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Mathematical Model of Depletion of Forestry Resource: Effect of Synthetic Based Industries

Manisha Chaudhary, Joydip Dhar and Govind Prasad Sahu

Abstract—A mathematical model is proposed considering the forest biomass density $B(t)$, density of wood based industries $W(t)$ and density of synthetic industries $S(t)$. It is assumed that the forest biomass grows logistically in the absence of wood based industries, but depletion of forestry biomass is due to presence of wood based industries. The growth of wood based industries depends on $B(t)$, while $S(t)$ grows at a constant rate, independent of $B(t)$. Further there is a competition between $W(t)$ and $S(t)$ according to market demand. The proposed model has four ecologically feasible steady states, namely, E_1: forest biomass free and wood industries free equilibrium; E_2: wood industries free equilibrium and two coexisting equilibria E_1^*, E_2^*. Behavior of the system near all feasible equilibria is analyzed using the stability theory of differential equations. In the proposed model, the natural depletion rate h_1 is a crucial parameter and system exhibits Hopf-bifurcation about the non-trivial equilibrium with respect to h_1. The analytical results are verified using numerical simulation.

Keywords—A mathematical model, Competition between wood based and synthetic industries, Hopf-bifurcation, Stability analysis.

I. Introduction

MATHEMATICAL modeling is an alternative tool to study the population dynamics with nutrient, under the influence of many facts, such as toxicants, diseases etc.[1]-[21]. Prior to the 20th century, most transportation structures in North America were made of wood. During this century, concrete and steel have replaced wood in many applications. In the global economy of the 21st century, knowledge, innovation, energy supply, and access to natural resources are keys to economic competitiveness and vitality. As communities within the Northern Forest region continue to face significant transition and globalization in the wood products industry (with related loss of local employment and income), wood-based biomass energy and bio-fuels are important components of the region's future wood-products economy. Forests are nature's greatest gift to mankind since origin, human population has depended upon forest for its various needs, be it food, fodder, fibre, fertilizers, medicine, construction material etc. Forests are now well-known for the wealth of services they provide: supplying timber and non-timber forest products, mitigating climate change, preserving biological diversity, maintaining indigenous livelihoods, and providing for recreational and spiritual purposes. It is a well-known accepted fact that forestry resource plays a vital role in the development of any country present and future. But it is being depleted by increased industrialization, over growth of population and associated pollution. A typical example is the Doon Valley in the northern part of India where the forestry resources are being depleted by limestone quarries, wood and paper based industries, growth of human and livestock populations, expansion of forest land for agriculture and settlement, etc., threatening the ecological stability of the entire region. It has been noted that the forest bio-diversity loss and changes in climate are closely linked with deforestation. All measures that are taken to ensure a long life of forestry resource would fall under the definition of wood preservation. To overcome the world wide problem of conservation of forestry resources, synthetic is good alternative of wood based product as it is cheap, needs not much maintenance and the one most important thing that it looks fresher than a wood based product. Synthetic wood simulates the color and grain of natural woods. The most factors in the purchase decision include: quality, durability and installed stability. Synthetic wood can be machined (cut, routed, drilled, etc.) and fastened like wood. It won't splint and because it doesn't absorb water, it won't shrink and swell, and almost recyclable [9], [12]. Forests play a very important role in maintaining the environment including atmospheric stability and in supplying the essential requirements of people all around the world. But due to global warming and due to diversion of forest lands to non-forest, forest suffers from depletion[1], [2], [6], [7], [19], [21]. In recent years, some investigations have been made to study the effect of deforestation and various factors (industrialization, increasing population, etc.) which affect the forest biomass the most. Depletion of forest biomass is mainly due to deforestation and industrialization [1], [2], [9], [16], [21]. There are several literatures which include the mathematical models simulating the effect of depletion of a renewable resource by population and industrialization. Shukla et al. [16]-[21], proposed few mathematical models for the depletion of forest biomass by industries, pollution, population, etc. Dhar et al. proposed various model to study the effect of industrialization on growth and existence of a biological species that depends partially or wholly on a given resource or act as a predator on the resource [2]-[5]. It has been noted here that to save forest based resources it is very important to search alternatives of the same. Keeping the aforementioned facts in mind, we propose and analyze a mathematical model based on forestry resource conservation with growth of synthetic industry. We consider an ecosystem

M. Chaudhary, Research scholar is with School of Mathematics and Allied Sciences Department, Jiwaji University , Gwalior, India (e-mail: manisha23june@gmail.com).

