1}(n-2i)\binom{i+1}{j-1} + (\frac{1}{2})^{n-2i}(\binom{0}{j} \\ \quad -\binom{0}{j-i-1} - (-1)^j\binom{i+1}{j})), & \text{for } i = \lfloor \frac{n}{2} \rfloor, \\[6pt] (-1)^{j-n+i}(n-2i)\binom{1}{j-n+i-1} + (\frac{1}{2})^{2i-n}(\binom{0}{j} \\ \quad -\binom{0}{j-n+i-1} - (-1)^j\binom{n-i+1}{j})), & \text{for } i = \lceil \frac{n}{2} \rceil, \\[6pt] (-1)^{j-i}\binom{1}{j-i}, & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n-1, \\[6pt] \binom{0}{j-n}, & \text{for } i = n. \end{cases} \qquad (8)$$

**Proposition 5.** *(Dejdumrong Monomial Form)[12] A Dejdumrong curve of degree $n$, denoted by $D^n(t)$, with $n+1$ control points, denoted by $\{\mathbf{d}_i\}_{i=0}^n$, can be computed by*

$$D^n(t) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{d}_i \cdot d_{i,j} \cdot t^j, \qquad (9)$$

*where*

$$d_{i,j} = \begin{cases} (-1)^{j-i}3^i\binom{i+3}{j-i} & \text{for } 0 \le i < \lceil \frac{n}{2} \rceil - 1, \\[6pt] (-1)^{j-i}3^i\binom{n-i}{j-i} & \text{for } i = \lceil \frac{n}{2} \rceil - 1, \\[6pt] (-1)^{j-i}2(3^{i-1})\binom{i}{j-i} & \text{for } i = \frac{n}{2} \text{ and } n \text{ is even}, \\[6pt] (-1)^{j-i}3^{n-i}\binom{n-i}{j-i} & \text{for } i = \lfloor \frac{n}{2} \rfloor + 1, \\[6pt] (-1)^{j-n+i-1}3^{n-i}\binom{n-i}{j-n+i-3} & \text{for } \lfloor \frac{n}{2} \rfloor + 1 < i \le n. \end{cases} \qquad (10)$$

**Proposition 6.** *(NB1 Monomial Form)[12] An NB1 curve of degree $n$, $N^n(t)$, with $n+1$ control points, denoted by $\{\mathbf{y}_i\}_{i=0}^n$, can be formed by the power basis form as follows:*

$$N^n(t) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{y}_i \cdot g_{i,j} \cdot t^j, \qquad (11)$$

*where*

$$g_{i,j} = \begin{cases} (-1)^{j-i}\binom{\lfloor\frac{n}{2}\rfloor-1+i}{i}\binom{\lfloor\frac{n}{2}\rfloor}{j-i} & 0 \le i \le \lfloor \frac{n}{2} \rfloor - 2, \\[6pt] (-1)^{j-i}\binom{2i}{i}\binom{i+2}{j-i} & i = \lfloor \frac{n}{2} \rfloor - 1, \\[6pt] (-1)^{j-i}2\binom{2i-2}{i-1}\binom{n-i}{j-i} & i = \lfloor \frac{n}{2} \rfloor, \\[6pt] (-1)^{j-i}2\binom{2(n-i-1)}{n-i-1}\binom{n-i}{j-i} & i = \lceil \frac{n}{2} \rceil, \\[6pt] (-1)^{j-n+i}\binom{2(n-i)}{n-i}\binom{n-i}{j-n+i-2} & i = \lceil \frac{n}{2} \rceil + 1, \\[6pt] (-1)^{j-\lfloor\frac{n}{2}\rfloor}\binom{\lfloor\frac{n}{2}\rfloor-1+n-i}{n-i}\binom{n-i}{j-\lfloor\frac{n}{2}\rfloor} & \lceil \frac{n}{2} \rceil + 2 \le i \le n. \end{cases} \qquad (12)$$

## III. MONOMIAL FORM APPROACH TO SURFACE MODELING

Regarding monomial matrix approach, it facilitates construction of CAGD curves and CAGD curve properties, e.g., the derivatives, the degree elevations, the degree reductions and the CAGD curve conversions. Besides of the cases of curves, the surface modeling can be utilized by using monomial matrices.

