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Lagrange’s Inversion Theorem and Infiltration

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Abstract**—**Implicit equations play a crucial role in Engineering. Based on this importance, several techniques have been applied to solve this particular class of equations. When it comes to practical applications, in general, iterative procedures are taken into account. On the other hand, with the improvement of computers, other numerical methods have been developed to provide a more straightforward methodology of solution. Analytical exact approaches seem to have been continuously neglected due to the difficulty inherent in their application; notwithstanding, they are indispensable to validate numerical routines. Lagrange’s Inversion Theorem is a simple mathematical tool which has proved to be widely applicable to engineering problems. In short, it provides the solution to implicit equations by means of an infinite series. To show the validity of this method, the tree-parameter infiltration equation is, for the first time, analytically and exactly solved. After manipulating these series, closed-form solutions are presented as H-functions.

Keywords**—**Green-Ampt Equation, Lagrange’s Inversion Theorem, Talsma-Parlange Equation, Three-Parameter Infiltration Equation

I. Introduction

NUMERICAL methods have been widely applied to solve all classes of equations present in engineering. Due to the improvement of computers, this methodology has been continuously enhanced, producing results once never thought to be achieved.

This unquestionable effectiveness of numerical routines, sometimes, implies engineers to put aside analytical exact approaches since the application of the latter is generally harder than the former. One must, on the other hand, keep in mind that such routines are validated by means of analytical solutions, making them dependent in some way.

In fact, engineering has evolved to such stage that even Mathematics has not always given an answer to the questions asked. This can be easily seen when highly nonlinear partial differential equations are taken into account. In some other cases, Mathematics has the answers but the application of the methodologies developed is unpractical for engineering purposes.

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At this point, one of the most famous principles which relates causes and effects in natural sciences has to be evoked: the Pareto’s Principle. The principle named after the Italian economist Vilfredo Pareto is also known as the 80-20 rule and states that, for some events, nearly 80% of the consequences come from just 20% of the causes. In social sciences this principle has a straightforward application in which it is easily verified that 20% of the richest people in the world control 82.7% of the world's income.

In exact sciences, such as engineering and Mathematics, this principle is also applicable. It can be expressed as the idea that the majority of the solution process follows by means of simple methods and only a small share is due to complex procedures.

In this sense, simple methods may be applied showing great success in solving hard problems. This is the case of Lagrange’s Inversion Theorem, which is a powerful tool to solve implicit equations and has been known for a long time by mathematicians.

In the present paper, Lagrange’s Inversion Theorem is going to be briefly discussed and applied to analytically and exactly solve the tree-parameter infiltration equation.

Finally, the series obtained by means of Lagrange’s Inversion Theorem are manipulated in order to obtain the solution in a closed-form fashion. This compact representation is in terms of the H-Function, a mathematical special function which has been greatly developed over the last years.

II. Lagrange’s Inversion Theorem

A. Theorem Statement

Let y be defined as the following function of constant χ, function φ, and a parameter δ:

$$y = \chi + \delta \varphi(y) \tag{1}$$

Then any function $\zeta(y)$ is expressed as the following power series in δ [1]:

$$\zeta(y) = \zeta(\chi) + \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \left\{ \frac{d\zeta(x)}{dx} \varphi^n(x) \right\} \Bigg|_{x=\chi} \tag{2}$$

It can be noticed that the right hand side of (2) contains y through δ defined in (1).

It is evident that the convergence issues concerning the series in (2) have to be taken into account for consistency of the solution.

Besides, it is worth noticing that the right hand side of (2) does not depend on y, this way if one takes $\zeta(y) = y$, the once implicit function y is now explicit.

In the presented paper, in order to use Lagrange's Inversion Theorem, a special function is needed. The gamma function is quite popular to scientists, notwithstanding, its definition and some properties are described below.