J. Dhar, Associate Professor, is with the Applied Sciences Department, ABV-Indian Institute of Information Technology and Management Gwalior, Morena Road, Gwalior-474015, India (Phone/Fax: +91-751-2449829; e-mail: jdhar.iiitmg@gmail.com).

G. P. Sahu, Research scholar is with the Applied Sciences Department, ABV-Indian Institute of Information Technology and Management, Morena Road, Gwalior-474015, India (e-mail: govind3012@iiitm.ac.in).

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where the wood based resource is being continuously depleted due to the industrialization.

II. Mathematical Model

We consider the density at any time t of the forest biomass $B(t)$, wood industries $W(t)$, synthetic industries $S(t)$ respectively and wish to conserve the forest by using wood alternative industries. The depletion of forest is mainly due to wood industries, population and pollution. We also assume that forest population grows logistically in the absence of wood based industries with intrinsic growth rate r and carrying capacity K. Again, c_1, c_2 shows the competition effect of $B(t)$ on $W(t)$ and $W(t)$ on $B(t)$, respectively. Also, we assume that wood based industries entirely depend on the forest biomass while the synthetic industries do not. We also consider that a sufficient amount of synthetic is provided to synthetic industries at constant rate Q. Depletion rate of forest biomass is α and α_1 is growth rate of wood based industries in presence of forestry biomass. Finally, h_1 and h_2 are natural depletion rate of wood industries and synthetic industries, respectively. The model is being formulated with the help of following system of non-linear differential equations:

$$\frac{dB}{dt} = rB(1 - \frac{B}{k}) - \alpha BW, \tag{1}$$

$$\frac{dW}{dt} = \alpha_1 BW - c_1 WS - h_1 W, \tag{2}$$

$$\frac{dS}{dt} = Q - c_2 WS - h_2 S, \tag{3}$$

where all initial population are positive, i.e., $B(0) > 0, W(0) > 0, S(0) > 0$. The above system (1) - (3) can be non-dimensionalised by substituting $x = \frac{B}{k}; y = \frac{W}{k}; z = \frac{S}{k}$, we get the following re-scaled system:

$$\frac{dx}{dt} = x(1 - x) - axy, \tag{4}$$

$$\frac{dy}{dt} = a_1 xy - a_2 yz - a_3 y, \tag{5}$$

$$\frac{dz}{dt} = Q_0 - a_4 yz - a_5 z, \tag{6}$$

where $a = \frac{\alpha k}{r}; a_1 = \frac{\alpha_1 k}{r}; a_2 = \frac{c_1 k}{r}; a_3 = \frac{h_1}{r}; a_4 = \frac{c_2 k}{r}; a_5 = \frac{h_2}{r}; Q_0 = \frac{q}{rk}$ and $x(0) > 0, y(0) > 0, z(0) > 0$. In the next section, we will study the existence of all possible steady states of the system and the boundedness of the solutions.

