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Isomorphism on Fuzzy Graphs

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***Abstract*—**In this paper, the order, size and degree of the nodes of the isomorphic fuzzy graphs are discussed. Isomorphism between fuzzy graphs is proved to be an equivalence relation. Some properties of self complementary and self weak complementary fuzzy graphs are discussed.

***Keywords*—**complementary fuzzy graphs, co-weak isomorphism, equivalence relation, fuzzy relation, weak isomorphism.

I. INTRODUCTION

THE concept of fuzzy relations which has a widespread application in pattern recognition was introduced by Zadeh in his classical paper in 1965. P. Bhattacharya in [1] showed that a fuzzy graph can be associated with a fuzzy group in a natural way as an automorphism group. K.R .Bhutani in [2] introduced the concept of weak isomorphism and isomorphism between fuzzy graphs. In this paper some properties of isomorphic fuzzy graphs are discussed.

II. PRELIMINARIES

A fuzzy graph with S as the underlying set is a pair $G:(\sigma,\mu)$ where $\sigma : S \to [0,1]$ is a fuzzy subset, $\mu : S \times S \to [0,1]$ is a fuzzy relation on the fuzzy subset σ, such that $\mu(\mathrm{x},\mathrm{y}) \le \sigma(\mathrm{x}) \wedge \sigma(\mathrm{y})\ \forall \mathrm{x},\mathrm{y} \in \mathrm{S}$.
Throughout this paper S is taken as a finite set. $\sup p(\sigma) = \{u / \sigma(u) > 0\}$ } and $\sup p(\mu) = \{(u,v) / \mu(u,v) > 0\}$. For the definitions that follow let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be the fuzzy graphs with underlying sets S and S' respectively.

Definition 2.1 [6] : Given a fuzzy graph $G:(\sigma,\mu)$ with the underlying set S, the order of G is defined and denoted as $p = \sum_{x \in S} \sigma(x)$ and size of G is defined and denoted as

$$q = \sum_{x,y \in S} \mu(x,y)$$

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III. ISOMORPHISM – BASIC PROPERTIES

Definition 3.1 [2]: A homomorphism of fuzzy graphs $h : G \to G'$ is a map $h : S \to S'$ which satisfies $\sigma(x) \le \sigma'(h(x)) \forall x \in S$ and

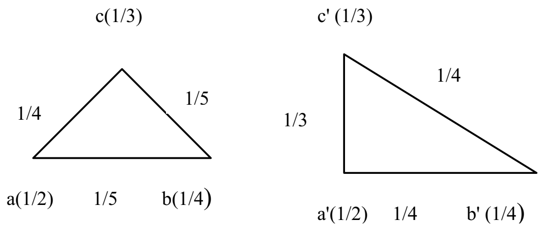
$$\mu(x,y) \le \mu'(h(x),h(y)) \forall x,y \in S$$

Definition 3.2[2]: A weak isomorphism $h : G \to G'$ is a map, $h : S \to S'$ which is a bijective homomorphism that satisfies $\sigma(x) = \sigma'(h(x)) \forall x \in S$.

Example 3.3: Let $G:(\sigma,\mu)$ and $G':(\sigma',\mu')$ be the fuzzy graphs with underlying sets $S = \{a,b,c\}$ and $S' = \{a',b',c'\}$ where $\sigma : S \to [0,1]$, $\mu : S \times S \to [0,1]$ $\sigma' : S' \to [0,1]$, $\mu' : S' \times S' \to [0,1]$ are defined as $\sigma(a) = 1/2$, $\sigma(b) = 1/4$, $\sigma(c) = 1/3$; $\mu(a,b) = 1/5$, $\mu(b,c) = 1/5$, $\mu(a,c) = 1/4$; $\sigma'(a') = 1/2$, $\sigma'(b') = 1/4$, $\sigma'(c') = 1/3$ and $\mu'(a',b') = 1/4$, $\mu'(b',c') = 1/4$, $\mu'(a',c') = 1/3$;

Defining $h : S \to S'$ as $h(a) = a'$, $h(b) = b'$, $h(c) = c'$ this h is a bijective mapping satisfying $\sigma(x) = \sigma'(h(x)) \forall x \in S$ $\mu(x,y) \le \mu'(h(x),h(y)) \forall x,y \in S$.

Fig. 1(a) is weak isomorphic to Fig.1(b)



(a) G: (σ , μ) (b) G': (σ', μ')

Fig. 1 Weak Isomorphism

Definition 3.4 [2]: A co- weak isomorphism $h : G \to G'$ *is a* map, $h : S \to S'$ which is a bijective homomorphism that satisfies $\mu(x,y) = \mu'(h(x),h(y)) \forall x,y \in S$.