### A. Monomial Forms

Applying monomial forms for constructing surfaces, the matrix products must be expressed and the coefficient matrix can be formulated for each surface as follows.

1) Bézier rectangular surface with Bézier control net, $P \in \{\{\mathbf{b}_{i,j}\}_{i=0}^{p}\}_{j=0}^{q}$ can be defined as

$$B(u,v) = (\mathcal{B}^p \cdot U) \cdot P \cdot (\mathcal{B}^q \cdot V)^\top \qquad (13)$$

where

$$\mathcal{B}^n = \begin{bmatrix} m_{0,0} & m_{0,1} & \dots & m_{0,n} \\ m_{1,0} & m_{1,1} & \dots & m_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n,0} & m_{n,1} & \dots & m_{n,n} \end{bmatrix}_{(n+1)\times(n+1)}, \qquad (14)$$

and $m_{i,j}$ is defined in (2).

2) Said-Ball rectangular surface with Said-Ball control net, $Q \in \{\{\mathbf{V}_{i,j}\}_{i=0}^{p}\}_{j=0}^{q}$ can be given by

$$S(u,v) = (\mathcal{S}^p \cdot U) \cdot Q \cdot (\mathcal{S}^q \cdot V)^\top \qquad (15)$$

where

$$\mathcal{S}^n = \begin{bmatrix} s_{0,0} & s_{0,1} & \dots & s_{0,n} \\ s_{1,0} & s_{1,1} & \dots & s_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n,0} & s_{n,1} & \dots & s_{n,n} \end{bmatrix}_{(n+1)\times(n+1)}, \qquad (16)$$

and $s_{i,j}$ is defined in (4).

3) Wang-Ball rectangular surface with Wang-Ball control net, $K \in \{\{\mathbf{p}_{i,j}\}_{i=0}^{p}\}_{j=0}^{q}$ can be shown as

$$A(u,v) = (\mathcal{A}^p \cdot U) \cdot K \cdot (\mathcal{A}^q \cdot V)^\top \qquad (17)$$

where

$$\mathcal{A}^n = \begin{bmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n} \\ a_{1,0} & a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,0} & a_{n,1} & \dots & a_{n,n} \end{bmatrix}_{(n+1)\times(n+1)}, \qquad (18)$$

and $a_{i,j}$ is defined in (6).

4) DP rectangular surface with DP control net, $Y \in \{\{\mathbf{q}_{i,j}\}_{i=0}^{p}\}_{j=0}^{q}$ can be expressed by

$$C(u,v) = (\mathcal{C}^p \cdot U) \cdot Y \cdot (\mathcal{C}^q \cdot V)^\top \qquad (19)$$

where

$$\mathcal{C}^n = \begin{bmatrix} c_{0,0} & c_{0,1} & \dots & c_{0,n} \\ c_{1,0} & c_{1,1} & \dots & c_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,0} & c_{n,1} & \dots & c_{n,n} \end{bmatrix}_{(n+1)\times(n+1)}, \qquad (20)$$

and $c_{i,j}$ is defined in (8).

5) Dejdumrong rectangular surface with Dejdumrong control net, $H \in \{\{\mathbf{d}_{i,j}\}_{i=0}^{p}\}_{j=0}^{q}$ can be determined as

$$D(u,v) = (\mathcal{D}^p \cdot U) \cdot H \cdot (\mathcal{D}^q \cdot V)^\top \qquad (21)$$

where

$$\mathcal{D}^n = \begin{bmatrix} d_{0,0} & d_{0,1} & \dots & d_{0,n} \\ d_{1,0} & d_{1,1} & \dots & d_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n,0} & d_{n,1} & \dots & d_{n,n} \end{bmatrix}_{(n+1)\times(n+1)}, \qquad (22)$$

and $d_{i,j}$ is defined in (10).

