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Iteration Acceleration for Nonlinear Coupled Parabolic-Hyperbolic System

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Abstract—A Picard-Newton iteration method is studied to accelerate the numerical solution procedure of a class of two-dimensional nonlinear coupled parabolic-hyperbolic system. The Picard-Newton iteration is designed by adding higher-order terms of small quantity to an existing Picard iteration. The discrete functional analysis and inductive hypothesis reasoning techniques are used to overcome difficulties coming from nonlinearity and coupling, and theoretical analysis is made for the convergence and approximation properties of the iteration scheme. The Picard-Newton iteration has a quadratic convergent ratio, and its solution has second order spatial approximation and first order temporal approximation to the exact solution of the original problem. Numerical tests verify the results of the theoretical analysis, and show the Picard-Newton iteration is more efficient than the Picard iteration.

Keywords—nonlinearity, iterative acceleration, coupled parabolic-hyperbolic system, quadratic convergence, numerical analysis.

I. Introduction

Coupled parabolic-hyperbolic system often appears in the study of biological problems, high temperature hydrodynamics and thermo-elasticity, magneto-elasticity problems [1],[2],[3]. Its numerical simulation is of specific importance [2],[4]. Fully implicit nonlinear schemes are desirable for nonlinear coupled problems and applicable for simulating transient problems, since no rigorous stability restriction on temporal steplength is needed for them, while it is needed by explicit or operator splitting schemes. For nonlinear schemes, proper nonlinear iterative algorithms are very important to fulfil fast and accurate resolving [5]. There is much research on the iteration techniques [5],[6],[7], but works on nonlinear iterations for coupled system of different types of equation can be found seldom [8].

The traditional way for solving nonlinear PDE is to discretize the PDE first and get a nonlinear algebraic system which is then linearized to get a linear algebraic system to be solved. It is very difficult to construct Newton linearization for complex practical applications in this way. Another way called LD (linearization-discretization) is suggested in [5] by first linearizing the origianl PDE and then discretizing the derived linear PDE to get linear algebraic system. By using LD approach, it is more convenient to construct new iteration schemes. Specially, Picard-Newton iteration can be built by adding higher-order approximation terms in existing Picard iteration to accelerate the convergence of the latter. Also various discrete iteration schemes can be designed by different discretizations for temporal and spatial operators.

In this paper, iteration acceleration for nonlinear coupled parabolic-hyperbolic system is studied through LD approach. By introducing intermediate variables to diminish the discrete template, and approximating the spatial and temporal operators with second-order and first-order discretization respectively, a Picard-Newton iteration scheme with quadratic convergence ratio is designed to accelerate the Picard iteration (being with linear convergence ratio) in [8]. Main attention is paid on the nonlinear coupling property for the two equations both in the scheme design and numerical analysis procedures. Numerical results are presented, which show the Picard-Newton iteration gives the same accuracy as the Picard iteration, while its computation cost is much less than the latter.

Consider the two-dimensional coupled parabolic-hyperbolic system as follows:

$$\begin{aligned}
&u_t - \nabla\cdot(A(X,t,u,v)\nabla u) \\
&= f(X,t,u,v,u_x,u_y,v_x,v_y), \\
&v_{tt} - \nabla\cdot(B(X,t,u,v)\nabla v) \\
&= g(X,t,u,v,u_x,u_y,v_x,v_y,v_t), && X\in\Omega, t\in J. \\
&u(X,t)=0,\ \ v(X,t)=0, && X\in\partial\Omega, t\in J. \\
&u(X,0)=u_0(X), v(X,0)=v_0(X), \\
&v_t(X,0)=v_{t0}(X), && X\in\Omega. \qquad (1)
\end{aligned}$$

where $u_t=\frac{\partial u}{\partial t}$, $u_x=\frac{\partial u}{\partial x}$, etc. $X=(x,y)$, $\Omega=(0,L_1)\times(0,L_2)$, $J=(0,T]$; A,B,f,g,u_0,v_0,v_{t0} are known functions. Consider the problem with the following assumptions:

(1) There exist positive constants A_*,A^*,B_*,B^*, such that $A_*\le A(X,t,\phi)\le A^*, B_*\le B(X,t,\phi)\le B^*, X\in\bar\Omega, t\in\bar J, \phi\in R^2$.

(2) The partial derivatives A_t,B_t are bounded; A_u,A_v, B_u,B_v are continuous, and their derivatives with respect to x,y,t,u and v are bounded; the derivatives of f (and g) with respect to u,v,u_x,u_y,v_x,v_y (and v_t) are continuous, and their derivatives with respect to u,v,u_x,u_y,v_x,v_y (and v_t) are bounded.

(3) Problem (1) is uniquely solvable, and its solution $u,v\in C^2(\bar\Omega\times\bar J)$.

II. Notations and Preparation work

By introducing a new variant $w=v_t$, system (1) can be rewritten as an equivalent form:

$$\begin{aligned}
&u_t - \nabla\cdot(A(X,t,u,v)\nabla u) \\
&= f(X,t,u,v,u_x,u_y,v_x,v_y), \\
&w_t - \nabla\cdot(B(X,t,u,v)\nabla v)
\end{aligned}$$

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$$
\begin{aligned}
&= g(X,t,u,v,u_x,u_y,v_x,v_y,w), \\
&w = v_t, \qquad X \in \Omega, t \in J. \\
&u(X,t)=0, v(X,t)=0, w(X,t)=0, \quad X \in \partial\Omega, t \in J. \\
&u(X,0)=u_0(X), v(X,0)=v_0(X), \\
&w(X,0)=v_{t0}(X), \qquad X \in \Omega. \qquad (2)
\end{aligned}
$$

We will start with (2) to design the new iteration scheme.