Example 3.5: Let G: (σ , μ) and G': (σ', μ') be the fuzzy graphs with underlying sets S and S' respectively with $S = \{a,b,c\}$ and $S' = \{a',b',c'\}$ where $\sigma : S \to [0,1]$, $\mu : S \times S \to [0,1]$ $\sigma' : S' \to [0,1]$, $\mu' : S' \times S' \to [0,1]$

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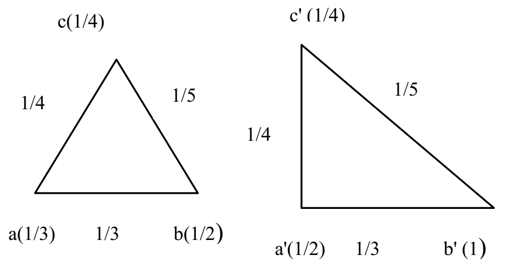
are defined as $\sigma(a) = 1/3$, $\sigma(b) = 1/2$, $\sigma(c) = 1/4$; $\mu(a,b) = 1/3$, $\mu(b,c) = 1/5$ $\mu(a,c) = 1/4$; $\sigma'(a') = 1/2$, $\sigma'(b') = 1$, $\sigma'(c') = 1/4$;and $\mu'(a',b') = 1/3$, $\mu'(b',c') = 1/5$, $\mu'(a',c') = 1/4$;
Defining $h: S \to S'$ as $h(a) = a'$, $h(b) = b'$ $h(c) = c'$; This h is a bijective mapping satisfying
$\sigma(x) \le \sigma'(h(x)) \forall x \in S$
$\mu(x,y) = \mu'(h(x),h(y)) \forall x,y \in S$
so that Fig 2(a) is co-weak isomorphic to Fig 2(b).



(a) G: (σ, μ) (b) G': (σ', μ')

Fig. 2 Co-weak Isomorphism

Definition 3.6[2]: An isomorphism $h: G \to G'$ is a map $h: S \to S'$ which is bijective that satisfies
$\sigma(x) = \sigma'(h(x)) \forall x \in S$
$\mu(x,y) = \mu'(h(x),h(y)) \forall x,y \in S$
We denote it as G ≅ G'.

Remark 3.7

1. A weak isomorphism preserves the weights of the nodes but not necessarily the weights of the edges.
2. A co- weak isomorphism preserves the weights of the edges but not necessarily the weights of the nodes.
3. An isomorphism preserves the weights of the edges and the weights of the nodes.
4. An endomorphism of a fuzzy graph G: (σ, μ) is a homomorphism of G to itself.
5. An automorphism of a fuzzy graph G: (σ, μ) is an isomorphism of G to itself
6. When the two fuzzy graphs G & G' are same the weak isomorphism between them becomes an isomorphism and similarly the co-weak isomorphism between them also becomes isomorphism.

In crisp graph, when two graphs are isomorphic they are of same size and order. The following theorem is analogous to this.

Theorem 3.8: For any two isomorphic fuzzy graphs their order and size are same.

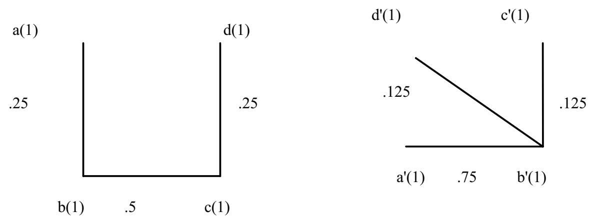
Proof: If $h: G \to G'$ is an isomorphism between the fuzzy graphs G & G' with the underlying sets S & S' respectively then $\sigma(x) = \sigma'(h(x)) \forall x \in S$
$\mu(x,y) = \mu'(h(x),h(y)) \forall x,y \in S$

(i) order (G) = $\sum_{x \in S} \sigma(x) = \sum_{x \in S} \sigma'(h(x))$ = order (G')

(ii)Size (G) = $\sum_{x,y \in S} \mu(x,y) = \sum_{x,y \in S} \mu'(h(x),h(y))$ = Size(G')

Corollary 3.9: Converse of the above theorem need not be true.

Example 3.10: Consider the fuzzy graphs G and G' with underlying sets S and S' as S= {a, b, c, d}, S' = {a', b', c', d'} $\sigma(x) = 1 \forall x \in S$, $\mu(a,b)$ =0.25, $\mu(b,c)$ =0.5, $\mu(c,d)$ = 0.25 $\sigma'(x) = 1 \forall x \in S$, $\mu'(a',b')$ =0.75, $\mu'(b',c')$ =0.125, $\mu'(b',d')$ =0.125. In these two graphs p = 4, q =1. But G is not isomorphic to G'.

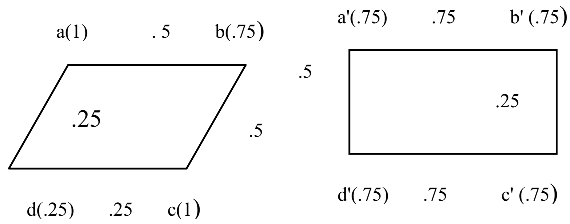


(a) G: (σ, μ) (b) G': (σ', μ')

Fig. 3 Graphs of same order and size but not isomorphic

Remark 3.11: If the fuzzy graphs are weak isomorphic then their orders are same. But the fuzzy graphs of same order need not be weak isomorphic.

