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Load Discontinuity in Shock Response and Its Remedies

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***Abstract*—**It has been shown that a load discontinuity at the end of an impulse will result in an extra impulse and hence an extra amplitude distortion if a step-by-step integration method is employed to yield the shock response. In order to overcome this difficulty, three remedies are proposed to reduce the extra amplitude distortion. The first remedy is to solve the momentum equation of motion instead of the force equation of motion in the step-by-step solution of the shock response, where an external momentum is used in the solution of the momentum equation of motion. Since the external momentum is a resultant of the time integration of external force, the problem of load discontinuity will automatically disappear. The second remedy is to perform a single small time step immediately upon termination of the applied impulse while the other time steps can still be conducted by using the time step determined from general considerations. This is because that the extra impulse caused by a load discontinuity at the end of an impulse is almost linearly proportional to the step size. Finally, the third remedy is to use the average value of the two different values at the integration point of the load discontinuity to replace the use of one of them for loading input. The basic motivation of this remedy originates from the concept of no loading input error associated with the integration point of load discontinuity. The feasibility of the three remedies are analytically explained and numerically illustrated.

***Keywords*—**Dynamic analysis, load discontinuity, shock response, step-by-step integration

I. Introduction

THE shock response from a short pulse is a special class of the dynamic analysis [1]–[3]. Although the loading time of the impulse is relatively short it might still play an important role in the design of certain classes of structures. The classical method to solve differential equations of motion or Duhamel's integral [2], [3] is often employed to compute the response to an impulse for a linear elastic system. However, they are, in general, not applicable to a nonlinear system. Hence, a step-by-step integration method [4]–[17] is often used to solve a nonlinear system alternatively. In the step-by-step solution of the shock response from an impulse, it might experience that a very small time step is needed for conducting the step-by-step integration although the impulse is in a simple form, such as a rectangular impulse or a rising triangular impulse. This difficulty has been investigated [18] and it was shown that the load discontinuity at the end of an impulse is responsible for using a very small time step. This is because this discontinuity at the end of an impulse will lead to an extra impulse, and then an extra displacement is introduced in addition to the correct displacement response in the step-by-step solution of the shock response to the impulse. It is clear that to reduce or even eliminate the extra impulse is the key to overcome the load discontinuity problem at the end of an impulse. For this purpose, three remedies were proposed and will be applied to overcome the difficulty caused by the load discontinuity at the end of an impulse. Although the analytical proofs of the three remedies will not be presented herein for brevity the concept of each remedy will be thoroughly explained. In addition, the feasibility each remedy will be confirmed by numerical examples.

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II. Load Discontinuity

A load discontinuity at the end of an impulse will lead to an extra impulse, and thus an extra displacement response. This result has been analytically verified and numerically confirmed [18]. In addition, a simple formula is also defined so that the relative amplitude error for the shock response from an impulse caused by a load discontinuity at the end of the impulse can be reliably estimated for a linear elastic system. In fact, the relative amplitude error A_{err} is defined as

$$A_{err} = \frac{A_{ei}}{A_{ii}} \quad , \quad A_{ei} = \frac{1}{2} q(\Delta t) \quad \text{and} \quad A_{ii} = \int_0^{t_d} f(t)dt \qquad (1)$$

where A_{ii} and A_{ei} represent the total area of the input impulse and the area of the extra impulse, respectively. Meanwhile, q is the value of the load discontinuity at the end of the impulse and Δt is the step size. This is manifested from the plot of figure 1.