Divide domain $\Omega\times J$ into $J_1\times J_2\times M$ equal small intervals, denote $h_1=\frac{L_1}{J_1}$, $h_2=\frac{L_2}{J_2}$, $\tau=\frac{T}{M}$, and $h=\max\{h_1,h_2\}$, $x_i=ih_1, y_j=jh_2$, $x_{ij}=x_{i,j}=(x_i,y_j)$, $\tau_n=n\tau$. For function ψ, denote $\psi^n=\psi(\tau_n)$ and $d_t\psi^{n+1}=\frac{1}{\tau}(\psi^{n+1}-\psi^n)$. For function ϕ, denote $\phi_{ij}=\phi(x_{ij})$, $\phi_{i+\frac{1}{2},j}=\frac{1}{2}(\phi_{ij}+\phi_{i+1,j})$, $\phi_{i,j+\frac{1}{2}}=\frac{1}{2}(\phi_{ij}+\phi_{i,j+1})$, $\delta_x\phi_{i+\frac{1}{2},j}=\frac{1}{h_1}(\phi_{i+1,j}-\phi_{ij})$, $\delta_y\phi_{i,j+\frac{1}{2}}=\frac{1}{h_2}(\phi_{i,j+1}-\phi_{ij})$, $\partial_x\phi_{ij}=\frac{1}{2h_1}(\phi_{i+1,j}-\phi_{i-1,j})$ and $\partial_y\phi_{ij}=\frac{1}{2h_2}(\phi_{i,j+1}-\phi_{i,j-1})$. For $n\geq 0$, for functions $\phi=U,u$; $\psi=V,v$; $\varphi=W,w$; and $\Phi=A,B$; denote

$$
\begin{aligned}
\Phi^n_{i+\frac{1}{2},j}(\phi,\psi) &= \Phi(x_{i+\frac{1}{2},j},\tau_n,\phi^n_{i+\frac{1}{2},j},\psi^n_{i+\frac{1}{2},j}), \\
\Phi^n_{i,j+\frac{1}{2}}(\phi,\psi) &= \Phi(x_{i,j+\frac{1}{2}},\tau_n,\phi^n_{i,j+\frac{1}{2}},\psi^n_{i,j+\frac{1}{2}}), \\
f^n_{ij}(\phi,\psi) &= f(x_{ij},\tau_n,\phi^n_{ij},\psi^n_{ij},\partial_x\phi^n_{ij},\partial_y\phi^n_{ij}, \\
&\qquad \partial_x\psi^n_{ij},\partial_y\psi^n_{ij}), \\
g^n_{ij}(\phi,\psi,\varphi) &= g(x_{ij},\tau_n,\phi^n_{ij},\psi^n_{ij},\partial_x\phi^n_{ij},\partial_y\phi^n_{ij}, \\
&\qquad \partial_x\psi^n_{ij},\partial_y\psi^n_{ij},\varphi^n_{ij}).
\end{aligned}
$$

Let $U^{n(s)}_{ij}$ stands for the value for U at (x_{ij},τ_n) after the s-th iteration, $U^{n(s)}_{i+\frac{1}{2},j}=\frac{1}{2}[U^{n(s)}_{i+1,j}+U^{n(s)}_{ij}]$, etc. Similarly for functions $\Phi=A,A_u,A_v,B,B_u$, B_v; $\Psi=f,f_u,f_v,f_{u_x}$, f_{u_y},f_{v_x},f_{v_y}; $\Theta=g,g_u,g_v,g_{u_x},g_{u_y},g_{v_x}$, g_{v_y},g_w; denote

$$
\begin{aligned}
\Phi^{n(s)}_{i+\frac{1}{2},j}(U,V) &= \Phi(x_{i+\frac{1}{2},j},\tau_n,U^{n(s)}_{i+\frac{1}{2},j},V^{n(s)}_{i+\frac{1}{2},j}), \\
\Phi^{n(s)}_{i,j+\frac{1}{2}}(U,V) &= \Phi(x_{i,j+\frac{1}{2}},\tau_n,U^{n(s)}_{i,j+\frac{1}{2}},V^{n(s)}_{i,j+\frac{1}{2}}), \\
\Psi^{n(s)}_{ij}(U,V) &= \Psi(x_{ij},\tau_n,U^{n(s)}_{ij},V^{n(s)}_{ij},\partial_xU^{n(s)}_{ij}, \\
&\qquad \partial_yU^{n(s)}_{ij},\partial_xV^{n(s)}_{ij},\partial_yV^{n(s)}_{ij}), \\
\Theta^{n(s)}_{ij}(U,V,W) &= \Theta(x_{ij},\tau_n,U^{n(s)}_{ij},V^{n(s)}_{ij},\partial_xU^{n(s)}_{ij}, \\
&\qquad \partial_yU^{n(s)}_{ij},\partial_xV^{n(s)}_{ij},\partial_yV^{n(s)}_{ij},W^{n(s)}_{ij}).
\end{aligned}
$$

Denote

$$
\begin{aligned}
& L(f'^{\,n(s+1)}(U,V)[\phi,\psi])_{ij} \\
=\ & f^{n(s)}_{u\,ij}[\phi^{n(s+1)}_{ij}-\phi^{n(s)}_{ij}]+f^{n(s)}_{v\,ij}[\psi^{n(s+1)}_{ij}-\psi^{n(s)}_{ij}] \\
&+f^{n(s)}_{u_x\,ij}[\partial_x\phi^{n(s+1)}_{ij}-\partial_x\phi^{n(s)}_{ij}] \\
&+f^{n(s)}_{u_y\,ij}[\partial_y\phi^{n(s+1)}_{ij}-\partial_y\phi^{n(s)}_{ij}] \\
&+f^{n(s)}_{v_x\,ij}[\partial_x\psi^{n(s+1)}_{ij}-\partial_x\psi^{n(s)}_{ij}] \\
&+f^{n(s)}_{v_y\,ij}[\partial_y\psi^{n(s+1)}_{ij}-\partial_y\psi^{n(s)}_{ij}],
\end{aligned}
$$

and

$$
\begin{aligned}
& L(g'^{\,n(s+1)}(U,V,W)[\phi,\psi,\varphi])_{ij} \\
=\ & g^{n(s)}_{u\,ij}[\phi^{n(s+1)}_{ij}-\phi^{n(s)}_{ij}]+g^{n(s)}_{v\,ij}[\psi^{n(s+1)}_{ij}-\psi^{n(s)}_{ij}] \\
&+g^{n(s)}_{u_x\,ij}[\partial_x\phi^{n(s+1)}_{ij}-\partial_x\phi^{n(s)}_{ij}] \\
&+g^{n(s)}_{u_y\,ij}[\partial_y\phi^{n(s+1)}_{ij}-\partial_y\phi^{n(s)}_{ij}] \\
&+g^{n(s)}_{v_x\,ij}[\partial_x\psi^{n(s+1)}_{ij}-\partial_x\psi^{n(s)}_{ij}] \\
&+g^{n(s)}_{v_y\,ij}[\partial_y\psi^{n(s+1)}_{ij}-\partial_y\psi^{n(s)}_{ij}]+g^{n(s)}_{w\,ij}[\varphi^{n(s+1)}_{ij}-\varphi^{n(s)}_{ij}],
\end{aligned}
$$