Example 3.12:



(a) G: (σ, μ) (b) G': (σ', μ')

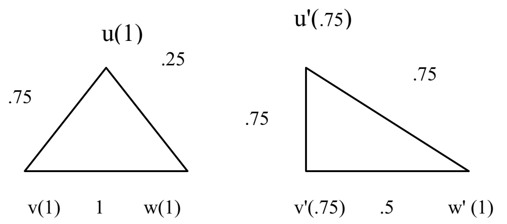
Fig. 4 Graphs of same order but not weak isomorphic

The above two fuzzy graphs are of order 3 but they are not weak isomorphic.

Remark 3.13: If the fuzzy graphs are co-weak isomorphic their sizes are same. But the fuzzy graphs of same size need not be co-weak isomorphic.

Example 3.14:

The following two fuzzy graphs in Fig 5 are of size 2, but they are not co-weak isomorphic



(a) G: (σ, μ) (b) G': (σ', μ')

Fig. 5 Graphs of same size but not co- weak isomorphic

Definition 3.15[3]: Let G: (σ, μ) be a fuzzy graph. The degree of a vertex 'u' is defined as $d(u) = \sum_{\substack{v \neq u \\ v \in S}} \mu(u,v)$

Theorem 3.16: If G and G' are isomorphic fuzzy graphs then the degrees of their nodes are preserved.

Proof: Let h: S $\rightarrow$ S' be an isomorphism of G onto G'. By the definition of isomorphism

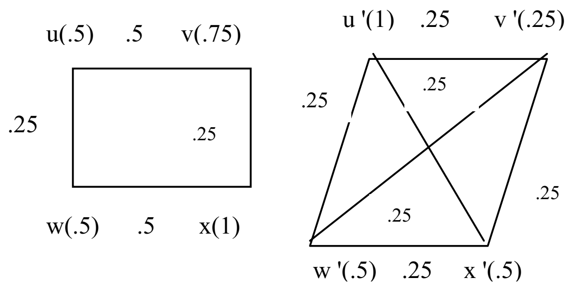
$\mu(x,y) = \mu'(h(x),h(y)) \forall x, y \in S$, So

d(u) = $\sum_{\substack{v \neq u \\ v \in S}} \mu(u,v) = \sum_{\substack{v \neq u \\ v \in S}} \mu'(h(u),h(v))$ = d(h(u)).

Corollary 3.17: Converse of the above theorem need not be true. Consider G :(σ, μ) with the underlying set S= {a, b} where σ (a) =1/2 & σ (b) =1/4 and μ (a,b)=1/4;
G': (σ', μ') with the underlying set S'= {a', b'} where σ'(a') =1/2 and σ' (b') =3/4 and μ' (a',b')=1/4 ;
Here d(a)= d(b)= d(a')=d(b')=1/4 but G and G' are not isomorphic, only co-weak isomorphic.

Remark 3.18: The degree of a vertex is measured only by adding the weights of the edges incident with that vertex. But fuzzy graphs preserving the degree of the vertices need not be co-weak isomorphic.

Example 3.19:



(a) G: (σ, μ) (b) G': (σ', μ')

Fig. 6 Graphs preserving the degree of the vertices

In the above two graphs each vertex is of degree .75. But those two are neither co-weak nor weak isomorphic graphs.

Theorem 3.20: Isomorphism between fuzzy graphs is an equivalence relation.

Proof: Let G: (σ, μ), G':(σ', μ'), G" :(σ", μ") be fuzzy graphs with underlying sets S, S' and S" respectively.

(i) Reflexive: Consider the identity map $h: S \rightarrow S$ $\ni$: h(x) = x for all x in S. This h is a bijective map satisfying

$\sigma(x) = \sigma(h(x)) \forall x \in S$.

$\mu(x,y) = \mu(h(x),h(y)) \forall x, y \in S$

Hence h is an isomorphism of the fuzzy graph to itself. Therefore it satisfies reflexive relation.

(ii) Symmetric: Let $h: S \rightarrow S'$ be an isomorphism of G onto G' then h is a bijective map

$h(x) = x', x \in S$ satisfying (1)

$\sigma(x) = \sigma'(h(x)) \forall x \in S$ and

$\mu(x,y) = \mu'(h(x),h(y)) \forall x, y \in S$ (2)

As h is bijective, by (1) $h^{-1}(x') = x \forall x' \in S'$;Using (2)

$\sigma(h^{-1}(x')) = \sigma'(x') \forall x' \in S'$

$\mu(h^{-1}(x'), h^{-1}(y')) = \mu'(x',y') \forall x', y' \in S'$ (3)

Hence we get a 1-1, onto map $h^{-1}: S' \rightarrow S$, which is an isomorphism from G' to G. i.e G $\cong$ G' $\Rightarrow$ G' $\cong$ G.