III. Boundedness of The System

LEMMA 1: All the solution of the system (4) - (6) that initiate in ..3+ are eventually bounded and enter into a region R, defined by $R = x, y, z : 0 < x_l \leq x \leq< x_u, 0 \leq y \leq y_u, 0 \leq z \leq z_u$ where $x_l = 1 - \frac{a_1 \theta}{a m_1}, x_u = 1, y_u = \frac{a_1 \theta}{a m_1}, z_u = \frac{Q_0 + \theta a_1}{m_2}$. Proof is given in APPENDIX-A

IV. Existence of Equilibrium Points

There are four equilibrium points namely E_1, E_2, E^{*1}, E^{*2} where $(i) E_1 = (1, 0, \frac{Q_0}{a_5}), E_2 = (0, 0, \frac{Q_0}{a_5}), E^{*i} = (x^*, y^*, z^*)$, where y^* is root of the following quadratic equation,

$$\begin{aligned} a a_1 a_4 y^{*2} &+ (a_3 a_4 - a_1 a_4 + a a_1 a_5) y^* + (a_3 a_4 - a_1 a_4 \\ &+ a a_1 a5) = 0. \end{aligned} \tag{7}$$

(i) Equation (7) has two positive roots if,
$a_3 a_4 - a_1 a_4 + a a_1 a5 < 0$ and $a_3 a_4 - a_1 a_4 + a a_1 a_5 > 0$
(ii) and has exactly one positive root if,
$a_3 a_4 - a_1 a_4 + a a_1 a5 < 0$ and $a_3 a_4 - a_1 a_4 + a a_1 a_5 < 0$, where
$x^* = \frac{a_2 Q_0 + a_3 a_4 y^* + a_3 a_5}{a_1 a_4 y^* + a_1 a_5}$,
$z^* = \frac{Q_0}{a_1 y^* + a_5}$,
${y_1}^* = \frac{-a_3 a_4 + a_1 (a_4 - a a_5) \pm \sqrt{((a_1 - a_3) a_4 + a a_1 a_5)^2 - 4 a a_1 a_2 a_4 Q_0}}{2 a a_1 a_4}$.
On substituting the values of y^* in x^* and z^*, we have two interior equilibrium in terms of x^*, y^*, z^*, say, E^{*1} and E^{*2} as given below

$$E^{*1} = \begin{pmatrix} \frac{(a_1 + a_3) a_4 + a a_1 a_5 - \sqrt{(a_3 a_4 - a_1 (a_4 + a a a_5)^2 - 4 a a_1 a_2 a_4 Q_0)}}{2 a_1 a_4}, \\ \frac{-a_3 a_4 + a_1 (a_4 - a a_5) + \sqrt{((a_1 - a_3) a_4 + a a_1 a_5)^2 - 4 a a_1 a_2 a_4 Q_0}}{2 a a_1 a_4}, \\ \frac{-a_3 a_4 + a_1 (a_4 - a a_5) + \sqrt{((a_1 - a_3) a_4 + a a_1 a_5)^2 - 4 a a_1 a_2 a_4 Q_0}}{2 a a_1 a_4} \end{pmatrix}$$

$$E^{*2} = \begin{pmatrix} \frac{(a_1 + a_3) a_4 + a a_1 a_5 + \sqrt{(a_3 a_4 - a_1 (a_4 + a a a_5)^2 - 4 a a_1 a_2 a_4 Q_0)}}{2 a_1 a_4}, \\ -\frac{(-a_1 + a_3) a_4 - a a_1 a_5 + \sqrt{(a_3 a_4 - a_1 (a_4 + a a_5))^2 - 4 a a_1 a_2 a_4 Q_0}}{2 a a_1 a_4}, \\ \frac{-a_3 a_4 - a_1 (a_4 - a a_5) + \sqrt{((a_1 - a_3) a_4 + a a_1 a_5)^2 - 4 a a_1 a_2 a_4 Q_0}}{2 a a_1 a_4} \end{pmatrix}$$

out of which interior equilibrium E^{*i} is biologically feasible equilibria for the system, for i=1, 2, provided all the components are non-negative.