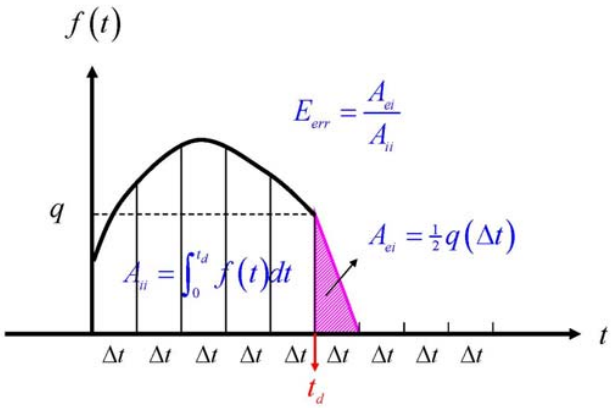

Fig. 1 A load discontinuity and its relative amplitude error

In this figure, the symbol E_{err} is the extra relative amplitude error due to the presence of the extra impulse. It is apparent that the area of the extra impulse is equal to the half of the product of the discontinuity value at the end of the impulse and the size of integration time step. This implies that the extra impulse is linearly proportional to the step size. Consequently, it is natural to use a small time step for a complete step-by-step integration procedure so that the extra impulse can be reduced. However, the time step used in the step-by-step integration for overcoming the difficulty caused by load discontinuity may be very small and is much smaller than that required by general considerations. Thus, the computational efforts significantly increase in a time history analysis.

III. Remedy 1

In structural dynamics, the equation of motion for a system is formulated based on the dynamic equilibrium of force, which includes four types of force and are the elastic spring force, damping force, inertial force and external force. As a result, it can be expressed as

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=f(t) \tag{2}$$

where m is the mass, c is the viscous damping coefficient, k is the stiffness and f is the external force; $u(t)$, $\dot{u}(t)$ and $\ddot{u}(t)$ are the displacement, velocity and acceleration.

It is generally recognized that the maximum displacement response to an impulse mainly depends upon the total amount of the applied impulse, i.e., external momentum, and is almost not affected by the shape of the loading impulse. Hence, it is very promising to obtain the shock response by considering external momentum directly. For this purpose, there is a great motive to construct the governing equation of motion from the dynamic equilibrium of momentum. Similar to the derivation of the force equation of motion, there are four types of momentum and they are the elastic spring momentum, damping momentum, inertial momentum and external momentum. Hence, the momentum equation of motion is found to be

$$m\dot{u}(t)+cu(t)+k\bar{u}(t)=\bar{f}(t) \tag{3}$$

where

$$\bar{u}(t)=\int_0^t u(\tau)d\tau \quad , \quad \bar{f}(t)=\int_0^t f(\tau)d\tau \tag{4}$$

Apparently, $\bar{u}(t)$ is the integral of displacement with respect to time once; and $\bar{f}(t)$ is the integral of force with respect to time once and is the external momentum.

Since the external momentum is used in the formulation of the momentum equation of motion, the difficulty caused by the load discontinuity at the end of an impulse will automatically disappear. The time integration of the external force $f(t)$ is schematically plotted in figure 2.

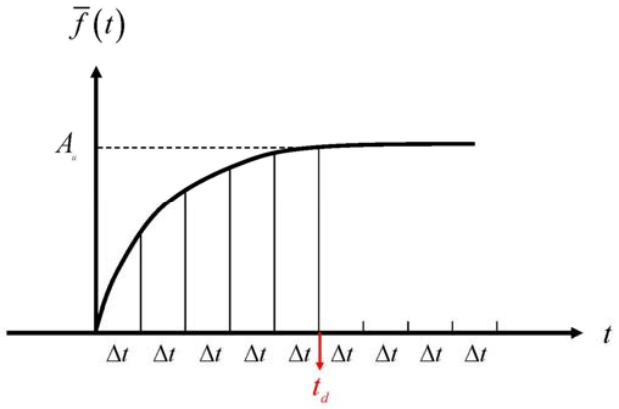

Fig. 2 No load discontinuity in external momentum

This figure reveals that the external momentum will become a constant after the loading duration and there is no discontinuity in the external momentum at the time of t_d. It is clear that the constant external momentum is equal to the total amount of the input impulse A_{ii}. This also implies that a time step larger than the loading duration t_d may still lead to a reliable solution since the total amount of the impulse is inputted into the system in the step-by-step solution of momentum equation of motion.