(iii) Transitive: Let $h: S \rightarrow S'$ and $g: S' \rightarrow S''$ be isomorphisms of the fuzzy graphs G onto G' and G' onto G" respectively.

Then g$\circ$h is a 1-1 onto map from S to S" where

$(g \circ h)(x) = g(h(x)) \forall x \in S$.

As $h: S \rightarrow S'$ is an isomorphism $h(x) = x', x \in S$

$\sigma(x) = \sigma'(h(x)) \forall x \in S$

$\mu(x,y) = \mu'(h(x),h(y)) \forall x, y \in S$

i.e, $\sigma(x) = \sigma'(x') \forall x \in S$ (4)

$\mu(x,y) = \mu'(x',y') \forall x, y \in S$ (5)

As g is an isomorphism from S' to S" we have

$g(x') = x'', x' \in S'$ &

$\sigma'(x') = \sigma''(g(x')) \forall x' \in S'$, (6)

$\mu'(x',y') = \mu''(g(x'),g(y')) \forall x', y' \in S'$ (7)

From (4) , (6) and using $h(x) = x', x \in S$

$$\begin{aligned}\sigma(x) &= \sigma'(x') = \sigma''(g(x')) \forall x' \in S' \\ &= \sigma''(g(h(x))) \forall x \in S.\end{aligned}$$

From (5) and (7) we have,

$$\begin{aligned}\mu(x,y) &= \mu'(x',y') \forall x, y \in S \\ &= \mu''(g(x'),g(y')) \forall x', y' \in S' \\ &= \mu''(g(h(x)),g(h(y))) \forall x, y \in S.\end{aligned}$$

Therefore g$\circ$h is an isomorphism between G and G". Hence isomorphism between fuzzy graphs is an equivalence relation.

Theorem 3.21: Weak isomorphism between fuzzy graphs satisfies the partial order relation.

Proof: Let G: (σ, μ), G':(σ', μ'), G": (σ", μ") be fuzzy graphs with underlying sets S, S', S" respectively.

(i) Reflexive: Consider the identity map $h: S \rightarrow S$, $\ni$: h(x) = x for all x in S. This h is a bijective map satisfying

$$\sigma(x) = \sigma(h(x)) \forall x \in S$$
$$\mu(x, y) = \mu(h(x), h(y)) \forall x, y \in S$$

Hence h is a weak isomorphism of the fuzzy graph to itself. Therefore G is weak isomorphic to itself

(ii) Anti symmetric: Let h be a weak isomorphism between G and G' and g be a weak isomorphism between G' and G. i.e h: S $\rightarrow$ S' is a bijective map $h(x) = x', x \in S$ satisfying

$$\sigma(x) = \sigma'(h(x)) \forall x \in S \quad \&$$
$$\mu(x, y) \le \mu'(h(x), h(y)) \forall x, y \in S \tag{8}$$

and g:S' $\rightarrow$ S is a bijective map satisfying

$$\sigma'(x') = \sigma(g(x')) \forall x' \in S' \quad \&$$
$$\mu'(x', y') \le \mu(g(x'), g(y')) \forall x', y' \in S' \tag{9}$$

The inequalities (8) and (9) hold good on the finite sets S and S' only when G and G' have the same number of edges and the corresponding edges have same weight. Hence G and G' are identical.

(iii) Transitive: Let $h: S \rightarrow S'$ and $g: S' \rightarrow S''$ be weak isomorphism of the fuzzy graphs G onto G' and G' onto G'' respectively.

Then g $\circ$ h is a 1-1 onto map from S to S'' where

$$(g \circ h)(x) = g(h(x)) \forall x \in S$$

As h is a weak isomorphism,

$$h(x) = x' \forall x \in S, \sigma(x) = \sigma'(h(x)) \forall x \in S \tag{10}$$
$$\mu(x, y) \le \mu'(h(x), h(y)) \forall x, y \in S \tag{11}$$

As g is a weak isomorphism from S' to S'' we have,

$$g(x') = x'', x' \in S', \sigma'(x') = \sigma''(g(x')) \forall x' \in S' \tag{12}$$
$$\mu'(x', y') \le \mu''(g(x'), g(y')) \forall x', y' \in S' \tag{13}$$

From (10) & (12),

$$\begin{aligned}\sigma(x) &= \sigma'(x') \forall x \in S \\ &= \sigma''(g(x')) \forall x' \in S' \\ &= \sigma''(g(h(x))) \forall x \in S\end{aligned}$$

From (10), (11),(12) & (13),

$$\begin{aligned}\mu(x, y) &\le \mu'(x', y') \forall x, y \in S \\ &\le \mu''(g(x'), g(y')) \forall x', y' \in S' \\ &= \mu''(g(h(x)), g(h(y))) \forall x, y \in S\end{aligned}$$

Therefore g $\circ$ h is a weak isomorphism between G and G''.
i.e weak isomorphism satisfies transitivity.

Hence weak isomorphism between fuzzy graphs is a partial order relation.