where $f^{n\,(s)}_{u\,ij}=(\frac{\partial f}{\partial u})^{n\,(s)}_{ij}(U,V)$, $g^{n\,(s)}_{u\,ij}=(\frac{\partial g}{\partial u})^{n\,(s)}_{ij}(U,V,W)$, etc. Denote

$$
\begin{aligned}
&\delta(\Theta^n(\phi,\psi)\delta\Phi^n)_{ij} \\
&= \frac{1}{h_1}[\Theta^n_{i+\frac{1}{2},j}(\phi,\psi)\delta_x\Phi^n_{i+\frac{1}{2},j}-\Theta^n_{i-\frac{1}{2},j}(\phi,\psi)\delta_x\Phi^n_{i-\frac{1}{2},j}] \\
&+\frac{1}{h_2}[\Theta^n_{i,j+\frac{1}{2}}(\phi,\psi)\delta_y\Phi^n_{i,j+\frac{1}{2}}-\Theta^n_{i,j-\frac{1}{2}}(\phi,\psi)\delta_y\Phi^n_{i,j-\frac{1}{2}}].
\end{aligned}
$$

Similar notations with superscripts $n(s)$ instead of n have analogous meanings.

Define the following discrete spatial norms:

$$
\begin{aligned}
\|\phi\| &= (\sum_{i=1}^{J_1-1}\sum_{j=1}^{J_2-1}|\phi_{ij}|^2h_1h_2)^{\frac{1}{2}}, \\
\|\delta\phi\| &= (\sum_{i=0}^{J_1-1}\sum_{j=1}^{J_2-1}|\delta_x\phi_{i+\frac{1}{2},j}|^2h_1h_2 \\
&\quad +\sum_{i=1}^{J_1-1}\sum_{j=0}^{J_2-1}|\delta_y\phi_{i,j+\frac{1}{2}}|^2h_1h_2)^{\frac{1}{2}}.
\end{aligned}
$$

A nonlinear fully implicit scheme for (1) is given in [8] to find U^{n+1}_{ij}, V^{n+1}_{ij}, W^{n+1}_{ij} such that

$$
\begin{aligned}
&d_tU^{n+1}_{ij}-\delta(A^{n+1}(U,V)\delta U^{n+1})_{ij}=f^{n+1}_{ij}(U,V), \\
&d_tW^{n+1}_{ij}-\delta(B^{n+1}(U,V)\delta V^{n+1})_{ij}=g^{n+1}_{ij}(U,V,W), \\
&d_tV^{n+1}_{ij}=W^{n+1}_{ij}, \\
&\quad i=1,2,\cdots,J_1-1;\ j=1,2,\cdots,J_2-1; \\
&U^{n+1}_{ij}=V^{n+1}_{ij}=W^{n+1}_{ij}=0, \\
&\quad i=0\ or\ J_1;\ j=0,1,\cdots,J_2; \\
&or\ \ i=0,1,\cdots,J_1;\ j=0\ or\ J_2;\ \ n=0,1,\cdots,M-1; \\
&U^0_{ij}=u_0(x_{ij}),\ V^0_{ij}=v_0(x_{ij}),\ W^0_{ij}=v_{t0}(x_{ij}), \\
&\quad i=0,1,\cdots,J_1;\ j=0,1,\cdots,J_2. \qquad (3)
\end{aligned}
$$

Denote $u^n_{ij}=u(x_{ij},\tau_n)$, $v^n_{ij}=v(x_{ij},\tau_n)$ and $w^n_{ij}=w(x_{ij},\tau_n)$. The truncation error for the exact solution of (1) in the fully implicit discretization is:

$$
\begin{aligned}
-R^{n+1}_{1ij} &=: d_tu^{n+1}_{ij}-\delta(A^{n+1}(u,v)\delta u^{n+1})_{ij}-f^{n+1}_{ij}(u,v) \\
&= O(h^2+\tau), \\
-R^{n+1}_{2ij} &=: d_tw^{n+1}_{ij}-\delta(B^{n+1}(u,v)\delta v^{n+1})_{ij} \\
&\quad -g^{n+1}_{ij}(u,v,w) \\
&= O(h^2+\tau), \\
-R^{n+1}_{3ij} &=: d_tv^{n+1}_{ij}-w^{n+1}_{ij}=O(\tau), \\
&\quad i=1,2,\cdots,J_1-1;\ j=1,2,\cdots,J_2-1. \qquad (4)
\end{aligned}
$$

Denote $\xi^n_{ij}=U^n_{ij}-u^n_{ij}$, $\zeta^n_{ij}=V^n_{ij}-v^n_{ij}$, $\eta^n_{ij}=W^n_{ij}-w^n_{ij}$, then $\xi^0_{ij}=\zeta^0_{ij}=\eta^0_{ij}=0$, and there is [8]

Lemma 1 The nonlinear fully discrete scheme (3) is unconditionally stable, and has the following approximation property.

$$
\|\xi^N\|+\|\zeta^N\|+\|\eta^N\|+\|\delta\xi^N\|+\|\delta\zeta^N\|
$$

$$+\tau(\sum_{n=0}^{N-1}\|d_t\eta^{n+1}\|^2)^{\frac{1}{2}}+\tau(\sum_{n=0}^{N-1}\|d_t\delta\xi^{n+1}\|^2)^{\frac{1}{2}}$$
$$+\tau(\sum_{n=0}^{N-1}\|d_t\delta\zeta^{n+1}\|^2)^{\frac{1}{2}}=O(h^2+\tau),$$

where $N\geq 1$.

