

# Mixed Convection Boundary Layer Flows Induced by a Permeable Continuous Surface Stretched with Prescribed Skin Friction

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**Abstract**—The boundary layer flow and heat transfer on a stretched surface moving with prescribed skin friction is studied for permeable surface. The surface temperature is assumed to vary inversely with the vertical direction  $x$  for  $n = -1$ . The skin friction at the surface scales as  $(x^{-1/2})$  at  $m = 0$ . The constants  $m$  and  $n$  are the indices of the power law velocity and temperature exponent respectively. Similarity solutions are obtained for the boundary layer equations subject to power law temperature and velocity variation. The effect of various governing parameters, such as the buoyancy parameter  $\lambda$  and the suction/injection parameter  $f_w$  for air ( $Pr = 0.72$ ) are studied. The choice of  $n$  and  $m$  ensures that the used similarity solutions are  $x$  independent. The results show that, assisting flow ( $\lambda > 0$ ) enhancing the heat transfer coefficient along the surface for any constant value of  $f_w$ . Furthermore, injection increases the heat transfer coefficient but suction reduces it at constant  $\lambda$ .

**Keywords**—Stretching surface, Boundary layers, Prescribed skin friction, Suction or injection, similarity solutions, buoyancy effects.

## I. INTRODUCTION

A continuously moving surface through an otherwise quiescent medium has many applications in manufacturing processes. Such processes are hot rolling, wire drawing, spinning of filaments, metal extrusion, crystal growing, continuous casting, glass fiber production, and paper production [1]-[3]. Since the pioneer study of Sakiadis [4] who developed a numerical solution for the boundary layer flow field of a stretched surface, many authors have attacked this problem to study the hydrodynamic and thermal boundary layers due to a moving surface [5]-[14]. Recently, Magyari and Keller [15] have initiated a new research field by solving the induced boundary layer flows of impermeable stretched surfaces using prescribed skin friction boundary condition instead of the usual prescribed velocity boundary condition. Their results have shown substantial deviation of velocity profiles, temperature profiles as well as the wall heat fluxes from that of defined velocity boundary condition when compared on the actual physical scales of the coordinates. Suction or injection of a stretched surface was introduced by Erickson et al. [16] and Fox et al. [17] for uniform surface velocity and temperature and by Gupta and Gupta [18] for linearly moving surface. Chen and Char [19] have studied the

suction and injection on a linearly moving plate subject to uniform wall temperature and heat flux. The more general case, using a power law velocity and temperature distributions at the surface was studied by Ali [20]. Recently, Magyari et al. [21] have reported analytical and computational solutions for a moving surface with rapidly decreasing velocities using the self-similar method. The flow part of the problem was considered analytically by Magyari and Keller [22] for permeable surface moving with a decreasing velocity for velocity parameters  $-1/3$  and  $-1/2$ . Ingham [23] studied the existence of solutions for the boundary layer equations of a uniformly moving vertical plate with temperature inversely proportional to the distance up the plate. Laminar mixed convection of uniformly moving vertical surface for different temperature boundary conditions are considered by Ali and Al-Yousef [24], [25] and by Ali [26] for stretched surface with rapidly decreasing velocities. Recently, Ali and Al-Salem [27] extended the work of [15] for permeable surface. In their results three different cases for prescribed skin friction have been studied for isothermal surface temperature. Their results showed also that increasing  $m$  enhances the dimensionless heat transfer coefficient for fixed  $f_w$  at the suction case and the reverse is true at the injection case.

The present paper extends the work of Ali and Al-Salem [27] for mixed convection where the buoyancy parameter  $\lambda$  plays a very important role. The surface is assumed to be permeable moving with prescribed skin friction of order  $(x^{-1/2})$ . However, the analysis is focused on the case of  $n = -1$  and  $m = 0$  and for  $Pr = 0.72$ .

The mathematical formulation of the problem is presented in Section II, followed by the Numerical solution procedure in Section III. Results and discussion are reported in Section IV and finally conclusions are given in Section V.