V. Dynamical behavior of the system

In the previous section, we have established that the system has three feasible equilibrium points, namely, $E_1(1, 0, \frac{Q_0}{a_5})$, $E_2(0, 0, \frac{Q_0}{a_5})$ and $E^{*i}(B^*, W^*, S^*)$. The general variation matrix of the system is given by-

$$J = \begin{bmatrix} 1 - 2x - ay & -ax & 0 \\ y a_1 & x a_1 - z a_2 - a_3 & -y a_2 \\ 0 & -z a_4 & -y a_4 - a_5 \end{bmatrix} \tag{8}$$

Now, corresponding to the equilibrium point E_1 Jacobean J has the following eigen values $\lambda_1 = -1, \lambda_2 = -a_5, \lambda_3 = -(\frac{a_3 a_5 + a_2 Q_0 - a_1 a_5}{a_5})$, i.e., E_1 is stable provided

$$a_1 < a_3 + \frac{a_2 Q_0}{a_5}. \tag{9}$$

It is clear from (9) that in long run the wood based industries will be extinct if the growth rate of these industries is less than a threshold.

The characteristic equation for the equilibrium E^{*i} is

$$P(\lambda) = \lambda^3 + b_1 \lambda^2 + b_2 \lambda + b_3 = 0, \tag{10}$$

where $b_1 = x^* + \frac{Q_0}{z^*}, b_2 = a a_1 x^* y^* + \frac{Q_0 x^*}{z^*} - a_2 a_4 y^* z^*, b_3 = \frac{a a_1 Q_0 x^* y^*}{z^*} - a_2 a_4 x^* y^* z^*$.

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$b_1 b_2 > b_3$, if

$$a_2 a_4 Q_0 y^* < a a_1 x^{*2} y^* + \frac{Q_0^2 x^*}{z^{*2}} + \frac{Q_0^2 x^*}{z^*} \frac{a a_1 Q_0 x^* y^*}{z^*} \quad (11)$$

Substituting the values of x^*,y^* and z^* it can be easily verified that $b_i > 0$, for i = 1, 2, 3. From the Routh-Hurwitz criterion, a set of necessary and sufficient conditions for all the roots of the equation (10) having negative real part are $b_i > 0$, i = 1, 2, 3 and

$$b_1 b_2 > b_3. \quad (12)$$

Hence, we can state the following theorem:

Theorem 1: The system of equation (4)-(6) is locally stable around the interior equilibrium E^{*i}, when inequality (12) holds.

VI. Hopf-bifurcation analysis

Now, we explore the possibility of Hopf-bifurcation of the above system (4)-(6), by taking "a_3" (i.e. the rate of natural depletion rate) as the bifurcation parameter. The necessary and sufficient conditions for the existence of the Hopf-bifurcation are, if there exists $a_3 = a_0$ such that (i)$b_i(a_0) > 0$,i=1, 2, 3 (ii)$b_1(a_0)b_2(a_0) - b_3(a_0) = 0$ i.e.,

$$-a_2 a_4 Q_0 y^* + a_0 a_1 y^* (1 - a_0 y^*)^2 + (1 - a_0 y^*)^2$$
$$(a_4 y^* + a_5) + (1 - a_0 y^*)(a_4 y^* + a_5)^2 = 0, \quad (13)$$

and (iii) if we consider the eigen-values of the characteristic equation (10) of the form $\lambda_i = u_i + v_i$ then $\frac{d}{d\lambda}(u_i) \neq 0$, $i = 1, 2, 3$. Putting $\lambda = u + iv$ in equation(10) we get

$$(u + iv)^3 + b_1(u + iv)^2 + b_2(u + iv) + b_3 = 0. \quad (14)$$

On separating the real and imaginary part of equation (14) and eliminating v between real and imaginary part, we get