III. Picard-Newton iteration Scheme

In [8], a simple Picard iteration with linear convergent ratio is proposed to solve (3). Here, to accelerate the resolving procedure, by using LD approach, a Picard-Newton iteration scheme is given by finding $U_{ij}^{n+1\,(s+1)}$, $V_{ij}^{n+1\,(s+1)}$, $W_{ij}^{n+1\,(s+1)}$ such that

$$\begin{aligned}
&\frac{U_{ij}^{n+1\,(s+1)}-U_{ij}^n}{\tau}-\delta(A^{n+1\,(s)}(U,V)\delta U^{n+1\,(s+1)})_{ij}\\
&-\theta\delta(\{A'^{\,n+1\,(s)}_u(U,V)[U^{n+1\,(s+1)}-U^{n+1\,(s)}]\\
&+A'^{\,n+1\,(s)}_v(U,V)[V^{n+1\,(s+1)}-V^{n+1\,(s)}]\}\delta U^{n+1\,(s)})_{ij}\\
&=f_{ij}^{n+1\,(s)}(U,V)+\theta L(f'^{\,n+1\,(s+1)}(U,V)[U,V])_{ij}, \quad (5)\\
&\frac{W_{ij}^{n+1\,(s+1)}-W_{ij}^n}{\tau}-\delta(B^{n+1\,(s)}(U,V)\delta V^{n+1\,(s+1)})_{ij}\\
&-\theta\delta(\{B'^{\,n+1\,(s)}_u(U,V)[U^{n+1\,(s+1)}-U^{n+1\,(s)}]\\
&+B'^{\,n+1\,(s)}_v(U,V)[V^{n+1\,(s+1)}-V^{n+1\,(s)}]\}\delta V^{n+1\,(s)})_{ij}\\
&=g_{ij}^{n+1\,(s)}(U,V,W)\\
&+\theta L(g'^{\,n+1\,(s+1)}(U,V,W)[U,V,W])_{ij}, \quad (6)\\
&\frac{V_{ij}^{n+1\,(s+1)}-V_{ij}^n}{\tau}=W_{ij}^{n+1\,(s+1)},\\
&\ i=1,2,\cdots,J_1-1;\ j=1,2,\cdots,J_2-1; \quad (7)\\
&U_{ij}^{n+1\,(s+1)}=V_{ij}^{n+1\,(s+1)}=V_{ij}^{n+1\,(s+1)}=0,\\
&\quad i=0\,or\,J_1;\ j=0,1,\cdots,J_2;\\
&\ or\ \ i=0,1,\cdots,J_1;\ j=0\,or\,J_2;\quad s=0,1,2\cdots; \quad (8)\\
&U_{ij}^{n+1\,(0)}=U_{ij}^n,\ V_{ij}^{n+1\,(0)}=V_{ij}^n,\ W_{ij}^{n+1\,(0)}=W_{ij}^n,\\
&\ i=0,1,\cdots,J_1;\ j=0,1,\cdots,J_2;\\
&\quad n=0,1,\cdots,M-1, \quad (9)
\end{aligned}$$

where $\theta=1$. If $\theta=0$, then the system (5)-(9) is the original Picard iteration.

For each time step from $\tau_n\rightarrow\tau_{n+1}$, the calculation proceeds as follows:

Step 1. Give initial values with (9), $U_{ij}^n,V_{ij}^n,W_{ij}^n\longrightarrow U_{ij}^{n+1\,(0)}$, $V_{ij}^{n+1\,(0)},W_{ij}^{n+1\,(0)}$.

Step 2. Execute iteration from s to $s+1$ $(s=0,1,2,\cdots)$ with (5)-(8), where

- 2.1 replace $W_{ij}^{n+1\,(s+1)}$ in (6) with (7),
- 2.2 with (5), (6), (8), $U_{ij}^n,V_{ij}^n,W_{ij}^n,U_{ij}^{n+1\,(s)}$, $V_{ij}^{n+1\,(s)}$, $W_{ij}^{n+1\,(s)}\longrightarrow U_{ij}^{n+1\,(s+1)},V_{ij}^{n+1\,(s+1)}$,
- 2.3 with (7), $V_{ij}^n,V_{ij}^{n+1\,(s+1)}\longrightarrow W_{ij}^{n+1\,(s+1)}$.

Step 3. Check for convergence – if the control tolerance satisfies, then $U_{ij}^{n+1\,(s+1)},V_{ij}^{n+1\,(s+1)},W_{ij}^{n+1\,(s+1)}\longrightarrow U_{ij}^{n+1}$, V_{ij}^{n+1}, W_{ij}^{n+1}, exit; otherwise, $s\leftarrow s+1$ and go to Step 2.

IV. Error Estimate for Iteration Scheme

Denote $\alpha_{ij}^{n\,(s)}=U_{ij}^{n\,(s)}-u_{ij}^n$, $\beta_{ij}^{n\,(s)}=V_{ij}^{n\,(s)}-v_{ij}^n$ and $\gamma_{ij}^{n\,(s)}=W_{ij}^{n\,(s)}-w_{ij}^n$, one has

Theorem 1 The solution of the Picard-Newton iteration scheme (5)-(9) has first order temporal and second order L^2 and H^1 norm spatial approximation to the exact solution of problem (1), and such approximation is uniform in s, i.e.,

$$\begin{aligned}
&\|\alpha^{n+1\,(s+1)}\|+\|\beta^{n+1\,(s+1)}\|+\|\gamma^{n+1\,(s+1)}\|\\
&+\|\delta\alpha^{n+1\,(s+1)}\|+\|\delta\beta^{n+1\,(s+1)}\|=O(h^2+\tau).
\end{aligned}$$

Proof: Denote

$$Y_{1ij}^n=-\tau d_tu_{ij}^n,Y_{2ij}^n=-\tau d_tv_{ij}^n,Y_{3ij}^n=-\tau d_tw_{ij}^n.$$

Subtracting (4) from (5)-(9), one has the following error equation.