B. Euler's Gamma Function

The gamma function can be defined by means of the following integral [2]:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \tag{3}$$

The integral above is valid for Re (z) > 0. By means of integration by parts, the following important property is demonstrated:

$$\Gamma(z+1) = z\Gamma(z) \tag{4}$$

III. Application of Lagrange's Inversion Theorem to Infiltration Equations

A. Problem Statement

Theoretical equations for predicting infiltration rate and cumulative infiltration are necessary in situations where infiltration into porous materials is considered. Specially, if one considers one-dimensional vertical infiltration, the development of such equations has a wide background [3]. It has been shown in [4] that infiltration behavior is located between two limiting cases, namely: a soil following Green-Ampt formula [5] and a soil which follows Talsma-Parlange hypothesis [6]. This way, [7] proposed an equation which interpolates between the two limiting cases by means of a transition parameter α. The general equation is expressed in terms of nondimensional cumulative infiltration (I_*) and nondimensional time (t_*) as follows:

$$t_* = I_* + (1-\alpha)^{-1} \ln\left[\frac{\alpha}{1-(1-\alpha)\exp(-\alpha I_*)}\right] \tag{5}$$

The transition parameter α pertains the interval [0,1]. For the lower extreme of the interval, by applying L'Hopital rule to (5) one gets:

$$\begin{aligned} t_* &= I_* \lim_{\alpha\to 0}\left\{(1-\alpha)^{-1}\ln\left[\frac{\alpha}{1-(1-\alpha)\exp(-\alpha I_*)}\right]\right\} \\ &= I_* + \lim_{\alpha\to 0}\left\{\ln\left[\frac{1}{(I_*+1-I_*\alpha)\exp(-\alpha I_*)}\right]\right\} \\ &= I_* + \ln\left[\frac{1}{I_*+1}\right] \end{aligned} \tag{6}$$

which is the exact formulation proposed in [5]. Thus, the lower boundary of the interval corresponds to Green-Ampt case.

On the other hand, when the transition parameter is taken as the highest value possible, i.e, $\alpha = 1$, the following relation can be obtained by means of standard techniques of finding limits:

$$\begin{aligned} t_* &= I_* + \lim_{x\to 1}\left\{\frac{-\ln\left[\dfrac{1-(1-\alpha)\exp(-\alpha I_*)}{\alpha}\right]}{(1-\alpha)}\right\} = \\ &= I_* + \lim_{x\to 1}\left\{\left[\frac{\alpha\left[\exp(-\alpha I_*) - I_*\exp(-\alpha I_*)(\alpha-1)\right]}{\alpha(1-(1-\alpha)\exp(-\alpha I_*))}\right]\right\} - \\ &\quad - \lim_{x\to 1}\left\{\left[\frac{\left[1+(\alpha-1)\exp(-\alpha I_*)\right]}{\alpha(1-(1-\alpha)\exp(-\alpha I_*))}\right]\right\} \\ &= I_* + \exp(-I_*) - 1 \end{aligned} \tag{7}$$

which is the formulation proposed in [6]. It has been shown that the upper boundary of the interval corresponds to Talsma-Parlange case.

Both (6) and (7) have exact explicit solutions by means of Lambert W function [8], which is a special function with great applicability in many branches of science. Notwithstanding, this function fails to provide the solution to (5), impelling engineers to ask if there exists an solution for $\alpha \in (0,1)$. In order to provide the sought solution, one shall apply Lagrange's Inversion Theorem.

B. Problem's Solution

Consider the following alternative representation of (5):

$$T^{\alpha} - hT + a = 0 \tag{8}$$

in which $T = \exp(I_*)$; $h = \alpha\exp(t_*(\alpha - 1))$; and $a = \alpha - 1$. Consider now the variable change $T^{-1} = i$, this way (8) becomes:

$$i^{\alpha-1} = \frac{a i^{\alpha}}{h} + \frac{1}{h} \tag{9}$$

By applying a new variable change $s = i^{\alpha - 1}$, it is easy to get:

$$s = \frac{a s^{\frac{\alpha}{\alpha-1}}}{h} + \frac{1}{h} \tag{10}$$

Note that in (10) s is implicitly defined, this way, by applying Lagrange's Inversion Theorem with $\zeta(s) = s^{1/(1-\alpha)}$, the following series is obtained:

$$T = (h)^{\frac{1}{\alpha-1}} + \sum_{n=1}^{\infty} \frac{[a/h]^n}{n!}\frac{d^{n-1}}{dx^{n-1}}\left\{\frac{1}{1-\alpha} x^{\frac{1}{1-\alpha}-1} x^{\frac{\alpha}{\alpha-1}n}\right\}\Bigg|_{x=1/h} \tag{11}$$