$$8u^3 + 8b_1 u^2 + 2({b_1}^2 + b_2)u + b_1 b_2 - b_3 = 0. \quad (15)$$

It is clear from above that $u(a_0) = 0$ if and only if $b_1(a_0)b_2(a_0) - b_3(a_0) = 0$. The existence of threshold value $a_3 = a_0$ is ensured by the positive root of (13). Here the discriminant of $8u^2 + 8b_1 u + 2(b_1^2 + b_2) = 0$ is $64b_1^2 - 64(b_1^2 + b_2) < 0$, which ensures that $\frac{d}{d\lambda}(b_1 b_2 - b_3) \neq 0$ at $a_3 = a_0$. Again differentiating (15) with respect to a_3 we have $(24u^2 + 16b_1 u + 2(b_1^2 + b_2))\frac{du}{d\lambda} + (8u^2 + 4b_1 u)\frac{db_1}{d\lambda} + 2u\frac{db_2}{d\lambda} + \frac{d}{d\lambda}(b_1 b_2 - b_3) = 0$. Now since at $a_3 = a_0, u(a_0) = 0$, we get $[\frac{du}{d\lambda}]_{a_3 = a_0} = \frac{-\frac{d}{d\lambda}(b_1 b_2 - b_3)}{2(b_1^2 + b_2)} \neq 0$, which satisfies transversality condition of the Hopf-bifurcation. It is tedious to find the analytic value of a_0 form (13), we show the existence of positive roots of (13) using particular set of parametric values in numerical section.

VII. Global Bifurcation

Theorem 2: The interior equilibrium $E^* = (x^*, y^*, z^*)$, if exists, is globally asymptotically stable provided the following inequalities are satisfied:

$$(a - a_1 y^*)^2 \leq (a_2 z_l - a_1 x_u), \quad (16)$$

$$(a_2 y^* - a_4 z_u)^2 \leq (a_2 z_l - a_1 x_u)(a_4 y^* + a_5). \quad (17)$$

Proof is given in appandix-II

VIII. Numerical Simulation

In this section we perform numerical simulation to check the feasibility of our analysis using Matlab taking following set of parameter values in model system (4)-(6); $a = 0.6, a_1 = 3.8, a_2 = 0.5, a_4 = 1.2, a_5 = 1.55, Q_0 = 5$. For the above set of parameter values, it can be easily verified that the condition of existence of interior equilibrium E* and the global stability conditions (14) and (15) are satisfied. The equilibrium values corresponding to the above parameters are obtained as $x^* = 0.352, y^* = 1.079$ and $z^* = 1.757$. Solving the

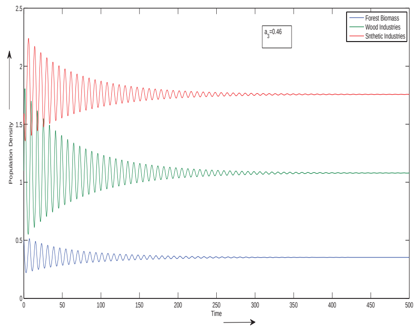

Fig. 1. Stable interior equilibrium point E^*, for $a_3 = 0.46$ and initial values $x(0) = 0.35, y(0) = 1.07, z(0) = 1.75$

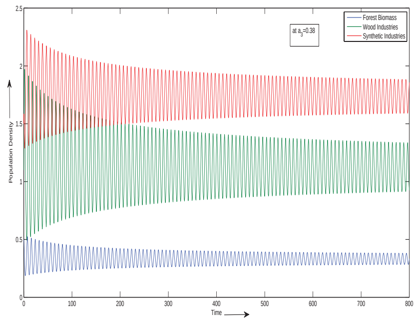

Fig. 2. Unstable(oscillating) interior equilibrium point E^*, for $a_3 = 0.38$

quadratic equation (13), we get a positive root $a_3 = 0.389$ which ensure that the above system has Hopf-bifurcation. The natural depletion rate of the wood based industries a_3 when remains below its threshold value $a_3 = a_0 = 0.389$, the system oscillates around the interior equilibrium point (see figure 2 and figure 3, $a_3 = 0.38$), and when it crosses the threshold value $a_3 = a_0$, the interior equilibrium is asymptotically stable (see figure. 1, $a_3 = 0.46$). It is further noted that all the necessary conditions required for the local and global stability behavior of E^* are satisfied for the above set of values of parameters.