To demonstrate that the load discontinuity at the end of an impulse will lead to an extra amplitude distortion in the solution of force equation of motion while no extra amplitude distortion is found in the solution of momentum equation of motion, the shock responses from the three different shapes of impulses for a single degree of freedom system are obtained from using the Newmark explicit method to solve the force and the momentum equations of motion. To avoid the difficulty arising from the linearization error of an inelastic system, a linear elastic system is considered. The three different shapes of the impulses are shown in figure 3.

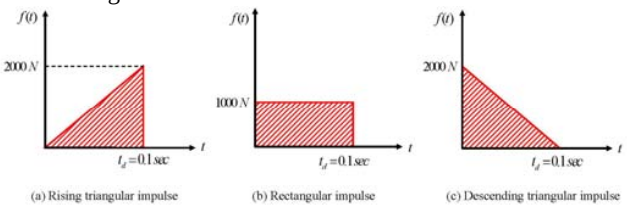

Fig. 3 Three different shapes of linear impulses

The lumped mass and stiffness of the system are simply taken to be $m=10000kg$ and $k=10000\ N/m$. The natural frequency of the system is found to be $1\ rad/sec$. Thus, the period of the system $T=2\pi\ sec$. Since $t_d/T=\frac{1}{20\pi}$ is smaller than 1/4, the three short pulses can be classified as impulses. The numerical results for the shock responses to these three impulses are plotted in figure 4.

The exact solution is obtained from the fundamental theory of structural dynamics for comparison purpose. Since the value of $\Delta t / T = \frac{1}{20\pi}$, which corresponds to the time step of $\Delta t = 0.1$ *sec*, is very small, the period distortion is insignificant based on accuracy consideration.

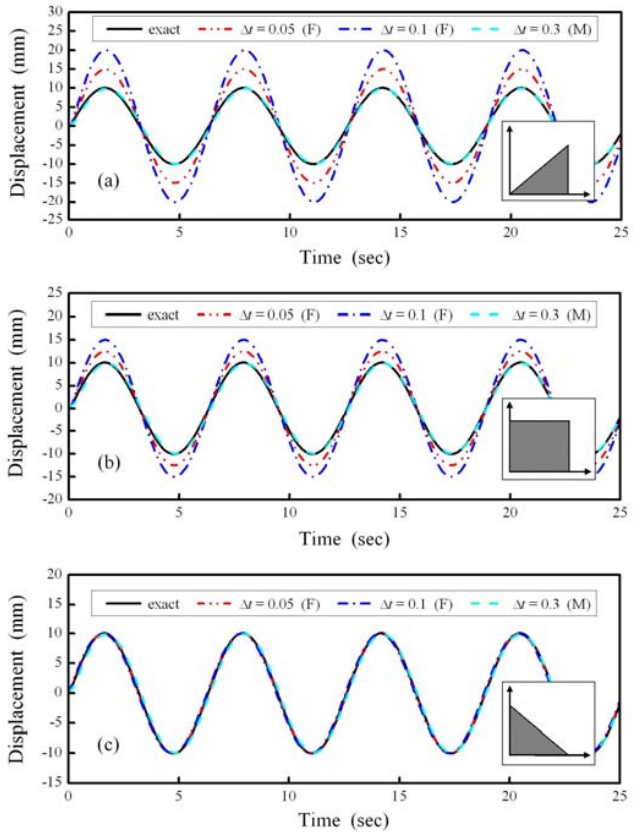

Fig. 4 Shock responses to SDOF system using Remedy 1

In figures (4-a) and (4-b), the shock responses obtained from using the force equation of motion (F) with $\Delta t = 0.05$ and 0.1 *sec* are unreliable while a reliable result can still be yielded by using the momentum equation of motion (M) with the time step as large as $\Delta t = 0.3$ *sec*. The inaccurate solutions obtained from the force equation of motion with $\Delta t = 0.05$ and 0.1 *sec* for the rising triangular and rectangular impulses are due to the load discontinuity at the end of the impulses. It is manifested from figure (4-c) that acceptable solutions can be obtained from the force equation of motion with $\Delta t = 0.05$ and 0.1 *sec*, and from the momentum equation of motion with $\Delta t = 0.3$ *sec*. This is because that there is no load discontinuity at the end of the descending triangular impulse and thus the use of the force equation of motion with $\Delta t = 0.05$ and 0.1 *sec* can provide accurate results. It is very important to find that the use of the momentum equation of motion with $\Delta t = 0.3$ *sec*, which is three times of the loading duration, still gives acceptable results for the three impulses. This implies that the momentum equation of motion can effectively capture the external momentum although the time step used is larger than the loading duration.