By undoing the variable changes, the value of the nondimensional cumulative infiltration can be explicitly obtained in terms of the nondimensional time as:

$$I_* = t_* + \ln\left[\frac{1}{\alpha^{\frac{1}{1-\alpha}}(1-\alpha)}\right] + \ln\left[\sum_{n=0}^{\infty}\left(\frac{(\alpha-1)}{\alpha^{\frac{\alpha}{\alpha-1}}\exp[t_*\alpha]}\right)^n \frac{\Gamma\left(\frac{\alpha n}{\alpha-1}+\frac{1}{1-\alpha}\right)}{\Gamma\left(\frac{n}{\alpha-1}+\frac{2-\alpha}{1-\alpha}\right)}\frac{1}{n!}\right] \quad (12)$$

The ratio test for the convergence of series in (12) provides $0 < \alpha < 1$. This way, the solution has been obtained by the aid of Lagrange's Inversion Theorem. It is worth noticing that the desired number of terms of the series in (12) will depend on the precision required.

C. Comparison to Known Results

Some other efforts have been made in order to solve (5) analytically. Recently, [3] proposed a curve-fitting solution which is fully applicable to practical situations since the errors inherent in their solution is less than 3%. Reference [9], on the other hand, applied Householder's method to numerically solve (5). The latter approach provided an iterative formula whose first iteration could be set as accurate as desired. The error, in this case, has to be evaluated for each case since an initial guess is required by the methodology. While considering theoretical and refined application to real-world problems, the solution provided here is more effective since if one takes more terms, the solution is at some point virtually exact.

In order to check the consistency of (12), a particular situation when $\alpha = ½$ is going to be explored. In [8] it has been shown that, only for this case, (5) can be manipulated in order to obtain the solution explicitly in terms of elementary functions. The solution itself is presented as:

$$I_* = t_* + 2\ln\left[1+\left(1-\exp\left(-\frac{t_*}{2}\right)\right)^{1/2}\right] \quad (13)$$

By letting $\alpha = ½$ in (12) one easily obtains:

$$I_* = t_* + \ln\left[8\sum_{n=0}^{\infty}\left(\frac{-(1/2)^2}{\exp[t_*/2]}\right)^n \frac{\Gamma(-n+2)}{\Gamma(-2n+3)}\frac{1}{n!}\right] \quad (14)$$

Next we show that (14) is an alternate form to (13). Note that, according to the singularities of the gamma function, the formula above is consistent only for the first two terms of the summation since if n is greater, negative integers appear as gamma functions arguments, which is unacceptable. This kind of concernment arises when one intends to alternate between the classic factorial function and the generalized factorial, represented by gamma function. One must find a way to transform the gamma function ratio in (14) to another one in which negative integers are not the arguments.

It is known that the quotient of gamma functions in (14) can be rewritten as:

$$\frac{\Gamma(-n+2)}{\Gamma(-2n+3)} = (-1)^{n-1}\frac{\Gamma(2n-2)}{\Gamma(n-1)} \quad (15)$$

The conversion presented in (15) is only valid when n is greater or equal to 2, thus, by taking out the first two terms of the summation, from (14) and (15) one obtains:

$$I_* = t_* + \ln\left[4 - 2e^{-t_*/2} - 8\sum_{n=2}^{\infty}\left(\frac{e^{-t_*/2}}{4}\right)^n \frac{\Gamma(2n-2)}{\Gamma(n-1)}\frac{1}{n!}\right] \quad (16)$$

On the other hand, the duplication formula for gamma function provides:

$$\Gamma(2(z-1)) = \frac{\Gamma(z-1)\Gamma\left(z-\frac{1}{2}\right)}{2^{3-2z}\sqrt{\pi}} \quad (17)$$

By successive application of (4) the following relation is obtained:

$$\Gamma\left(n-\frac{1}{2}\right) = -\frac{2(2n)!}{4^n n!(1-2n)}\sqrt{\pi} \quad (18)$$