## II. MATHEMATICAL ANALYSIS

Consider the steady two-dimensional motions of convective boundary layer flow induced by a moving vertical surface with suction or injection at the surface. For incompressible viscous fluid environment with constant properties, the equations governing this convective flow can be written as:

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$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g\beta(T - T_\infty) + v \frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (3)$$

Subject to the following prescribed skin friction coefficient boundary conditions:

$$\begin{aligned} \tau_w(x) &= \tau_w(x), v(x) = v_w(x), \text{ at } y = 0 \\ T - T_\infty &= T_w - T_\infty = Cx^n \quad \text{at } y = 0 \\ u(x) &= 0, T(x) = T_\infty = \text{const.}, \text{ as } y \rightarrow \infty \end{aligned} \quad (4)$$

The  $x$  coordinate is measured along the moving surface from the point where the surface originates, and the  $y$  coordinate is measured normal to it (Fig. 1). Positive or negative  $v$  imply injection or suction at the surface respectively, and  $u$  and  $v$  are the velocity components in  $x$  and  $y$  directions respectively. Similarity solutions arise when

$$u = U_0 x^m f'(\eta), \quad T - T_\infty = Cx^n \theta(\eta) \quad (5)$$

$$\eta = y \sqrt{\frac{m+1}{2}} \sqrt{\frac{U_0 x^m}{\nu x}} = \frac{y}{x} \sqrt{\frac{m+1}{2}} \sqrt{\text{Re}_x} \quad (6)$$

$$v = -\sqrt{\frac{2\nu U_0}{m+1}} x^{\frac{m-1}{2}} \left( \frac{m+1}{2} f + \frac{m-1}{2} f' \eta \right), m \neq -1 \quad (7)$$

and the shear stress and the heat flux at the surface are given by

$$\tau_w(x) = \mu U_0 x^{m-1} \sqrt{\frac{m+1}{2}} \text{Re}_x^{1/2} f''(0) \sim x^{(3m-1)/2} f''(0) \quad (8)$$

$$q_w(x) = -k C x^{n-1} \text{Re}_x^{1/2} \sqrt{\frac{m+1}{2}} \theta'(0) \sim x^{(m-1+2n)/2} \theta'(0) \quad (9)$$

It should be mentioned that, positive or negative  $m$  indicates that the surface is accelerated or decelerated from the extruding slit respectively. Where  $f'$  and  $\theta$  are the dimensionless velocity and temperature respectively, and  $\eta$  is the similarity variable. Substitution in the governing equations gives rise to the following two-point boundary-value problem.

$$f''' + ff'' - \frac{2m}{m+1} f'^2 + \frac{g\beta C}{U_0^2} \left( \frac{2}{m+1} \right) \theta x^{n-2m+1} = 0 \quad (10)$$

$$\theta'' + \text{Pr} \left( f\theta' - \left( \frac{2n}{m+1} \right) f'\theta \right) = 0 \quad (11)$$

For (10) to be  $x$  independent, for the similarity solution to be satisfied, then  $n$  must equal to  $2m-1$ . For this assumption of  $n = 2m-1$  (10) can be written as:

$$f''' + ff'' - \frac{2m}{m+1} f'^2 + \left( \frac{2}{m+1} \right) \theta \lambda = 0 \quad (12)$$

where  $\lambda = \frac{g\beta C}{U_0^2}$  is a constant not function of  $x$ . Therefore the

following study is focused on  $n = -1$  and  $m = 0$ . The transformed boundary conditions for a prescribed skin friction coefficient following Magyari and Keller [15] and Ali and Al-Salem [27].

$$f''(0) = -1, \quad f(0) = f_w, \quad f'(\infty) = 0 \quad (13)$$

$$\theta(0) = 1, \quad \theta(\infty) \rightarrow 0 \quad (14)$$

In deriving the second boundary condition in (13) the horizontal injection or suction speed  $v_w$  must be a function of the distance from the leading edge (for  $m \neq 1$ ). Consequently,  $v_w$  is given by:

$$v(x) = -\sqrt{\frac{\nu U_0 (m+1)}{2}} x^{\frac{m-1}{2}} f(0), m \neq -1 \quad (15)$$

The quantity  $f(0) = f_w$  will be referred to as the dimensionless suction/injection velocity. In this way  $f_w = 0$  corresponds to an impermeable surface,  $f_w > 0$  to suction (i. e.  $v_w(x) < 0$ ) and  $f_w < 0$  to lateral injection (i. e.  $v_w(x) > 0$ ) of the fluid through a permeable surface. The temperature of the injected fluid is assumed to coincide with the local temperature  $T_w(x)$  of the stretching surface.