IV. Remedy 2

Since the extra impulse arising from the load discontinuity at the end of an impulse is linearly proportional to the step size, it is promising to perform a single small time step immediately upon the termination of the applied impulse while the other time steps can be still conducted by using the step size determined from general considerations. The basic concept of this remedy is illustrated in figure 5.

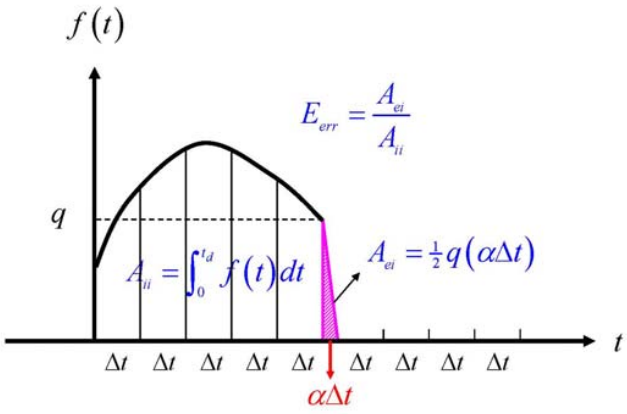

Fig. 5 Single small time step to overcome load discontinuity

In this figure, a time step of Δt is used for the complete step-by-step integration procedure except that the time step right after the end of the impulse is performed by using a small time step of $\alpha\Delta t$, where α might be chosen to be between 0 and 1. Thus, the extra impulse caused by the load discontinuity at the end of the impulse will be reduced from $A_{ei} = \frac{1}{2} q(\Delta t)$ to $A_{ei} = \frac{1}{2} q(\alpha \Delta t)$, and then the relative amplitude error is reduced accordingly. It is apparent that the smaller the value of α is, the less the relative amplitude error is.

To confirm the feasibility of Remedy 2, the problem solved in the section of Remedy 1 is also used herein and the numerical results are displayed in figure 6. In general, the time step of $\Delta t = 0.1$ *sec*, which is equal to the loading duration, is used for the whole step-by-step integration procedure except that a small time step of $\alpha(\Delta t)$, where $\alpha = 1$, 0.1, and 0.01 are considered, is conducted for the time step right after the end of the impulse. In figure (6-a), the amplitude distortion is very significant for the case of using $\alpha = 1$. This case is numerically equivalent to conducting step-by-step integration without using Remedy 2. In general, amplitude distortion is decreased with the decrease of α and a very reliable solution can be obtained from using $\alpha = 0.01$ for the rising triangular impulse. Similar results are also found in figure (6-b) for the rectangular impulse. Unlike the numerical solutions found in figures (6-a) and (6-b), all the shock responses to the descending triangular impulse are almost overlapped together for the cases of using

$\alpha = 1$, 0.1, and 0.01 as shown in figure (6-c). This is because that there is no load discontinuity at the end of the descending triangular impulse and thus it involves no extra amplitude distortion.