This way, from (17) and (18) one gets:

$$\frac{\Gamma(2n-2)}{\Gamma(n-1)} = -\frac{(2n)!}{4(1-2n)n!} \quad (19)$$

The combination of (16) and (19) provides:

$$I_* = t_* + \ln\left[4 - 2e^{-\frac{t_*}{2}} + 2\sum_{n=2}^{\infty}\left(e^{-\frac{t_*}{2}}\right)^n \frac{(2n)!}{4^n(1-2n)(n!)^2}\right] \quad (20)$$

Finally, by adding and diminishing the first two terms of the summation inside the logarithm in (20), the following is obtained:

$$I_* = t_* + \ln\left[2 - e^{-\frac{t_*}{2}} + 2\sum_{n=0}^{\infty}\left(e^{-\frac{t_*}{2}}\right)^n \frac{(2n)!}{4^n(1-2n)(n!)^2}\right] \quad (21)$$

At this point one shall consider the following MacLaurin series of the shifted square root function:

$$\sqrt{1+x} = \sum_{n=0}^{\infty}(-x)^n \frac{(2n)!}{4^n(1-2n)}\frac{1}{(n!)^2} \quad (22)$$

It is clear from the comparison of (21) and (22) that:

$$I_* = t_* + \ln\left[2 - \exp\left(-\frac{t_*}{2}\right) + 2\left(1 - \exp\left(-\frac{t_*}{2}\right)\right)^{1/2}\right] \quad (23)$$

By taking the power inside the logarithm in (13), it is easy to see that:

$$\begin{aligned} I_* &= t_* + \ln\left\{\left[1 + \left(1 - \exp\left(-\frac{t_*}{2}\right)\right)^{1/2}\right]^2\right\} \\ &= t_* + \ln\left\{2 - \exp\left(-\frac{t_*}{2}\right) + 2\left(1 - \exp\left(-\frac{t_*}{2}\right)\right)^{1/2}\right\} \end{aligned} \quad (24)$$

This finally proves that the solution obtained by means of Lagrange's Inversion Theorem is, in fact, the correct solution to the problem.

D.Alternative Representation In Terms of H-Function

In order to give the alternative representation, one shall define H-function. The H-function is defined by means of a Mellin–Barnes type integral in the following manner [10]:

$$H_{p,q}^{m,n}\left[\left.\begin{matrix}(a_1, A_1),...,(a_n, A_n),(a_{n+1}, A_{n+1}),...,(a_p, A_p)\\(b_1, B_1),...,(b_m, B_m),(b_{m+1}, B_{m+1}),...,(b_q, B_q)\end{matrix}\right| z\right] = \frac{1}{2\pi i_*}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)} z^s ds \quad (25)$$

where:
(a) $i_* = \sqrt{-1}$;
(b) z ($\neq 0$) is a complex variable;
(c) $z^s = \exp(s(\ln|z| + i_* \arg z))$;
(d) An empty product is interpreted as unity;
(e) *m, n, p* and *q* are non-negative integers satisfying $0 \le n \le p, 0 \le m \le q$ (both *n* and *m* are not zeros);
(f) A_j (j=1,…,p) and B_j (j=1,…,q) are assumed to be positive quantities;
(g) a_j (j=1,…,p) and b_j (j=1,…,q) are complex numbers such that none of the poles of $\Gamma(b_j - B_j s)$ (j = 1,…,m) coincide with the poles of $\Gamma(1 - a_j + A_j s)$ (j = 1,…,n) i.e. $A_k(b_n + \nu) \neq B_h(a_k - \lambda - 1)$ for $\nu, \lambda = 0,1,...$; h = 1,…,m; k=1,…,n;
(h) The contour L runs from $-i_*\infty$ to $+i_*\infty$ such that the poles of $\Gamma(b_j - B_j s)$ (j = 1,…,m) lie to the left of L and the poles of $\Gamma(1 - a_j + A_j s)$ (j = 1,…,n) lie to the right of L.