$$\begin{aligned}
&\frac{\alpha_{ij}^{n+1\,(s+1)}-\xi_{ij}^n}{\tau}-\delta(A^{n+1\,(s)}(U,V)\delta\alpha^{n+1\,(s+1)})_{ij}\\
=\ &\delta([A^{n+1\,(s)}(U,V)-A^{n+1}(u,v)]\delta u^{n+1})_{ij}\\
&+\delta(\{[A'^{\,n+1\,(s)}_u(U,V)[\alpha^{n+1\,(s+1)}-\alpha^{n+1\,(s)}]\\
&+A'^{\,n+1\,(s)}_v(U,V)[\beta^{n+1\,(s+1)}-\beta^{n+1\,(s)}]\}\\
&\delta[\alpha^{n+1\,(s)}+u^{n+1}])_{ij}\\
&+[f_{ij}^{n+1\,(s)}(U,V)-f_{ij}^{n+1}(u,v)]\\
&+L(f'^{\,n+1\,(s+1)}(U,V)[\alpha,\beta])_{ij}+R_{1ij}^{n+1}, \quad (10)\\
&\frac{\gamma_{ij}^{n+1\,(s+1)}-\eta_{ij}^n}{\tau}-\delta(B^{n+1\,(s)}(U,V)\delta\beta^{n+1\,(s+1)})_{ij}\\
=\ &\delta([B^{n+1\,(s)}(U,V)-B^{n+1}(u,v)]\delta v^{n+1})_{ij}\\
&+\delta(\{B'^{\,n+1\,(s)}_u(U,V)[\alpha^{n+1\,(s+1)}-\alpha^{n+1\,(s)}]\\
&+B'^{\,n+1\,(s)}_v(U,V)[\beta^{n+1\,(s+1)}-\beta^{n+1\,(s)}]\}\\
&\delta[\beta^{n+1\,(s)}+v^{n+1}])_{ij}\\
&+[g_{ij}^{n+1\,(s)}(U,V,W)-g_{ij}^{n+1}(u,v,w)]\\
&+L(g'^{\,n+1\,(s+1)}(U,V,W)[\alpha,\beta,\gamma])_{ij}+R_{2ij}^{n+1}, \quad (11)\\
&\frac{\beta_{ij}^{n+1\,(s+1)}-\zeta_{ij}^n}{\tau}=\gamma_{ij}^{n+1\,(s+1)}+R_{3ij}^{n+1},\\
&\ i=1,2,\cdots,J_1-1;\ j=1,2,\cdots,J_2-1; \quad (12)\\
&\alpha_{ij}^{n+1\,(s+1)}=\beta_{ij}^{n+1\,(s+1)}=\gamma_{ij}^{n+1\,(s+1)}=0,\\
&\quad i=0\,or\,J_1;\ j=0,1,\cdots,J_2;\\
&or\,i=0,1,\cdots,J_1;\ j=0\,or\,J_2;\ s=0,1,2,\cdots; \quad (13)\\
&\alpha_{ij}^{n+1\,(0)}=\xi_{ij}^n+Y_{1ij}^{n+1},\ \beta_{ij}^{n+1\,(0)}=\zeta_{ij}^n+Y_{2ij}^{n+1},\\
&\gamma_{ij}^{n+1\,(0)}=\eta_{ij}^n+Y_{3ij}^{n+1},\\
&\ i=0,1,\cdots,J_1;j=0,1\cdots,J_2;\\
&\ n=0,1,\cdots,M-1. \quad (14)
\end{aligned}$$

Multiplying formulas (10) and (11) with $\frac{\alpha_{ij}^{n+1\,(s+1)}-\xi_{ij}^n}{\tau}h_1h_2\tau$ and $\gamma_{ij}^{n+1\,(s+1)}h_1h_2\tau$ respectively, summing for $i=1,2,\cdots,J_1-1$ and $j=1,2,\cdots,J_2-1$, after a complex derivation procedure, one gets

$$\begin{aligned}
&\|\frac{\alpha^{(s+1)}-\xi^n}{\tau}\|^2\tau+\|\alpha^{(s+1)}\|^2+\|\beta^{(s+1)}\|^2\\
&+\|\gamma^{(s+1)}\|^2+\|\delta\alpha^{(s+1)}\|^2+\|\delta\beta^{(s+1)}\|^2\\
\leq\ &\bar{K}_1[\|\alpha^{(s)}\|_\infty+\|\beta^{(s)}\|_\infty+\|\delta\alpha^{(s)}\|_\infty+\|\delta\beta^{(s)}\|_\infty]
\end{aligned}$$

$$[\|\alpha^{(s+1)}\|^2 + \|\beta^{(s+1)}\|^2 + \|\delta\alpha^{(s+1)}\|^2 + \|\delta\beta^{(s+1)}\|^2]$$
$$+\bar{K}_2[\|\alpha^{(s)}\|_\infty + \|\beta^{(s)}\|_\infty + \|\delta\alpha^{(s)}\|_\infty + \|\delta\beta^{(s)}\|_\infty$$
$$+\|\alpha^{(s)}\|_\infty^2 + \|\beta^{(s)}\|_\infty^2 + \|\gamma^{(s)}\|_\infty^2 + \|\delta\alpha^{(s)}\|_\infty^2$$
$$+\|\delta\beta^{(s)}\|_\infty^2][\|\alpha^{(s)}\|^2 + \|\beta^{(s)}\|^2 + \|\gamma^{(s)}\|^2 + \|\delta\alpha^{(s)}\|^2$$
$$+\|\delta\beta^{(s)}\|^2 + \|\delta\xi^n\|^2 + \|\delta\zeta^n\|^2 + \|\delta R_3^{n+1}\|^2\tau^2]$$
$$+\bar{K}_3(\|\xi^n\|^2 + \|\zeta^n\|^2 + \|\eta^n\|^2 + \|\delta\xi^n\|^2 + \|\delta\zeta^n\|^2$$
$$+\|R_1^{n+1}\|^2\tau + \|R_2^{n+1}\|^2\tau + \|R_3^{n+1}\|^2\tau^2$$
$$+\|\delta R_3^{n+1}\|^2\tau^2),$$

where $\phi^{(s+1)}$ is the abbreviation for $\phi^{n+1\,(s+1)}$. Then by using Lemma 1 and inductive hypothesis reasoning, Theorem 1 is proved.

V. Convergence Rate for Iteration Scheme

Now consider the convergence property of the iterative scheme. Denote $\xi_{ij}^{n\,(s)} = U_{ij}^{n\,(s)} - U_{ij}^n$, $\zeta_{ij}^{n\,(s)} = V_{ij}^{n\,(s)} - V_{ij}^n$ and $\eta_{ij}^{n\,(s)} = W_{ij}^{n\,(s)} - W_{ij}^n$.