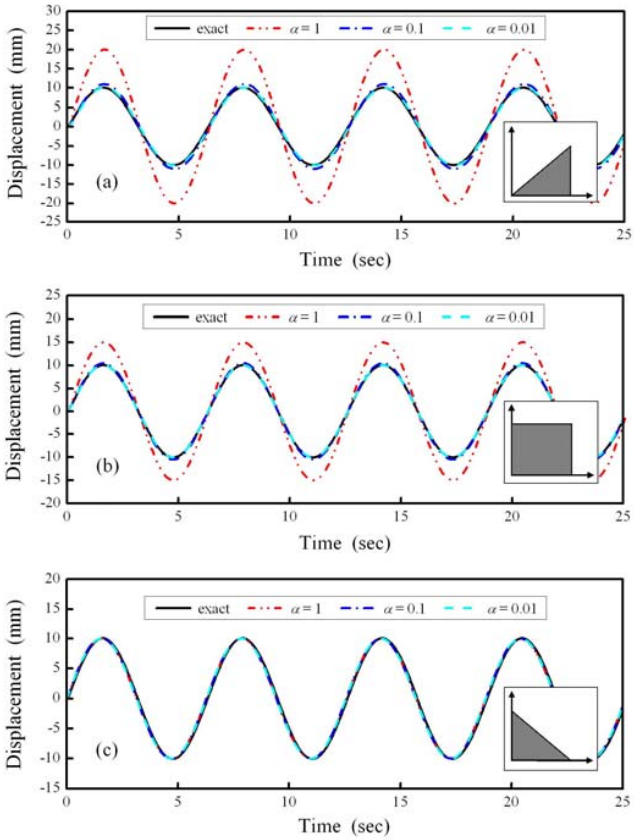

Fig. 6 Shock responses to SDOF system using Remedy 2

Consequently, to overcome the load discontinuity occurred at the end of an impulse it seems appropriate to recommend the choice of $\alpha = 0.01$ or smaller for the single small time step in practice while the other time steps are determined from general considerations. This is because that the single small time step will lead to an insignificant amplitude distortion.

V. Remedy 3

To elucidate the input error caused by the load discontinuity at the end of an impulse, the possible arrangements of the input data for the impulse are schematically sketched in figure 7. The impulse can be reliably captured if the line-connected segments at the integration points are taken to be in the sequence of $\cdots EF\hat{F}G\cdots$. However, it is impossible to input both data of F and $\hat{F}$ at an integration point. In general, the data sequence of $\cdots EFG\cdots$ is often adopted for data input in practice although the sequence of $\cdots E\hat{F}G\cdots$ might also be used. Either using the input sequence of $\cdots EFG\cdots$ or $\cdots E\hat{F}G\cdots$, an input error will be introduced in representing the dynamic loading. In fact, the input sequence of $\cdots EFG\cdots$ will lead to a positive input error, i.e., the area of the triangle $FF\hat{F}G$, while a negative input error, i.e., the area of the triangle $EFF\hat{F}$, is introduced if the input sequence of $\cdots E\hat{F}G\cdots$ is adopted. Apparently, the input error caused by the load discontinuity immediately upon termination of the applied impulse will lead to a displacement response error.

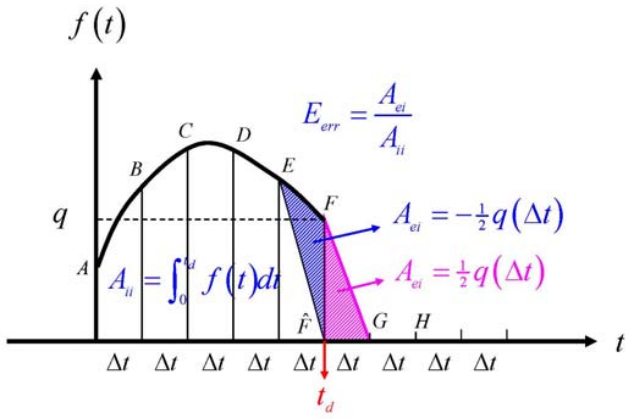

Fig. 7 Input error caused by load discontinuity

Since the maximum shock response to an impulse is almost linearly proportional to the total amount of the impulse, the amplitude distortion can be significantly reduced if the input error at the time instant of discontinuity is largely reduced. Thus, it seems very promising to cautiously adjust the input data at the time instant of the load discontinuity so that there is no input error. In order to have a zero input error at the end of an impulse, the adjusted dynamic loading at the integration point of load discontinuity can be simply taken as the average value of the two discontinuity values of the input data F and $\hat{F}$. This is depicted in figure 8, where the common use of input data F is replaced by the input data of Z.