IX. Conclusion

In this paper, we proposed a mathematical model for the conservation of forest biomass with wood based industries

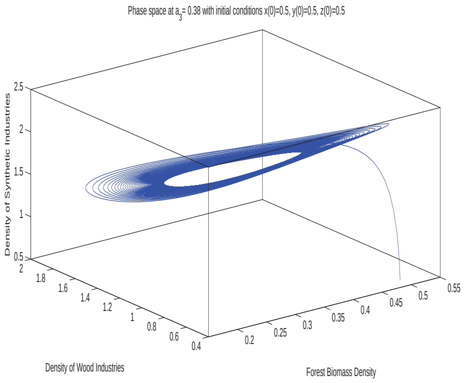

Fig. 3. Unstable(oscillating) interior equilibrium point E^*, for $a_3 = 0.38$

and synthetic industries. The density of biomass grows logistically and its growth rate is depleted by wood industries. The synthetic industries grow at a constant rate; synthetic industries is promoted over the wood based industries due to human awareness to save our forest environment. There is a competition between wood based industries due to market demand. Numerical simulation has also been performed to study the effects of various parameters on the dynamics of the system. In existing literature much work has been done to study the dynamics of wood industries and synthetic industries to make environment eco-friendly. Forests serve as a source of life for the forest based small and large scale industries. However, due to shrinking forests area the industries are facing wood crisis. To overcome from wood crisis synthetic is one of the good alternatives, which is cheap and widely available and is also helpful to preserve forest biomass. It is clear from 9 that in future the wood based industries will be extinct if the growth rate of these industries is less than a threshold depends on growth of synthetic industries. As synthetic products are good alternative for wood based products and choosing particular level of deforestation and control the wood based industries by human awareness or by some government action to preserve our forestry biomass. Hence, a mathematical model is proposed and analyzed to study the depletion of resource biomass due to wood based industries. Criteria for local stability, instability, and global stability of non-negative equilibria are obtained. Numerical simulations are carried out to investigate the dynamics of the system. Through this paper we determine criteria for Hopf-bifurcation using natural depletion rate of wood based industries a_3 as bifurcation parameter. We found that as the natural depletion rate exceeds its threshold value the system becomes stable while below the threshold value system oscillates around the interior equilibrium.

Appendix A

PROOF OF LEMMA 1: We have from equations (4) - (6)
$\frac{dx}{dt} = x(1-x) - axy \Rightarrow \frac{dx}{dt} \le x(1-x)$.
$\Rightarrow \lim_{t\to\infty} x \le 1$, therefore $x_u = 1$. $\frac{dx}{dt} \ge x(1-x) - axy_u$,
$\frac{dx}{dt} \ge x(1-ay_u) - x^2$, taking $(1-ay_u) = k_1$.
$\Rightarrow lim_{t\to\infty} x \ge k_1, x_l = (1-ay_u)$.
From set of equation (4) - (6)
$a_1\frac{dx}{d\tau} + a\frac{dy}{d\tau} = a_1x - a_1x^2 - aa_3y$,
$\frac{d}{d\tau}(a_1x + ay) \le a_1x - aa_3y$
$\le a_1x + \theta a_1x - \theta a_1x - aa_3y$,
$\le \theta a_1x_u - xa_1(\theta - 1) - aa_3y$,
$\le \theta a_1x_u - mxa_1 - aa_3y - aa_3y$,
taking $m_1 = min(m, a_3)$,
$\le \theta a_1 - m_1(xa_1 + ay) \le \theta a_1 - m_x$,
$\Rightarrow lim_{t\to\infty} xa_1 + ay \le \frac{a_1\theta}{m_1}$.
$ay_u = \frac{a_1\theta}{m_1}$.
Therefore $y_u = \frac{a_1\theta}{am_1}$, again from equation (4) - (6)
$a_1\frac{dx}{d\tau} + a\frac{dy}{d\tau} + \frac{dz}{d\tau} = a_1x - a_1x^2 - aa_3y + Q_0 - a_5z$, $\le a_1x - aa_3y + Q_0 - a_5x$, $\le a_1x + \theta a_1x - \theta a_1x - aa_3y + Q_0 - a_5z$,
$\frac{d}{d\tau}(a_1x + ay + z) \le Q_0 + \theta a_1x - ma_1x - aa_3y - a_5z$,
choosing m_2=min(m, a_3, a_5),
$\le Q_0 + \theta a_1x - m_2(a_1x + ay + z)$,
$\le Q_0 + \theta a_1x - m_2(Y)$, where $Y = (a_1x + ay + z)$.
$\Rightarrow \lim_t \to \infty Y \le \frac{Q_0 + \theta a_1x_u}{m_2}$,
$\Rightarrow \lim_t \to \infty(a_1x + ay + z) \le \frac{Q_0 + \theta a_1x_u}{m_2}$,
$z_u = \frac{Q_0 + \theta a_1x_u}{m_2}$.
Therefore $z_u = \frac{Q_0 + \theta a_1}{m_2}$, which completes the proof and hence we can say that all the solutions of the system lies in region $R = \{(x, y, z) : 0 < x_l \le x \le x_u, \theta < y \le y_u, 0 < z \le z_u\}$