The local skin friction coefficient, local Reynolds number and the Nusselt number are now given at the surface by

$$C_f \sqrt{\frac{\text{Re}_x}{2}} = f''(0) \sqrt{(m+1)} = -\sqrt{(m+1)} \quad (16)$$

$$\text{Re}_x = \frac{u_w x}{\nu} = \frac{U_0 x^{m+1}}{\nu} \quad (17)$$

$$\frac{\text{Nu}_x}{\sqrt{\text{Re}_x}} = -\sqrt{\frac{m+1}{2}} \theta'(0) \quad (18)$$

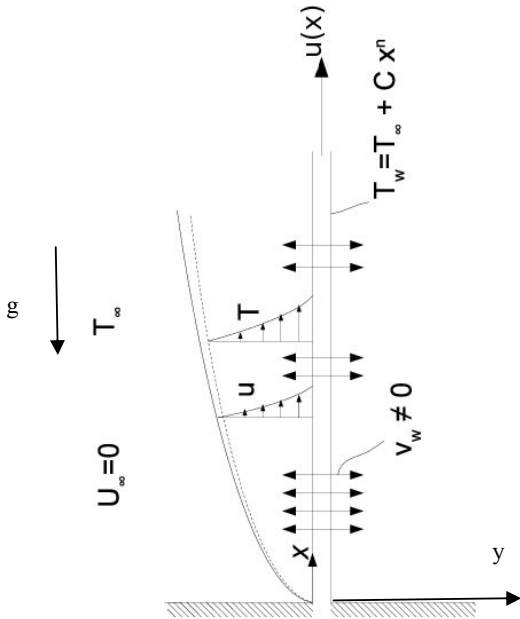


Fig. 1 Schematic of boundary layers induced close to a moving surface with prescribed shear stress

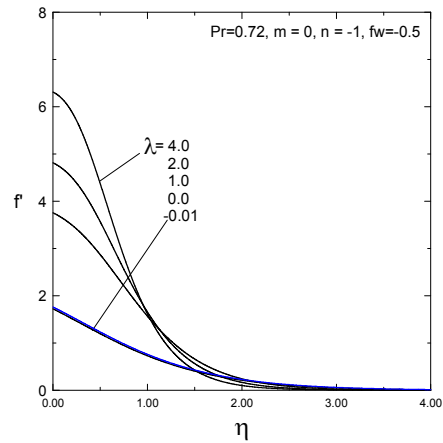
### III. NUMERICAL SOLUTION PROCEDURE

The coupled nonlinear ordinary differential equations (11) and (12) are solved for  $m = 0$  and  $n = -1$  numerically by using the fourth order Runge-Kutta method. Solutions of the differential equations (11) and (12), subject to the boundary conditions (13) and (14) were obtained for different values of  $\lambda$  and  $f_w$ . At each  $f_w$  and  $\lambda$ , the values of  $f'(0)$  and  $\theta'(0)$  are guessed and the differential equations (11) and (12) are integrated until the boundary conditions at infinity  $f'(\eta)$  and  $\theta(\eta)$  decay exponentially to zero (at least of order  $10^{-5}$ ). If the boundary conditions at infinity are not satisfied then the numerical routine uses a half interval method to calculate corrections to the estimated values of  $f'(0)$  and  $\theta'(0)$ . This process, which known as shooting method, is repeated iteratively until the boundary conditions at infinity are satisfied for  $f'$  and  $\theta$ . The value of  $\eta_\infty$  was chosen as large as possible depending upon the suction/injection parameter  $f_w$  and  $\lambda$ , without causing numerical oscillations in the values of  $f'$  and  $\theta$ . It should be noted that since the  $f'$  and  $\theta$  must exponentially decay to zero to satisfy the boundary conditions at infinity,  $f''$  and  $\theta'$  must also approach zero at infinity. Numerical solutions are obtained for  $f_w = -0.5$  (injection), 0 (permeable surface), 0.5 and 1 (suction) for a range of  $\lambda$  corresponding to assisting and opposing flow.

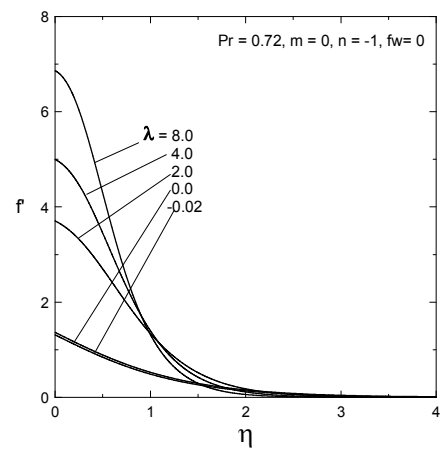
### IV. RESULTS AND DISCUSSION

Equations (11) and (12) are solved numerically for surface temperature inversely proportion with the  $x$  direction ( $n = -1$ ) and  $m = 0$  corresponding to skin friction at the surface scales as  $(x^{-1/2})$ . The solutions are obtained for Prandtl number,  $Pr = 0.72$  corresponding to air. Figs. 2 (a)-(c) show the velocity