IV. ISOMORPHIC GRAPHS AND THEIR COMPLEMENTS

Definition 4.1 [7]: Let G: (σ, μ) be a fuzzy graph .The complement of G is defined as $\overline{G}$: (σ, $\overline{\mu}$) where

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \forall x, y \in S$$

Theorem 4.2: Two fuzzy graphs are isomorphic if and only if their complements are isomorphic.

Proof: G :(σ, μ) & G': (σ', μ') be the two fuzzy graphs given.

Assume G $\cong$ G'.

There exists a bijective map $h: S \rightarrow S'$ satisfying

$$\sigma(x) = \sigma'(h(x)) \forall x \in S \quad \&$$
$$\mu(x, y) = \mu'(h(x), h(y)) \forall x, y \in S$$ .By definition

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \forall x, y \in S$$
$$\begin{aligned}\overline{\mu}(x, y) &= \sigma'(h(x)) \wedge \sigma'(h(y)) - \mu'(h(x), h(y)) \\ &= \overline{\mu'}(h(x), h(y)) \forall x, y \in S\end{aligned}$$

i.e $\overline{G} \cong \overline{G'}$

Conversely, assume that $\overline{G} \cong \overline{G'}$;

i.e there exists a bijective map $g: S \rightarrow S'$ satisfying

$$\sigma(x) = \sigma'(g(x)) \forall x \in S \tag{14}$$
$$\overline{\mu}(x, y) = \overline{\mu'}(g(x), g(y)) \forall x, y \in S \tag{15}$$

Using the definition of complement

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \forall x, y \in S$$
$$\overline{\mu'}(g(x), g(y)) = \sigma'(g(x)) \wedge \sigma'(g(y)) - \mu'(g(x), g(y)$$

Using the above two equations in (15) and by (14)

$$\mu(x, y) = \mu'(g(x), g(y)) \forall x, y \in S \tag{16}$$

Hence from (14) and (16) $g: S \rightarrow S'$ is an isomorphism between G and G'. i.e. $G \cong G'$

Theorem 4.3: If there is a weak isomorphism between G and G' then there is a weak isomorphism between $\overline{G'}$ and $\overline{G}$.

Proof: If h is a weak isomorphism between G & G' then,

$h: S \rightarrow S'$ is a bijective map that satisfies

$$h(x) = x', x \in S, \sigma(x) = \sigma'(h(x)) \forall x \in S \tag{17}$$
$$\mu(x, y) \le \mu'(h(x), h(y)) \forall x, y \in S \tag{18}$$

As $h^{-1}: S' \rightarrow S$ is also bijective for every x' in S' there is an x $\in$ S such that $h^{-1}(x') = x$. Using this in (17)

$$\sigma'(x') = \sigma(h^{-1}(x')) \forall x' \in S' \tag{19}$$

Also, by using (17) & (18) in

$$\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \forall x, y \in S$$
$$\begin{aligned}&\overline{\mu}(h^{-1}(x'), h^{-1}(y') \\ &\ge \sigma'(h(x)) \wedge \sigma'(h(y)) - \mu'(h(x), h(y)) \\ &= \sigma'(x') \wedge \sigma'(y') - \mu'(x', y') \forall x', y' \in S' \\ &= \overline{\mu'}(x', y') \forall x', y' \in S'\end{aligned}$$

$$\text{i.e } \overline{\mu'}(x', y') \le \overline{\mu}(h^{-1}(x'), h^{-1}(y')) \tag{20}$$

Thus $h^{-1}: S' \rightarrow S$ is a bijective map, which is a weak isomorphism between $\overline{G'}$ & $\overline{G}$ by (19) and (20).

Theorem 4.4: If there is a co-weak isomorphism between G & G' then there can be a homomorphism between $\overline{G}$ and $\overline{G'}$.

Proof: Let h be co-weak isomorphism between G and G' respectively ie, $h: S \to S'$ is a bijective map that satisfies

$$\sigma(x) \le \sigma'(h(x)) \forall x \in S \quad (21)$$

$$\mu(x, y) = \mu'(h(x), h(y)) \forall x, y \in S \quad (22)$$

In $\overline{G}$, using (21) and (22)

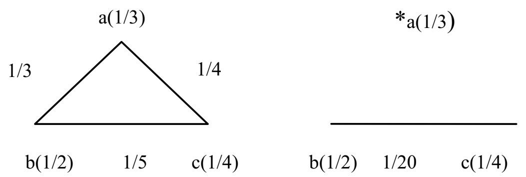
$$\begin{aligned} \overline{\mu}(x, y) &= \sigma(x) \wedge \sigma(y) - \mu(x, y) \forall x, y \in S \\ &\le \sigma'(h(x)) \wedge \sigma'(h(y)) - \mu'(h(x), h(y)) \\ &= \overline{\mu'}(h(x), h(y)) \forall x, y \in S \end{aligned}$$

Hence $\overline{\mu}(x, y) \le \overline{\mu'}(h(x), h(y)) \forall x, y \in S$ (23)

By (21) and (23) h is a bijective homomorphism between $\overline{G}$ and $\overline{G'}$.