Theorem 2 The solution of the Picard-Newton iteration (5)-(9) converges to the solution of the nonlinear fully implicit scheme (3) in L^2 and H^1 norm

$$\lim_{s\to\infty}[\frac{1}{\sqrt{\tau}}\|\xi^{n+1\,(s)}\| + \frac{1}{\tau}\|\zeta^{n+1\,(s)}\| + \|\eta^{n+1\,(s)}\|$$
$$+\|\delta\xi^{n+1\,(s)}\| + \|\delta\zeta^{n+1\,(s)}\|] = 0,$$

and the convergence rate is quadratic, i.e., there exists a positive constant C independent of h and τ such that

$$\limsup_{s\to\infty}\frac{\|\eta^{n+1\,(s+1)}\| + \|\delta\xi^{n+1\,(s+1)}\| + \|\delta\zeta^{n+1\,(s+1)}\|}{\|\eta^{n+1\,(s)}\|^2 + \|\delta\xi^{n+1\,(s)}\|^2 + \|\delta\zeta^{n+1\,(s)}\|^2} \le C.$$

Proof: Subtracting (3) from (5)-(9), one has the following relation.

$$\begin{aligned}
&\frac{\xi_{ij}^{n+1\,(s+1)}}{\tau} - \delta(A^{n+1\,(s)}(U,V)\delta\xi^{n+1\,(s+1)})_{ij}\\
=\;& \delta([A^{n+1\,(s)}(U,V) - A^{n+1}(U,V)]\delta U^{n+1})_{ij}\\
&+\delta(\{A_u'^{\,n+1\,(s)}(U,V)[\xi^{n+1\,(s+1)} - \xi^{n+1\,(s)}]\\
&+A_v'^{\,n+1\,(s)}(U,V)[\zeta^{n+1\,(s+1)} - \zeta^{n+1\,(s)}]\}\delta U^{n+1\,(s)})_{ij}\\
&+[f_{ij}^{n+1\,(s+1)}(U,V) - f_{ij}^{n+1}(U,V)]\\
&+L(f'^{\,n+1\,(s+1)}(U,V)[\xi,\zeta])_{ij}, \qquad (15)\\
&\frac{\eta_{ij}^{n+1\,(s+1)}}{\tau} - \delta(B^{n+1\,(s)}(U,V)\delta\zeta^{n+1\,(s+1)})_{ij}\\
=\;& \delta([B^{n+1\,(s)}(U,V) - B^{n+1}(U,V)]\delta V^{n+1})_{ij}\\
&+\delta(\{B_u'^{\,n+1\,(s)}(U,V)[\xi^{n+1\,(s+1)} - \xi^{n+1\,(s)}]\\
&+B_v'^{\,n+1\,(s)}(U,V)[\zeta^{n+1\,(s+1)} - \zeta^{n+1\,(s)}]\}\delta V^{n+1\,(s)})_{ij}\\
&+[g_{ij}^{n+1\,(s+1)}(U,V,W) - g_{ij}^{n+1}(U,V,W)]\\
&+L(g'^{\,n+1\,(s+1)}(U,V,W)[\xi,\zeta,\eta])_{ij}, \qquad (16)\\
&\frac{\zeta_{ij}^{n+1\,(s+1)}}{\tau} = \eta_{ij}^{n+1\,(s+1)},\\
&\;i = 1,2,\cdots,J_1-1;\ j = 1,2,\cdots,J_2-1; \qquad (17)\\
&\xi_{ij}^{n+1\,(s+1)} = \zeta_{ij}^{n+1\,(s+1)} = \eta_{ij}^{n+1\,(s+1)} = 0,\\
&\quad i = 0\,or\,J_1;\ j = 0,1,\cdots,J_2;\\
&or\,i = 0,1,\cdots,J_1;\ j = 0\,or\,J_2;\ s = 0,1,2,\cdots; \qquad (18)
\end{aligned}$$

$$\begin{aligned}
&\xi_{ij}^{n+1\,(0)} = -\tau d_t U_{ij}^{n+1},\ \zeta_{ij}^{n+1\,(0)} = -\tau d_t V_{ij}^{n+1},\\
&\eta_{ij}^{n+1\,(0)} = -\tau d_t W_{ij}^{n+1},\\
&\;i = 0,1,\cdots,J_1; j = 0,1\cdots,J_2;\\
&\;n = 0,1,\cdots,M-1. \qquad (19)
\end{aligned}$$

Multiply (15) and (16) with $\xi_{ij}^{n+1\,(s+1)}h_1h_2$ and $\zeta_{ij}^{n+1\,(s+1)}$ h_1h_2 respectively, and sum up the products over $1 \le i \le J_1-1$ and $1 \le j \le J_2-1$. By using discrete inverse inequality and Lemma 1, after a long deduction procedure, one can obtain

$$\begin{aligned}
&\frac{1}{\tau}\|\xi^{(s+1)}\|^2 + \|\eta^{(s+1)}\|^2 + \|\delta\xi^{(s+1)}\|^2 + \|\delta\zeta^{(s+1)}\|^2\\
\le\;& K[\|\eta^{(s)}\|^4 + \|\delta\xi^{(s)}\|^4 + \|\delta\zeta^{(s)}\|^4]\\
&[1 + \|\xi^{(s+1)}\|^2 + \|\zeta^{(s+1)}\|^2].
\end{aligned}$$

Noticing that with Lemma 1, one has $\|\eta^{(0)}\| + \|\delta\xi^{(0)}\| + \|\delta\zeta^{(0)}\| = O(h^2+\tau)$. Hence, by inductive hypothesis reasoning, Theorem 2 is proved.