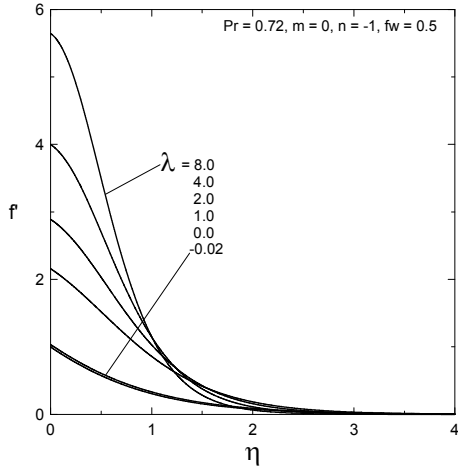
profiles for  $f_w = -0.5, 0, 0.5$  corresponding to injection, impermeable surface, and suction respectively and for different buoyancy parameter  $\lambda$ . It is clear from this figure that for the same  $\lambda$  suction reduces the dimensionless velocity  $f'(0)$  at the surface. Furthermore, increasing the buoyancy parameter  $\lambda$  always increasing  $f'(0)$  at the surface (assisting flow). On the other hand, opposing flow corresponding to negative  $\lambda$  reduces  $f'(0)$  with respect to the impermeable surface as seen for  $\lambda = -0.02$ . Figs. 3 (a)-(c) shows the temperature profiles for injection, impermeable, and suction at the surface corresponding to  $f_w = -0.5, 0, 0.5$  respectively. As seen in this figure for the same  $\lambda$  injection increases the slope at the surface and suction reduces it. However, for all values of  $f_w$  increasing  $\lambda$  (assisting flow) reduces the thermal boundary layer thickness and also increases the gradient of heat transfer at the surface. Fig. 4 shows the non-dimensional velocity profiles  $f'(\eta)$  for different values of  $f_w$ . As seen in this figure  $f'(\eta)$  increases as  $\lambda$  increases (assisting flow) for constant  $f_w$ . On the other hand, the negative  $\lambda$  (opposing flow) has little effect on  $f'(\eta)$  which approaches an asymptotic value independent of  $f_w$ .



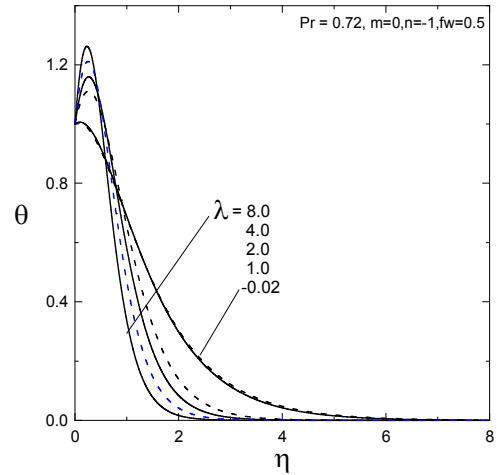
(a)



(b)



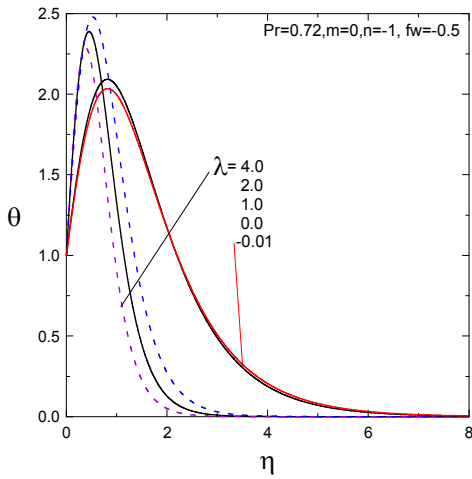
(c)



(c)

Fig. 2 Non-dimensional velocity profiles for various values of  $\lambda$  (a) for injection at the surface  $f_w = -0.5$ , (b) for impermeable surface  $f_w = 0$ , and (c) for suction  $f_w = 0.5$

Fig. 3 Non-dimensional temperature profiles for various values of  $\lambda$  (a) for injection at the surface  $f_w = -0.5$ , (b) for impermeable surface  $f_w = 0$ , and (c) for suction  $f_w = 0.5$



(a)