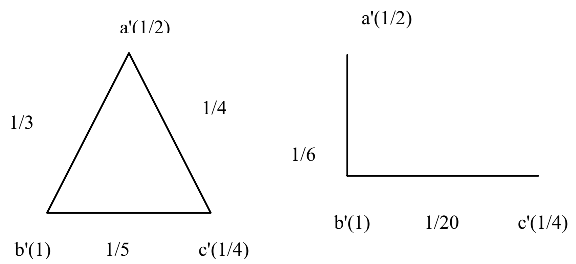
Remark 4.5: If G is co–weak isomorphic to G', then $\overline{G}$ and $\overline{G'}$ need not be co-weak isomorphic.

Example 4.6: Consider the example 3.5, in which there is a co-weak isomorphism between G and G'. i.e Here Fig 7(a) is co-weak isomorphic with fig 8(a)



(a) G : (σ, μ) (b) $\overline{G}$: (σ, $\overline{\mu}$)

Fig. 7 Fuzzy Graph G and its complement $\overline{G}$



(a) G': (σ', μ') (b) $\overline{G'}$: ($\overline{\sigma'}$, $\overline{\mu'}$)

Fig. 8 Fuzzy Graph G' and its complement $\overline{G'}$

But there is no co-weak isomorphism between Fig 7 (b) and Fig 8(b) whereas there is a homomorphism between $\overline{G}$ and $\overline{G'}$.

Theorem 4.7 [3]: Let (σ_1, μ_1) and (σ_2, μ_2) be the two partial fuzzy subgraphs of the fuzzy graphs G_1 and G_2 respectively. If 'h' is an isomorphism of (σ_1, μ_1) and (σ_2, μ_2) then it is an isomorphism of (supp σ_1, supp μ_1) onto (supp σ_2, supp μ_2) .

Definition 4.8[6]: A fuzzy graph G: (σ, μ) with underlying set S is said to be a complete fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y) \forall x, y \in S$ and denoted as K_σ .

Theorem 4.9: Any simple fuzzy graph G: (σ, μ) of order 'n' is isomorphic to a partial fuzzy subgraph of the complete fuzzy graph K_σ of order 'n'

Proof: Let G: (σ, μ) be the given fuzzy graph of order 'n'. Now construct a K_σ of order 'n', as a fuzzy graph (σ, μ')

where $\mu'(x, y) = \sigma(x) \wedge \sigma(y) \forall x, y \in S$ where S is the under lying set of G.

As in general $\mu(x, y) \le \sigma(x) \wedge \sigma(y) \forall x, y \in S$

$$\mu(x, y) \le \mu'(x, y) \forall x, y \in S$$

Hence G is a fuzzy spanning subgraph of K_σ i.e G is isomorphic to a partial fuzzy sub graph of the complete fuzzy graph K_σ of order 'n'.

Definition 4.10[3]: Let G:(σ, μ) be a fuzzy graph with the underlying graph (V,E).The fuzzy line graph of G is L(G): (ω, λ) with the underlying graph (Z,W) where the node set Z is

$$\{S_x = \{x\} \cup \{u_x, v_x\} / x \in E, u_x, v_x \in V, x = (u_x, v_x)\}$$

$$W = \{(S_x, S_y) / S_x \cap S_y \ne \phi, x, y \in E, x \ne y\}$$

$\omega(S_x) = \mu(x) \forall S_x \in Z$ and

$$\lambda(S_x, S_y) = \mu(x) \wedge \mu(y) \forall (S_x, S_y) \in W$$

Theorem 4.11: If G_1 :(σ_1, μ_1) and G_2: (σ_2, μ_2) are the two isomorphic fuzzy graphs then their fuzzy line graphs are also isomorphic

Proof: Given G_1 and G_2 are the two isomorphic fuzzy graphs with the underlying sets S_1& S_2 respectively. i.e there exists a bijective map $h: S_1 \to S_2$ satisfying

$$\sigma_1(x) = \sigma_2(h(x)) \forall x \in S_1$$

$$\mu_1(x, y) = \mu_2(h(x), h(y)) \forall x, y \in S_1$$

Let $L(G_1)$:(ω_1, λ_1)& $L(G_2)$: (ω_2, λ_2) be the line graphs of G_1 & G_2 respectively.

Consider an x $\in E_1$. Let $x = (u_x, v_x)$ As $h: S_1 \to S_2$ is 1-1, onto, $h(x) = (h(u_x), h(v_x)) \in E_2$.