6) NB1 rectangular surface with NB1 control net, $G \in \{\{\mathbf{y}_{i,j}\}_{i=0}^{p}\}_{j=0}^{q}$ can be specified by

$$N(u,v) = (\mathcal{N}^p \cdot U) \cdot G \cdot (\mathcal{N}^q \cdot V)^\top \qquad (23)$$

where

$$\mathcal{N}^n = \begin{bmatrix} g_{0,0} & g_{0,1} & \dots & g_{0,n} \\ g_{1,0} & g_{1,1} & \dots & g_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n,0} & g_{n,1} & \dots & g_{n,n} \end{bmatrix}_{(n+1)\times(n+1)}, \qquad (24)$$

and $g_{i,j}$ is defined in (12).

## IV. SOME APPLICATIONS OF USING MONOMIAL MATRICES

By using monomial matrices, surfaces can be constructed simply and efficiently. Furthermore, it can enhance the speed of construction by using parallel programming. Besides surface construction, there are other benefits of using monomial form approach as follows:

### A. Surface Constructions

Monomial matrices can be applied to construct surfaces, e.g. Bézier , Said-Ball, Wang-Ball, DP, Dejdumrong and NB1 surfaces as shown in Figures 1, 2, 3, 4, 5 and 6, respectively.

### B. Curve Conversions

Monomial matrices can be used to convert from one surface into the other model. This transformation can be applied by the same concept as shown in [12].

### C. Degree Elevations and Degree Reductions

Monomial matrices can be used to compute the degree elevation and the degree reduction of surfaces [13].

In the following section, an example of the degree elevation will be introduced only for the case of Wang-Ball surfaces. The other surface cases can be readily applied from the Wang-Ball case.

**Definition 1.** *An extended Wang-Ball monomial matrix, denoted by $\mathcal{A}_n^{n+1}$, is expressed by attaching 1 column vector of $n+1$ zeros to the Wang-Ball monomial matrix, denoted by*
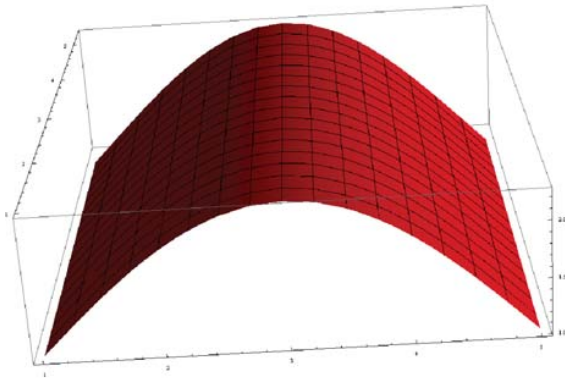
Fig. 1 Bézier Surface



Fig. 2 Said-Ball Surface



Fig. 3 Wang-Ball Surface

$\mathcal{A}_n$. *The extended monomial matrix with $n+1$ rows and $n+2$ columns, denoted by $\mathcal{A}_n^{n+1}$, can be written by*

$$\mathcal{A}_n^{n+1} = \begin{bmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n} & 0 \\ a_{1,0} & a_{1,1} & \dots & a_{1,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,0} & a_{n,1} & \dots & a_{n,n} & 0 \end{bmatrix}_{(n+1)\times(n+2)} \quad (25)$$

It can be defined the extended monomial matrices for the Said-Ball, DP, Dejdumrong and NB1 surfaces respectively, denoted by $\mathcal{S}_n^{n+1}$, $\mathcal{C}_n^{n+1}$, $\mathcal{D}_n^{n+1}$, and $\mathcal{N}_n^{n+1}$, respectively.

**Theorem 1.** *A degree elevation matrix for raising an $n^{th}$ degree of a Wang-Ball surface to an $(n+1)^{th}$ degree of the same surface, denoted by $E_n^{(1)}$ , can be defined as follows:*

$$E_n^{(1)} = \mathcal{A}_n^{n+1} \cdot (\mathcal{A}_{n+1})^{-1} \quad (26)$$

**Proposition 7.** *Given a set of Wang-Ball control nets, denoted by $K$, the degree elevation of this surface in terms of the new Wang-Ball control points, $K^{(1)}$, can be explicitly obtained from*

$$K^{(1)} = \left[\mathcal{A}_n^{n+1} \cdot (\mathcal{A}_{n+1})^{-1}\right] \cdot K \cdot \left[\mathcal{A}_n^{n+1} \cdot (\mathcal{A}_{n+1})^{-1}\right]^\top \quad (27)$$