For convergence issues, in [10] it has been demonstrated that the H function is consistent in the following two cases:
(i) $\delta > 0, z \neq 0$, where

$$\delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \quad (26)$$

(ii) $\delta = 0$ and $0 < |z| < D^{-1}$ where

$$D = \prod_{j=1}^{p} A_j^{A_j} \Big/ \prod_{j=1}^{q} B_j^{B_j} \quad (27)$$

The values of H function does not depend on the choice of L.

The H function can be represented in a computable form, depending on the kinds of the poles of its Gamma functions, in other words:
(*)When the poles of $\prod_{j=1}^{m}\Gamma(b_j - B_j s)$ are simple:

$$H_{p,q}^{m,n}(z) = \sum_{h=1}^{m}\sum_{\nu=0}^{\infty}\left[\frac{\prod_{j=1\neq h}^{m}\Gamma\left(b_j - B_j\frac{b_h+\nu}{B_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1 - b_j + B_j\frac{b_h+\nu}{B_h}\right)} \frac{\prod_{j=1}^{n}\Gamma\left(1 - a_j + A_j\frac{b_h+\nu}{B_h}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j - A_j\frac{b_h+\nu}{B_h}\right)} \frac{(-1)^\nu z^{(b_h+\nu)/B_h}}{\nu! B_h}\right] \quad (28)$$

(**)When the poles of $\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)$ are simple

$$H_{p,q}^{m,n}(z) = \sum_{h=1}^{n}\sum_{\nu=0}^{\infty}\left[\frac{\prod_{j=1\neq h}^{n}\Gamma\left(1 - a_j - A_j\frac{1-a_h+\nu}{A_h}\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j + A_j\frac{1-a_h+\nu}{A_h}\right)} \frac{\prod_{j=1}^{m}\Gamma\left(b_j + B_j\frac{1-a_h+\nu}{A_h}\right)}{\prod_{j=m+1}^{q}\Gamma\left(1 - b_j - B_j\frac{1-a_h+\nu}{A_h}\right)} \frac{(-1)^\nu \left(\frac{1}{z}\right)^{\frac{1-a_h+\nu}{A_h}}}{\nu! A_h}\right] \quad (29)$$

As exemplified in (15), gamma function quotients can be rearranged in order to change the signal of its arguments. This procedure was taken into account while deriving the series representation for α = ½ and must be considered in order to proper manipulate the series in (12). In general, the following relation holds:

$$\frac{\Gamma(-x+n)}{\Gamma(-x)} = (-1)^n\frac{\Gamma(x+1)}{\Gamma(x-n+1)} \quad (30)$$

It is clear that (30) only holds when all gamma function arguments are not negative integers, this way, the series in (12) can be rearranged as:

$$\frac{\Gamma\left(\frac{\alpha n}{\alpha-1}+\frac{1}{1-\alpha}\right)}{\Gamma\left(\frac{n}{\alpha-1}+\frac{2-\alpha}{1-\alpha}\right)}=(-1)^{n-1}\frac{\Gamma\left(\frac{n}{1-\alpha}+\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha n}{1-\alpha}+\frac{\alpha}{\alpha-1}\right)} \tag{31}$$

Note that (31) holds only when n ≥ 2, thus the first two terms of the series in (12) have to be taken out. This way, the latter takes the form:

$$\begin{aligned}&1-\alpha-\frac{(\alpha-1)^2}{\alpha^{\frac{\alpha}{\alpha-1}}\exp[t_*\alpha]}+\lim_{n\to 0}\frac{\Gamma\left(\frac{n}{1-\alpha}+\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha n}{1-\alpha}+\frac{\alpha}{\alpha-1}\right)}\\&-\sum_{n=0}^{\infty}\left(\frac{(1-\alpha)}{\alpha^{\frac{\alpha}{\alpha-1}}\exp[t_*\alpha]}\right)^n\frac{\Gamma\left(\frac{n}{1-\alpha}+\frac{1}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha n}{1-\alpha}+\frac{\alpha}{\alpha-1}\right)}\frac{1}{n!}\end{aligned} \tag{32}$$