Appendix B

Proof of theorem 2: Let us consider the positive definite function V about positive equilibrium point $E^*(x^*, y^*, z^*)$, as $V = \frac{1}{2}[(x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2]$, whose time derivative is given by
$\frac{dv}{d\tau} = (x-x^*)\frac{dx}{d\tau} + (y-y^*)\frac{dy}{d\tau} + (z-z^*)\frac{dz}{d\tau}$, using equation (4) - (6) and assuming
$z_1 = (x-x^*), z_2 = (y-y^*), z_3 = (z-z^*)$ doing some mathematical manupulation
$\frac{dV}{d\tau} = -[z_1^2 + z_1z_2(a - a_1y^*) + {z_2}^2(a_2z - a_1x) + z_2z_3(a_2y^* - a_4z) + {z_3}^2(a_4y^* + a_5)]$,
$\frac{dV}{d\tau} \le -[z_1^2 + z_1z_2(a - a_1y^*) + {z_2}^2(a_2z_l - a_1x_u) + z_2z_3(a_2y^* - a_4z_u) + {z_3}^2(a_4y^* + a_5)]$,
$\frac{dV}{d\tau} \le -[(z_1^2a_{11} + z_1z_2a_{12} + {z_2}^2a_22) + ({z_2}^2b_{11} + z_2z_3b_{12} + {z_3}^2b_{22})]$,
$a_{11} = 1$, $a_{12} = a - a_1y^*$, $a_{22} = \frac{1}{2}(a_2z_l - a_1x_u) = b_{11} =$, $b_{12} = a_2y^* - a_4z_u$, $b_{22} = a_4y^* + a_5$.
Hence $\frac{dV}{d\tau}$ is negative definite under conditions $a_12 \le 2a_11a_22$ and $b_12^2 \le 2b_11b_22$.
Therefore, V(t) is negative definite. Hence by Lyapunov's direct method it is proved that $E^*(x^*, y^*, z^*)$ is globally asymptotically stable.
The above conditions can be further rewritten as
$(a - a_1y^*)^2 < (a_2z_l - a_1x_u)$,
$(a_2y^* - a_4z_u)^* < (a_2z_l - a_1x_u)(a_4y^* + a_5)$.
This completes the proof.

References

[1] M. Agarwal, Tazeem Fatima, H.I.Freedman, Depletion of forestry resources biomass due to industrialization pressure: a ratio dependent mathematical model, Vol. 4 No. 4, pp. 381-396, 2010.