VI. Numerical Experiments

In this section, some numerical experiments are presented to demonstrate the good accuracy and high efficiency of the Picard-Newton iteration. Consider the nonlinear coupled system (1) in $\Omega\times J = (0,1)\times(0,1)\times(0,2]$ with the following coefficients and functions:

$$\begin{aligned}
&A(x,y,t,u,v)\\
=\;& 0.4\sin[0.5 + e^{-t}\sin(\pi x)\sin(\pi y) + u - 2.0v] + 0.5,\\
&B(x,y,t,u,v)\\
=\;& 0.4\sin[0.5 + e^{-t}\sin(\pi x)\sin(\pi y) - 2.0u + v] + 0.5,\\
&f(x,y,t,u,v,u_x,u_y,v_x,v_y)\\
=\;& 0.5\pi^2(0.5+e^{-t})\sin(\pi x)\sin(\pi y) + 0.5\pi^2 u - v\\
&+0.5\sin(\pi x)\sin(\pi y) + \sin(\pi x)\cos(\pi y)(u_x + v_x)\\
&-\cos(\pi x)\sin(\pi y)(u_y + v_y),\\
&g(x,y,t,u,v,u_x,u_y,v_x,v_y,v_t)\\
=\;& 0.5\pi^2(0.5+e^{-t})\sin(\pi x)\sin(\pi y) + u + 0.5\pi^2 v\\
&-(0.5+e^{-t})\sin(\pi x)\sin(\pi y)\\
&-\sin(\pi x)\cos(\pi y)(u_x - v_x)\\
&+\cos(\pi x)\sin(\pi y)(u_y - v_y) - v_t.
\end{aligned}$$

The boundary conditions and initial values are as follows:

$$\begin{aligned}
u(x,y,t) = v(x,y,t) &= 0, && (x,y)\in\partial\Omega, t\in J,\\
u(x,y,0) &= 1.5\sin(\pi x)\sin(\pi y), && (x,y)\in\Omega,\\
v(x,y,0) &= 1.5\sin(\pi x)\sin(\pi y), && (x,y)\in\Omega,\\
v_t(x,y,0) &= -\sin(\pi x)\sin(\pi y), && (x,y)\in\Omega.
\end{aligned}$$

The exact solution of this system can by expressed as:

$$u(x,y,t) = (0.5+e^{-t})\sin(\pi x)\sin(\pi y),$$
$$v(x,y,t) = (0.5+e^{-t})\sin(\pi x)\sin(\pi y).$$

Four groups of spatial and temporary step lengths are used in the tests, which are $J_1\times J_2\times M = 6\times6\times18$, $12\times12\times72$, $24\times24\times288$, $48\times48\times1152$,; hence their corresponding expected error bounds $h_1^2+h_2^2+\tau$ are $1.6667e-1$, $4.1667e-$

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TABLE I
APPROXIMATION ERRORS OF THE PICARD-NEWTON ITERATION

	6×6	12×12	24×24	48×48	ord
ue	2.1318e-2	4.9865e-3	1.2749e-3	4.2145e-5	2.99
ve	2.6487e-2	6.0132e-3	1.5757e-3	8.7774e-5	2.75
uhe	9.4234e-2	2.2218e-2	5.6975e-3	2.8454e-4	2.79
vhe	1.1640e-1	2.6665e-2	7.0010e-3	4.0948e-4	2.72
we	2.4300e-2	7.6719e-3	2.1520e-3	7.6021e-4	1.67
uem	2.8081e-2	7.2240e-3	1.8807e-3	1.9413e-4	2.39
vem	3.6077e-2	9.5224e-3	2.5247e-3	6.2539e-4	1.95
$uhem$	1.2475e-1	3.2450e-2	8.4751e-3	1.2135e-3	2.23
$vhem$	1.5862e-1	4.2198e-2	1.1231e-2	2.7971e-3	1.94
$utet$	3.3095e-2	9.0572e-3	2.3860e-3	6.4035e-4	1.90
$vtem$	4.0094e-2	1.1337e-2	3.0456e-3	1.0232e-3	1.76
wem	2.4300e-2	7.7522e-3	2.1828e-3	1.2515e-3	1.43

2, $2.6042e-3$ and $2.6042e-3$. Take the iterative control tolerance as 1×10^{-8}, and take 100 as the maximum iterative number between two adjoining temporal steps.

Use uem, vem, $uhem$, $vhem$, $utet$, $vtem$, wem to express the errors in different forms between the approximation solution obtained by the iterative procedure and that of the original problem (1), where $uem = \max\limits_{0\le n\le N}\|U^{n\,(s)}-u^n\|$, $vem = \max\limits_{0\le n\le N}\|V^{n\,(s)}-v^n\|$, $uhem = \max\limits_{0\le n\le N}\|\delta U^{n\,(s)}-\delta u^n\|$, $vhem = \max\limits_{0\le n\le N}\|\delta V^{n\,(s)}-\delta v^n\|$, $utet = [\sum\limits_{n=1}^{N}\|d_tU^{n\,(s)}-d_tu^n\|\tau]^{\frac{1}{2}}$, $vtem = \max\limits_{1\le n\le N}\|d_tV^{n\,(s)}-d_tv^n\|$ and $wem = \max\limits_{0\le n\le N}\|W^{n\,(s)}-w^n\|$, $N\le M$.

For Picard-Newton iteration (5)-(9) with the four group step lengths, those errors in the above seven norms are up bounded by $1.5862e-1$, $4.2198e-2$, $1.1231e-2$ and $2.7971e-3$ respectively, which are accordant with the theoretical expected error bounds.

Table I gives the data and order of the approximation errors of the Picard-Newton iteration. Herein ue, ve, uhe, vhe and we stand for the errors at the end of the computation, i.e., $ue = \|U^{M\,(s)}-v^M\|$, $ve = \|V^{M\,(s)}-v^M\|$, $uhe = \|\delta U^{M\,(s)}-\delta u^M\|$, $vhe = \|\delta V^{M\,(s)}-\delta v^M\|$ and $we = \|W^{M\,(s)}-w^M\|$. ord, the approximation order, is calculated as the average of three prediction orders obtained with applying formula $log_2(e_h/e_{\frac{h}{2}})$ on error data in two neighbor columns, and is shown around 2.

Table II compares the accuracy and efficiency of the Picard-Newton iteration and the Picard iteration scheme in [8]. Herein, $outtotal$, $intotal$ and $time$ respectively stand for the total numbers of outer iterations and inner iterations carried out and the total computation time needed. $outave$ and $inave$ are respectively the average outer and inner iteration numbers in each time step. $estop$ is the average error bound at each iterative stopping moment. It shows that less outer and inner iterations and time cost are needed to get similar accurate results for the Picard-Newton iteration than for Picard iteration. Hence the Picard-Newton iteration is more efficient than the latter.