By comparing (32) to (28), it is easy to see that the series in (32) corresponds to:

$$-H_{1,2}^{1,1}\left[\frac{(\alpha-1)}{\alpha^{\frac{\alpha}{\alpha-1}}\exp[t_*\alpha]}\middle|\begin{matrix}\left(\frac{\alpha-2}{\alpha-1},\frac{1}{1-\alpha}\right)\\(0,1),\left(\frac{1}{1-\alpha},\frac{\alpha}{1-\alpha}\right)\end{matrix}\right] \tag{33}$$

Finally, (12) turns to:

$$\begin{aligned}I_*=t_*+\ln\Bigg[&\frac{1}{\alpha^{\frac{1}{1-\alpha}}}+\frac{(\alpha-1)}{\alpha\exp[t_*\alpha]}+\frac{\sin\left(\frac{\alpha\pi}{\alpha-1}\right)}{\sin\left(\frac{\pi}{\alpha-1}\right)\alpha^{\frac{1}{1-\alpha}}}\\&+\frac{H_{1,2}^{1,1}\left[\frac{(\alpha-1)}{\alpha^{\frac{\alpha}{\alpha-1}}\exp[t_*\alpha]}\middle|\begin{matrix}\left(\frac{\alpha-2}{\alpha-1},\frac{1}{1-\alpha}\right)\\(0,1),\left(\frac{1}{1-\alpha},\frac{\alpha}{1-\alpha}\right)\end{matrix}\right]}{\alpha^{\frac{1}{1-\alpha}}(\alpha-1)}\Bigg]\end{aligned} \tag{34}$$

One may notice that (34) is the closed-form solution of (12) in terms of H-function.

In order to compare (34) to (13), let us take α = ½ in the former. The following is obtained:

$$I_*=t_*+\ln\left[2-e^{-t_*\alpha}-8H_{1,2}^{1,1}\left[-\frac{e^{-t_*\alpha}}{4}\middle|\begin{matrix}(3,2)\\(0,1),(2,1)\end{matrix}\right]\right] \tag{35}$$

In order to give (35) in terms of elementary function, consider the H-function in (35) given in its integral form (25):

$$H_{1,2}^{1,1}\left[\frac{-e^{-t_*\alpha}}{4}\middle|\begin{matrix}(3,2)\\(0,1),(2,1)\end{matrix}\right]=\int_L\frac{\Gamma(-s)\Gamma(2s-2)e^{-t_*\alpha s}}{2\pi i_*\Gamma(s-1)(-4)^s}ds \tag{36}$$

By means of (17) it is easy to see that:

$$\begin{aligned}H_{1,2}^{1,1}\left[\frac{-e^{-t_*\alpha}}{4}\middle|\begin{matrix}(3,2)\\(0,1),(2,1)\end{matrix}\right]&=\int_L\frac{\Gamma(-s)\Gamma\left(s-\frac{1}{2}\right)e^{-t_*\alpha s}}{2\pi i_*2^{3-2s}\sqrt{\pi}(-4)^s}ds\\&=\int_L\frac{\Gamma(-s)\Gamma\left(s-\frac{1}{2}\right)e^{-t_*\alpha s}}{2\pi i_*8\sqrt{\pi}(-1)^s}ds\\&=\frac{1}{8\sqrt{\pi}}H_{1,1}^{1,1}\left[-e^{-t_*\alpha}\middle|\begin{matrix}(3/2,1)\\(0,1)\end{matrix}\right]\end{aligned} \tag{37}$$

On the other hand, the following relation is given in [10]:

$$H_{1,1}^{1,1}\left[z\middle|\begin{matrix}(1-\vartheta,1)\\(0,1)\end{matrix}\right]=\Gamma(\vartheta)(1+z)^{-\vartheta},|z|<1 \tag{38}$$

In the case of (37), by taking $\vartheta=-1/2$ and noticing that $\Gamma(-1/2)=-2\sqrt{\pi}$, one gets:

$$-\frac{1}{2\sqrt{\pi}}H_{1,1}^{1,1}\left[-e^{-t_*\alpha}\middle|\begin{matrix}(3/2,1)\\(0,1)\end{matrix}\right]=\sqrt{1-e^{-t_*\alpha}} \tag{39}$$

This way, from (35), (37) and (39) one may get:

$$I_*=t_*+\ln\left[2-e^{-t_*\alpha}+2\sqrt{1-e^{-t_*\alpha}}\right] \tag{40}$$

Note that (40) is equal to (13), which confirms the correctness of (34).