Define g: $Z_1 \to Z_2$ as $g(S_x) = S_{h(x)}$

As h is one to one onto, g is well defined and one to one onto mapping. Consider,

$\omega_1(S_x) = \mu_1(x) \forall S_x \in Z_1$ (by definition of line graphs)

$$\omega_1(S_x) = \mu_1(u_x, v_x) = \mu_2(h(u_x), h(v_x)) = \mu_2(h(x))$$

$$\omega_1(S_x) = \omega_2(S_{h(x)}) = \omega_2(g(S_x)) \forall S_x \in Z_1 \quad (24)$$

$$\begin{aligned} \lambda_1(S_x, S_y) &= \mu_1(x) \wedge \mu_1(y) \forall (S_x, S_y) \in W_1 \\ &= \mu_1(u_x, v_x) \wedge \mu_1(u_y, v_y) \end{aligned}$$

$$= \mu_2(h(u_x), h(v_x)) \wedge \mu_2(h(u_y), h(v_y))$$
$$= \mu_2(h(x)) \wedge \mu_2(h(y))$$
$$= \lambda_2(S_{h(x)}, S_{h(y)})$$
$$= \lambda_2(g(S_x), g(S_y)) \forall x, y \in E_1$$
$$\lambda_1(S_x, S_y) = \lambda_2(g(S_x), g(S_y)) \forall S_x, S_y \in Z_1 \qquad (25)$$

From equations (24) and (25) $L(G_1)$:(ω_1, λ_1) and $L(G_2)$: (ω_2, λ_2) are isomorphic fuzzy line graphs when G_1 and G_2 are the two isomorphic fuzzy graphs.

Remark 4.12: In crisp graph we have L (C_n)≅ C_n , C_n denotes cycle of length n. But this does not hold good in fuzzy graph

Example 4.13: Consider the fuzzy cycle G: (σ, μ) with the under lying graph (V,E), represented as $V = \{u, v, w\}$ $E = \{(u,v),(v,w),(w,u)\}$ with $\sigma(u) = 3/4$, $\sigma(v) = 1$, $\sigma(w) = 1$, $\mu(u,v) = 1/2$, $\mu(u,w) = 1/2$, $\mu(v,w) = 1$

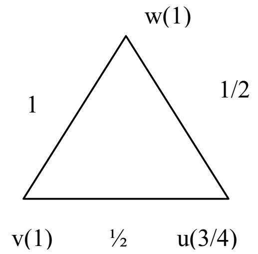


Fig. 9 *G: (σ, μ)*

The fuzzy line graph of the fuzzy cycle G:(σ, μ) is L(G): (ω, λ) with the underlying graph (Z,W) where $Z = \{S_1, S_2, S_3\}$ & $S_1 = \{(w,u),\{w\},\{u\}\}$, $S_2 = \{(u,v),\{u\},\{v\}\}$ $S_3 = \{(w,v), \{w\}, \{v\}\}$ is in the following figure with $W = \{(S_x, S_y) / S_x \cap S_y \neq \phi, x, y \in E, x \neq y\}$

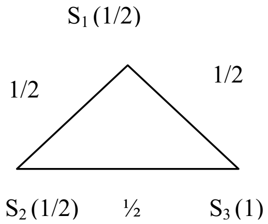


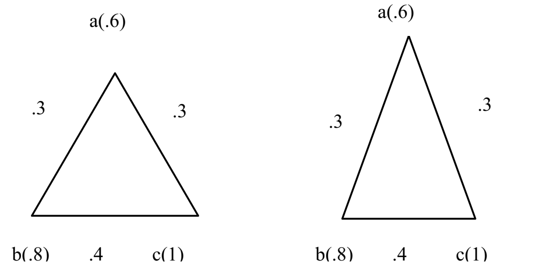
Fig. 10 L(G): (ω, λ)

As a map from V→Z preserving the weight of the vertices and weight of the arcs cannot be defined, L(G)≇G

V. SELF COMPLEMENTARY FUZZY GRAPHS

Definition 5.1 [7]: A fuzzy graph is said to be self complementary if G≅ G'

Example 5.2

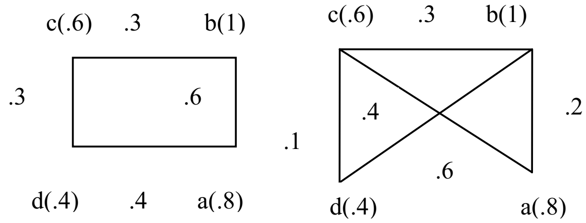


(a) G: (σ, μ) (b) $\overline{G}$: (σ, $\overline{\mu}$)

Fig. 11Self complementary fuzzy graph

In this example G ≅ $\overline{G}$

Example 5.3



(a)G: (σ, μ) (b) $\overline{G}$: (σ, $\overline{\mu}$)

Fig. 12 Fuzzy graph not self complementary.