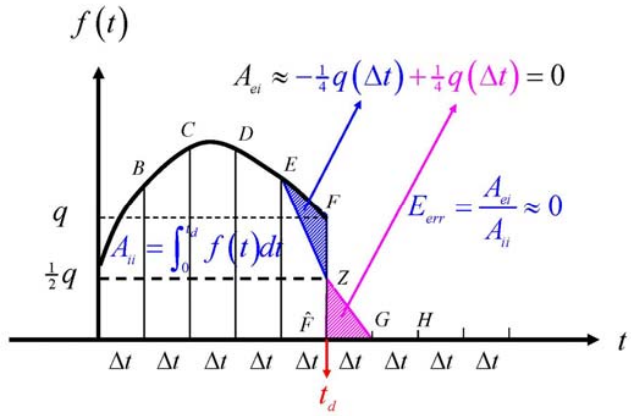

Fig. 8 Adjusting input data to overcome load discontinuity

In this figure, the area of triangle EFZ is almost equal to the area of triangle $Z\hat{F}G$. Hence, there is almost no input error. This is because that the missing input impulse of the area of triangle EFZ is compensated by the extra input impulse of the area of triangle $Z\hat{F}G$. This remedy is simple and involves no extra computational efforts. The input sequences of $\cdots EFG\cdots$ and $\cdots E\hat{F}G\cdots$ are referred to as S1 and S2, respectively, while that of $\cdots EZG\cdots$ is considered as S3.

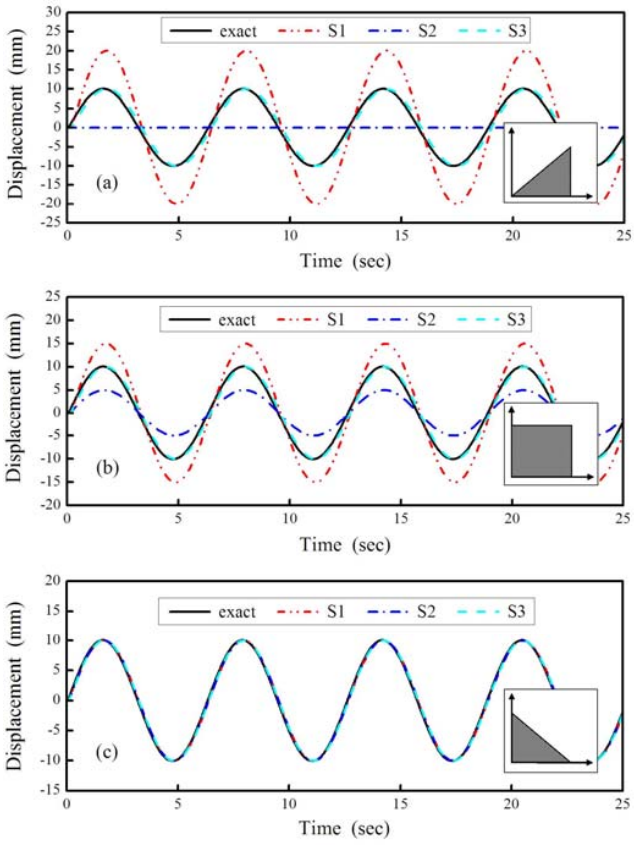

Fig. 9 Shock responses to SDOF system using Remedy 3

The feasibility of Remedy 3 is also verified by the example considered for Remedy 1 and Remedy 2. The problem is solved by the Newmark explicit method with S1, S2 and S3 for using $\Delta t = 0.1\ sec$ and numerical results are plotted in figure 9. It is found in figure (9-a) that the response obtained from using S1 is larger than the theoretical solution. In fact, the amplitude of the theoretical solution is only $10\ mm$ while it is $20\ mm$ for the result obtained from S1. This is because that a positive extra impulse, whose amount is equal to the total amount of the rising triangular impulse, is introduced into the system. Hence, the relative amplitude error is almost equal to 1. On the other hand, the shock response obtained from the use of S2 leads to a zero solution. This is because that a negative extra impulse, whose amount is equal to the total amount of the input rising triangular impulse, is introduced into the system. Hence, a zero solution is achieved. Apparently, this is also manifested from the fact that a zero load is inputted for each time integration point. On the other hand, it is amazing to find that a very accurate solution can be obtained from S3 since it coincides with the theoretical solution. This figure reveals that S1 will lead to an amplitude growth effect and S2 will result in an amplitude decay effect while S3 can have a very reliable solution. These numerical results are highly consistent with the analytical results.