Figures 1 and 2 illustrate the error development as time advances with a 48×48 spatial mesh for the Picard and Picard-Newton iteration respectively, and show they have similar accuracy. Herein $UERR = \|U^{n\,(s)}-v^n\|$, $VERR = \|V^{n\,(s)}-v^n\|$, $UHERR = \|\delta U^{n\,(s)}-\delta v^n\|$, $VHERR = \|\delta V^{n\,(s)}-\delta v^n\|$, $UTERR = \|d_tU^{n\,(s)}-d_tu^n\|$ and $WERR = \|W^{n\,(s)}-w^n\|$.

Figures 3 and 4 present the iteration number with a 48×48 spatial mesh for the Picard and Picard-Newton iteration respectively. Figures 5 and 6 respectively give the error bound at the iterative stopping moment in each time step for these

TABLE II
COMPARISON OF ACCURACY AND EFFICIENCY OF DIFFERENT ITERATIONS

	$h=1/24$		$h=1/48$	
	Picard	Picard-Newton	Picard	Picard-Newton
$outtotal$	1727	866	5115	3403
$intotal$	4317	2350	11889	9104
$outave$	6.0	3.0	4.4	3.0
$inave$	15.0	8.2	10.3	7.9
$estop$	3.2523e-9	1.6009e-11	2.8519e-9	2.8857e-10
$time$	80.375	60.610	5076.547	3624.750
ue	1.2198e-3	1.2749e-3	3.0273e-4	4.2145e-5
ve	1.4541e-3	1.5757e-3	3.5953e-4	8.7774e-5
uhe	5.4436e-3	5.6975e-3	1.3514e-3	2.8454e-4
vhe	6.4612e-3	7.0010e-3	1.5998e-3	4.0948e-4
we	2.0818e-3	2.1520e-3	5.3325e-4	7.6021e-4
uem	1.8356e-3	1.8807e-3	4.6051e-4	1.9413e-4
vem	2.4225e-3	2.5247e-3	6.0667e-4	6.2539e-4
$uhem$	8.2619e-3	8.4751e-3	2.0732e-3	1.2135e-3
$vhem$	1.0775e-2	1.1231e-2	2.7032e-3	2.7971e-3
$utet$	2.3722e-3	2.3860e-3	6.0265e-4	6.4035e-4
$vtem$	3.0004e-3	3.0456e-3	7.6127e-4	1.0232e-3
wem	2.1275e-3	2.1828e-3	5.4920e-4	1.2515e-3

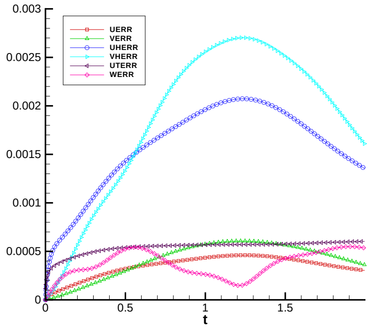

Fig. 1. Error development for Picard iteration with a 48×48 mesh

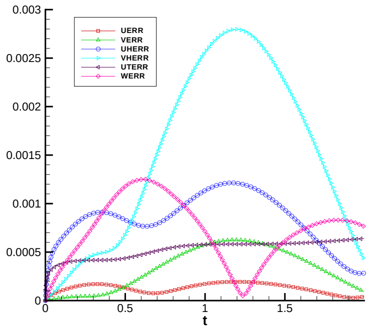

Fig. 2. Error development for Picard-Newton iteration with a 48×48 mesh

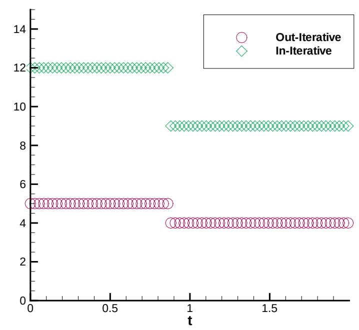

Fig. 3. Iteration number for Picard iteration with a 48 × 48 mesh

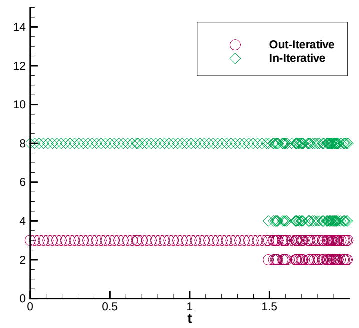

Fig. 4. Iteration number for Picard-Newton iteration with a 48 × 48 mesh

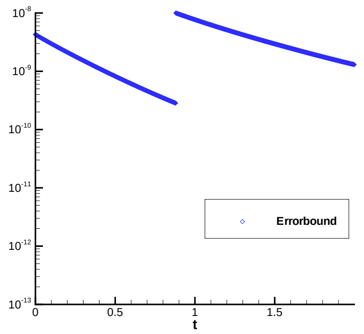

Fig. 5. Error bound at the iterative stopping moment in each time step for Picard iteration with a 48 × 48 mesh

two iteration schemes. Again it shows the good accuracy and better efficiency of the Picard-Newton iteration.

VII. Conclusion

In this paper, a Picard-Newton iteration is proposed to accelerate the resolving of a two-dimensional nonlinear coupled parabolic-hyperbolic system. It is constructed by adding some higher-order terms of small quantity on an existing Picard iteration through a linearization-discretization approach. Theoretical analysis is given on the approximation and convergence properties of the iteration, which shows its solution has second order spatial approximation and first order temporal approximation to the exact solution of the original problem, and converges to the solution of the nonlinear fully discrete scheme with a quadratic ratio. Numerical experiments verify the results of theoretical analysis and show this Picard-Newton iteration is more efficient than the Picard iteration scheme with linear convergent ratio. The idea can be extended to three-dimensional problems. Further works on more efficient iteration acceleration with second order accuracy both in spatial and temporal variants are in consideration.

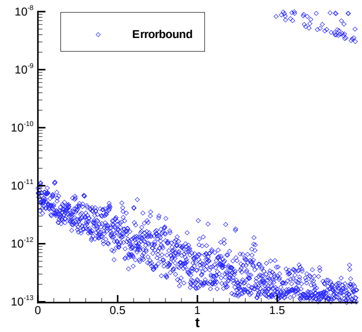

Fig. 6. Error bound at the iterative stopping moment in each time step for Picard-newton iteration with a 48 × 48 mesh

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