In this example, $G \ncong \overline{G}$

Theorem 5.4 [7]: Let G: (σ, μ) be a self complementary fuzzy graph. Then $\sum_{x \neq y} \mu(x,y) = \frac{1}{2} \sum_{x \neq y} (\sigma(x) \wedge \sigma(y))$

Remark 5.5: As a consequence of the above theorem if G is a self complementary fuzzy graph, then

Size of G = q = $\frac{1}{2} \sum_{x \neq y} (\sigma(x) \wedge \sigma(y))$

But the converse of the above theorem is not true, because in example 5.3, Size of G = 1.6 and $\frac{1}{2} \sum_{x \neq y}$ (σ(x) ∧ σ(y)) = 1.6. But $G \ncong \overline{G}$

VI. SELF WEAK COMPLEMENTARY FUZZY GRAPHS

Definition 6.1: A fuzzy graph G is said to be self weak complementary fuzzy graph if G is weak isomorphic with its complement $\overline{G}$

Remark 6.2: The fuzzy graph G, Fig 12 (a) given in example5.3 is not a self weak complementary fuzzy graph

Example 6.3

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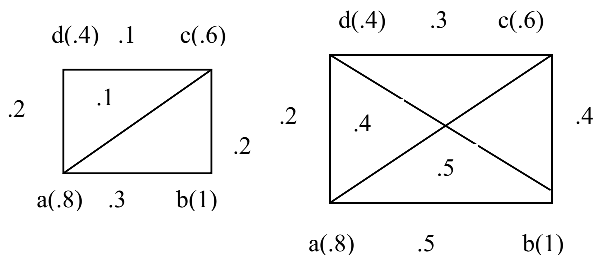

(a) G : (σ , μ) (b) $\overline{G}$: (σ , $\overline{\mu}$)

Fig. 13 Self weak complementary fuzzy graph

In this example G is weak isomorphic with $\overline{G}$, hence G is a self weak complementary fuzzy graph.

Theorem 6.4: Let G be a self weak complementary fuzzy graph then $\sum_{x\neq y}\mu(x,y)\leq\frac{1}{2}\sum_{x\neq y}(\sigma(x)\wedge\sigma(y))$

Proof: G is self weak complementary fuzzy graph .Hence G is weak-isomorphic with $\overline{G}$ .So, there exists a weak isomorphism, $h:S\rightarrow S$, a bijective mapping satisfying $\sigma(x)=\sigma(h(x))\forall x\in S$ and

$$\mu(x,y)\leq\overline{\mu}(h(x),h(y))\forall x,y\in S$$

Using the definition of complement, in the above inequality,

$$\mu(x,y)\leq\sigma(h(x))\wedge\sigma(h(y))-\mu(h(x),h(y))$$
$$=\sigma(x)\wedge\sigma(y)-\mu(h(x),h(y))$$
$$\mu(x,y)+\mu(h(x),h(y))\leq\sigma(x)\wedge\sigma(y)$$

Taking summation,

$$\sum_{x\neq y}\mu(x,y)+\sum_{x\neq y}\mu(h(x),h(y))\leq\sum_{x\neq y}\sigma(x)\wedge\sigma(y)$$

$$2\sum_{x\neq y}\mu(x,y)\leq\sum_{x\neq y}\sigma(x)\wedge\sigma(y)$$ (since Sis a finite set)

Hence $\sum_{x\neq y}\mu(x,y)\leq\frac{1}{2}\sum_{x\neq y}\sigma(x)\wedge\sigma(y)$ in S.

Theorem6.5: Let G be a fuzzy graph. If $\mu(x,y)\leq\frac{1}{2}(\sigma(x)\wedge\sigma(y))$ for all x, y in S then G is a self weak complementary fuzzy graph.

Proof: Consider the identity map $h:S\rightarrow S$

$$\sigma(x)=\sigma(h(x))\forall x\in S$$

By definition of $\overline{\mu}$

$$\overline{\mu}(x,y)=\sigma(x)\wedge\sigma(y)-\mu(x,y)\forall x,y\in S$$
$$\overline{\mu}(x,y)\geq\sigma(x)\wedge\sigma(y)-\frac{1}{2}(\sigma(x)\wedge\sigma(y)$$
$$=\frac{1}{2}(\sigma(x)\wedge\sigma(y))$$
$$\geq\mu(x,y)\forall x,y\in S$$

i.e $\mu(x,y)\leq\overline{\mu}(h(x),h(y))\forall x,y\in S$

Hence G is weak isomorphic with $\overline{G}$. Therefore G is a self weak complementary fuzzy graph.

Remark 6.6:

1. When G is co-weak isomorphic with $\overline{G}$, then G is a self complementary fuzzy graph.
2. When G is a self weak complementary fuzzy graph then order (G) =order ($\overline{G}$) and size (G) $\leq$ size ($\overline{G}$)

But the converse of the above is not true.

As in example 5.3, we have order (G) =order ($\overline{G}$) = 2.8 and size (G) = 1.6 and size ($\overline{G}$) =1.6 but G is not a self weak complementary fuzzy graph.

VII. CONCLUSION AND FUTURE STUDIES

In this paper isomorphism between fuzzy graphs is proved to be an equivalence relation and weak isomorphism is proved to be a partial order relation. Similarly it is expected that co-weak isomorphism can be proved to be a partial order relation. A necessary and then a sufficient condition for a fuzzy graph to be self weak complementary are studied. The results discussed may be used to study about various fuzzy graph invariants.

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