Very similar results are shown in figure (9-b) except that less amplitude distortions are found for using S1 and S2 when compared to those shown in figure (9-a). This is confirmed by the relative amplitude error A_{err} predicted by equation (1). The relative amplitude errors for S1 and S2 for the rising triangular impulse are found to be about 1 and –1 while those for S1 and S2 for the rectangular impulse are found to be 0.5 and –0.5. Again, there is almost no amplitude distortion in figure (9-c) since it involves no load discontinuity.

VI. Comparisons Of Remedies

The previous investigations reveal that all the three remedies can effectively overcome the difficulty caused by the load discontinuity at the end of an impulse without using a very small time step to reduce amplitude distortion. However, it is still of interest to compare and discuss the differences among the three remedies in conducting a step-by-step integration procedure. As a result, the best remedy can be identified and be recommended for practical applications.

Remedy 1 is to use the momentum equations of motion instead of the force equations of motion for time integration. The momentum equations of motion can be easily derived from dynamic equilibrium of momentums or time integration of force equations of motion. Since the external momentum is used instead of the external force in the step-by-step solution of momentum equations of motion, the difficulty arising from the load discontinuity at the end of an impulse will automatically disappear. However, it is needed to involve extra computational efforts in calculating the time integration of restoring force in each time step for a nonlinear system. In addition, it is also needed to calculate the external momentum by integrating the external force with respect to time once before conducting the step-by-step integration. On the other hand, although the load discontinuity problem can be easily overcome by Remedy 2, where a very small, single time step is performed immediately upon the termination of the applied impulse, it will complicate the programming of dynamic analysis codes. This implies that either Remedy 1 or Remedy 2 has its disadvantage in practice. It is apparent that Remedy 3 is very simple in the step-by-step solution of shock response. In fact, its step-by-step integration procedure is the same as that used for general applications. The only change is the loading input data at the time instant of load discontinuity in performing a whole step-by-step integration procedure. The average value of the two discontinuity values at the integration point of load discontinuity is applied to replace the use of one of them for loading input. Since Remedy 3 will not consume extra computational efforts and will not

complicate the dynamic analysis codes, it seems to be the best remedy for practical applications in view of computational aspects.

It is worth noting that although the numerical illustrations presented herein are only for single degree of freedom systems and for linear elastic systems for brevity, all the three remedies are applicable to any multiple degree of freedom systems and any nonlinear systems. Meanwhile, although only the Newmark explicit method is used to conduct the step-by-step integration in this study, it is apparent that all the three remedies are also applicable to the other step-by-step integration methods.

VII. Conclusions

A load discontinuity occurred at the end of an impulse will lead to an extra impulse and then an extra amplitude distortion in the displacement response. In order to effectively overcome this difficulty, three remedies were introduced to compute the shock responses. The concept of each remedy was thoroughly described in this work and a numerical example was used to confirm the feasibility of the three remedies. Remedy 1 is to solve the momentum equation of motion instead of the force equation of motion since the external force is integrated with time once and hence these load discontinuity will automatically disappear. Remedy 2 is proposed to perform a single, small time step immediately upon the termination of the applied impulse to effectively reduce the extra impulse while the other time steps are conducted by using the time step determined from the general considerations. It is also found that the selection of the time step for the single, small time step immediately upon the termination of the applied impulse equal to or smaller than one-hundredth of the other time steps is good enough to obtain an accurate solution. Remedy 3 is to adjust the input data at the integration point of the load discontinuity so that there is no input error.

Since Remedy 3 will not consume any extra computational efforts, which is generally needed for Remedy 1, and will not complicate the dynamic analysis codes, which is inevitable for Remedy 2, it might be the best remedy for practical applications in view of computational aspects.

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