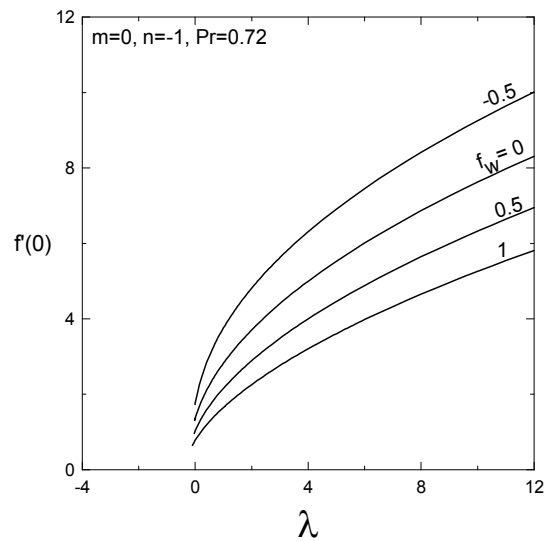
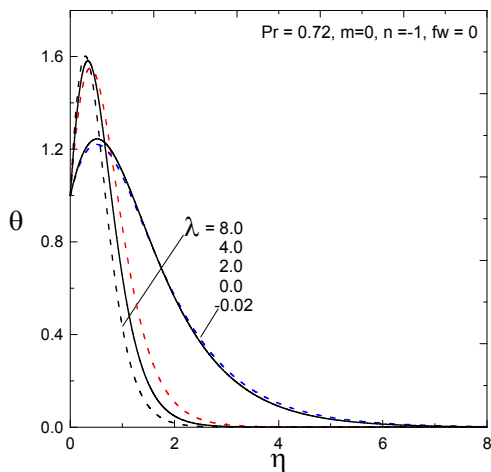


Fig. 4 Non-dimensional velocity profiles for various values of  $f_w$



(b)

Furthermore, for fixed values of  $\lambda$  injection increases the velocity at the surface and suction reduces it. The dimensionless heat transfer coefficient  $\theta'(0)$  is shown for different values of  $f_w$  in Fig. 5. This figure shows that assisting flow enhances the heat transfer coefficient along each constant  $f_w$  profile but for opposing flow ( $\lambda < 0$ ) they intend to reach asymptotic values independent of  $f_w$ . Moreover, for fixed  $\lambda$  suction reduces the heat transfer coefficient while injection increases it as expected. The dashed line presents a 5% increase of the  $\theta'(0)$  relative to the value at  $\lambda = 0$ , therefore, beyond this values natural convection dominates.

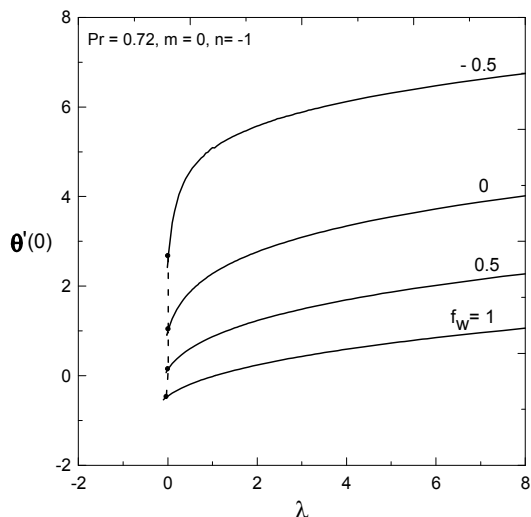


Fig. 5 Non-dimensional heat transfer coefficient (temperature gradient at the surface) profiles for various values of  $f_w$ . On the right of the dashed line natural convection dominates and on the left forced convection dominates

#### V. CONCLUSIONS

Heat transfer and flow field characteristics of a stretched permeable surface subject to mixed convection with prescribed skin friction are studied for skin friction boundary conditions corresponding to  $m = 0$  and for thermal boundary conditions corresponding to  $n = -1$ . The results show that at the surface; the dimensionless stretching velocity increases as the buoyancy parameter  $\lambda$  increases for any  $f_w$ . It is also shown that as  $\lambda$  increases the dimensionless heat transfer coefficient increases and also for fixed  $\lambda$ , suction reduces the heat transfer coefficient but injection enhances it. Furthermore, for any  $f_w$  and as  $\lambda$  approaches zero or negative values (opposing flow) the  $f'(0)$  tends to an asymptotic value independent of  $f_w$  and this case is almost true for  $\theta'(0)$ .

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