**Theorem 2.** *A degree reduction matrix for reducing an $n^{th}$ degree of a Wang-Ball surface into an $(n+1)^{th}$ degree of the same surface, denoted by $K_n^{(-1)}$ , can be defined as follows:*

$$K_n^{(1)} = \mathcal{A}_n \cdot (\mathcal{A}_{n-1}^n)^\top \cdot \left[\mathcal{A}_{n-1}^n \cdot (\mathcal{A}_{n-1}^n)^\top\right]^{-1} \quad (28)$$

**Proposition 8.** *Given a set of Wang-Ball control nets, denoted by $K$, the degree reduction of this surface in terms of the new Wang-Ball control points, $K^{(-1)}$, can be explicitly obtained from*

$$K^{(-1)} = \left[K_n^{(-1)}\right] \cdot K \cdot \left[K_n^{(-1)}\right]^\top \quad (29)$$



Fig. 4 DP Surface



Fig. 5 Dejdumrong Surface

*D. Normal and Derivatives of Surfaces*

Tangent vectors of a surface can be easily obtained by using monomial matrices.



Fig. 6 NB1 Surface

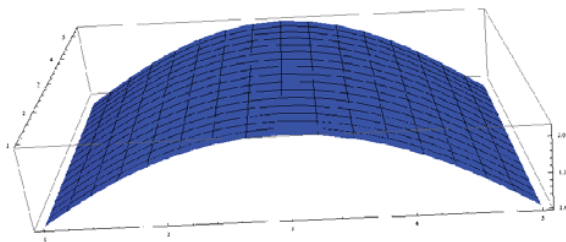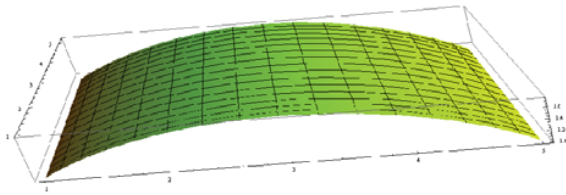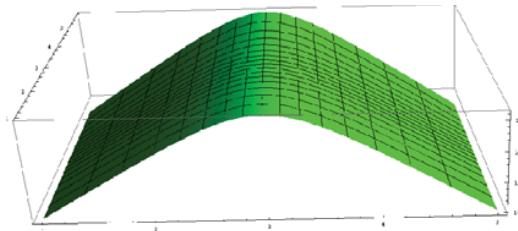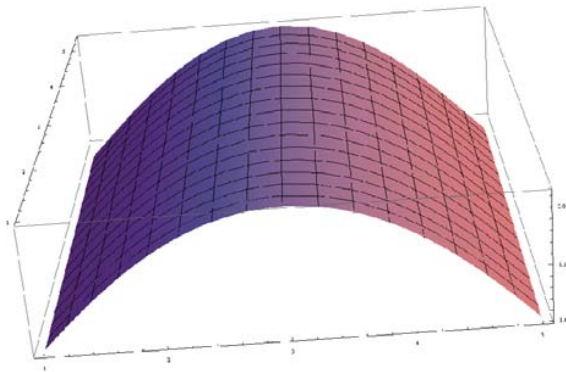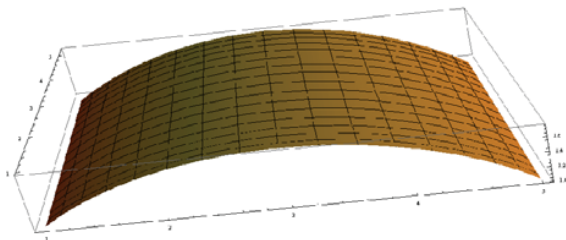## V. CONCLUSION

The monomial form approach is an efficient method for surfaces construction and properties. This approach can process simply by using matrix multiplication. Moreover, it is applied in parallel computation because each point on surface can computed independently. Adopting in parallel programming, the higher degree of surfaces can be represented with less computational time than other approaches. Additionally, some properties of surfaces can be simplified by using monomial form. For example, two surfaces with different polynomials can be simply converted into each other by matrix product. Thus, the monomial form can be utilize for any applications concerning curves and surfaces with simple and efficient process.

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