E. Evaluation of The Solutions Presented

It is worth noticing that the H-Function is not implemented in computational softwares, such as Maple and Mathematica. On the other hand, when the A_j (j=1,…,p) and B_j (j=1,…,q) are rational numbers, it is possible to convert the H-Function to Meijer G-function, being the latter easily evaluated by the commercial softwares cited.

Sometimes when implementing the solutions in Maple or Mathematica, the gamma function ratio in (12) may destabilize the numerical routine. This drawback, on the other hand, can be overcome by using the Pochhammer notation, which is better evaluated by the softwares.

Let the Pochhammer notation be stated as [10]:

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)} \tag{41}$$

This way, (12) can be rewritten as:

$$I_* = t_* + \ln\left[\frac{1}{\alpha^{\frac{1}{1-\alpha}}(1-\alpha)}\right] + \ln\left[1-\alpha+\sum_{n=1}^{\infty}\left(\frac{(\alpha-1)e^{-t_*\alpha}}{\alpha^{\frac{\alpha}{\alpha-1}}}\right)^n \frac{\left(\frac{n}{\alpha-1}+\frac{2-\alpha}{1-\alpha}\right)_{n-1}}{n!}\right] \tag{42}$$

Computation is much smoother with (42) than with (12) since the softwares internally interpret argument problems and give the correct result.

IV. Conclusions

A simple but powerful method, the Lagrange's Inversion Theorem, has been applied to solve the implicit tree-parameter infiltration equation. The solution provided is in terms of an infinite summation, this way, according to the accuracy desired, one may take as many as needed terms of the series.

The solution reduces to well-known cases, which corroborates to its validity. Besides, for the first time, the explicit exact solution to the tree-parameter infiltration equation can be stated for the whole domain of the interpolation variable α.

Besides the series form, a compact form of the solution is given in terms of H-Function, making the manipulation of the results easier.

Due to the importance of implicit equations in engineering, Lagrange's Inversion Theorem shows its value as a powerful problem solver.

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References

[1] E. T. Whittaker , G. N. Watson, *A Course of Modern Analysis*. 4th Edition, Cambridge Univ. Press, Cambridge, U.K.,1990; pg. 133.

[2] E. Artin, *The Gamma Function*. Rinehart and Winston, NY: Holt, New York, 1964.

[3] P. K. Swamee, P. N. Rathie, L. C. de S. M. Ozelim, "Explict equations for infiltration," *Journal of Hydrology*, 426–427, 2012, pp. 151–153.

[4] R. E. Smith, J. Y. Parlange, "A parameter-efficient hydrologic infiltration model," *Water Resour. Res.*, 14 (3), 1977, pp. 533–538.

[5] W. H. Green, G. A. Ampt, 1911. "Studies in soil physics I: the flow of air and water through soils," *J. Agric. Sci.*, 4 (1), 1911, pp. 1–24.

[6] T. Talsma, J. Y. Parlange, "One-dimensional vertical infiltration," *Aust. J. Soil Res.*, 10 (2), 1972, pp. 143–150.

[7] J. Y. Parlange, I. Lisle, R. D. Braddock, R. E. Smith, "The three parameter infiltration equation," *Soil Sci.*, 133 (6), 1982, pp. 337–341.

[8] J. Y. Parlange, D. A. Barry, R. Haverkamp, "Explicit infiltration equations and the Lambert W-function," *Adv. Water Resour.*, 25 (8–12), 2002, pp. 1119–1124.

[9] L. C. de S. M. Ozelim, A. L. B. Cavalcante, P. K. Swamee, P. N. Rathie, "Métodos Numéricos Iterativos Aplicados a Equações de Infiltração," In: VII Simpósio Brasileiro de Solos Não Saturados, 2011, Pirenópolis. *Anais do VII Simpósio Brasileiro de Solos Não Saturados*. Goiânia, Editora Kelps, v. 1, 2011, p. 209-213.

[10] A. M. Mathai, R. K. Saxena, H. J. Haubold, *The H-Function: Theory and Applications*. Springer, New York, 2010, pp. 2-23.