[2] J. Dhar, H. Singh, Modelling the depletion of forestry resource by wholly dependent industrialization in two adjoining habitat, Kobe J. Mathematics, Japan, 21(2-1), pp. 1-13, 2004.
[3] J. Dhar, H. Singh, Diffusion of population under the influence industrialization in a twin-city environment, Mathematical Modelling and Analysis, 9(3), pp.124-133, 2004.
[4] J. Dhar, Population model with diffusion and supplementary forestry resource in a two-patch habitat, applied mathematical modelling, Vol.32, pp. 1219-1235, 2008.
[5] J. Dhar, A.K. Sharma, The role of viral infection in phytoplankton dynamics with the inclusion of incubation class, Nonlinear Analysis: Hybrid Systems, Vol. 4 (1) , pp. 9-15, 2010.
[6] B. Dubey, S. Sharma, P. Sinha, J. B. Shukla, Modelling the depletion of forestry resources by population and population augmented industrialization, Vol. 33, Issue 7, pp. 3002-3014, 2009.
[7] D. Ghosh, A. K. Sarkar, Stability and oscillation in a resource based model of two interacting species with nutrient cycling, J. Ecological Modelling, 107, pp. 25-33, 1998.
[8] D. McHugh (Ed.) Forests and climate change. Special Report No. 6, World Wide Fund for Nature, WWF International, Gland Switzerland, pp. 16-18, 1991.
[9] O. Joshi, S. R. Mehmood, Factors affecting nonindustrial private forestry landowners willingness to supply woody biomass for bioenergy, biomass and bioenergy 35, pp. 186-192, 2011.
[10] S. Khare, O.P. Misra, J. Dhar, Role of toxin producing phytoplankton on a plankton ecosystem, Nonlinear Analysis: Hybrid Systems, Vol. 4 (3), pp. 496-502, 2010.
[11] W. Li, M. Ju, Le Liu, Y. Wang, T. Li, The Effects of Biomass Solid Waste Resources Technology in Economic Development, Energy Procedia 5 pp. 2455-2460, 2011.
[12] S.L. Naeem, J. Thompson, S.P. Lawler, G.H. Lawton and R.M. Wood fin, Decline biodiversity can alter the performance of ecosystem, Nature 368, pp. 734-737, 1994.
[13] Bob Persched, A. Evans, M. DeBonis, Forestry biomass Retention and harvesting guidelines for the northeast, Forestry guild, 2010.
[14] G.P. Sahu, J. Dhar, Analysis of an SVEIS epidemic model with partial temporary immunity and saturation incidence rate, Applied Mathematical Modelling, Vol. 36 (3), pp. 908-923, 2012.
[15] S. Sinha, O. P. Misra, J. Dhar, Modelling a predator-prey system with infected prey in polluted environment, Applied Mathematical Modelling 34 (7) , pp. 1861-1872, 2010.
[16] J. B. Shukla, O. P. Misra, M. Agarwal, A. Shukla, Effect of pollution and industrial development on degradation of biomass resource:A mathematical model with reference to Doon Valley, Mathl. Comput. Modell. Vol. 11, pp. 910-913, 1988.
[17] J. B. Shukla, H. I. Freedman, V. N. Pal, O.P. Misra,M. Agrawal,A. Shukla, Degradation and subsequent regeneration of a forestry resource: A mathematical model. Ecol. Modell. 44, pp. 219-229, 1989.
[18] J. B. Shukla, B. Dubey and H.I. Freedman, Effect of changing habitat on survival of species, Ecol. Model.87, pp. 205-216, 1996.
[19] J. B. Shukla and B. Dubey, Modelling the depletion and conservation of forestry resources: Effects of population and pollution, J. Math. Biol.36, pp. 7194, 1997.
[20] J. B. Shukla, A. K. Agrawal, P. Sinha, B. Dubey, Modelling effects of primary and secondary toxicants on renewable resources, Natural Resource Modelling, Vol. 16(1), pp. 1-22, 2003.
[21] J. B. Shukla, K. Lata, A. K. Misra, Modelling the depletion of a renewable resource by population and industrialisation: Effect of technology on its conservation, Natural Resource modelling, Vol. 24, 2011.